QUOTIENT RULE:

1) IF $g(x)$ IS DIFFERENTIABLE ON AN INTERVAL (a,b) AND $g(x)\neq 0$ IN (a,b), THEN

$\frac{1}{g(x)}$ IS DIFFERENTIABLE IN (a,b) AND

$$\left(\frac{1}{g(x)}\right)' = \frac{-g'(x)}{g(x)^2}$$

2) IF $g(x), f(x)$ ARE DIFFERENTIABLE ON (a,b) AND $g(x)\neq 0$ IN (a,b) THEN

$\frac{f(x)}{g(x)}$ IS DIFFERENTIABLE ON (a,b) AND

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

THE SECOND PART CAN BE OBTAINED BY THE FIRST USING THE PRODUCT RULE.

EXAMPLE:

$$f(x) = \frac{\log x}{5x^2}$$

$$f'(x) = \frac{(\log x)' 5x^2 - (5x^2)' \log x}{(5x^2)^2} = \frac{\frac{5x^2}{x} - 20x \log x}{25x^4}$$

$$= \frac{1 - 4\log x}{5x^3}$$

EXAMPLE:

WE CAN USE THE QUOTIENT RULE TO FIND THE DERIVATIVE OF THE TANGENT

$$\frac{d \tan x}{dx} = \frac{d \frac{\sin x}{\cos x}}{dx} = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2(x)}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

EXAMPLE:

USING THE QUOTIENT RULE WE CAN COMPUTE THE DERIVATIVE OF $\frac{1}{X^m}$:

$$\left(\frac{1}{X^m}\right)' = \frac{-(X^m)'}{(X^m)^2} = \frac{-m X^{m-1}}{X^{2m}} = \frac{-m}{X^{m+1}}$$

SO IF WE WRITE $\frac{1}{X^m} = X^{-m}$ WE GET THE USUAL RULE

$$(X^{-m})' = -mX^{-m-1} \; !$$

LET'S CONVINCE OURSELVES FOR A MOMENT THAT ALL THESE RULES ARE COMPATIBLE.

EXAMPLE: FIND $\left(\frac{X^2-5X+1}{X}\right)'$ BY APPLYING THE QUOTIENT RULE AND BY APPLYING THE PRODUCT RULE (ON $(X^2-5X+1)X^{-1}$)

- $$\left(\frac{X^2-5X+1}{X}\right)' = \frac{(X^2-5X+1)' \cdot X - (X)'(X^2-5X+1)}{X^2} = \frac{X^2+X-1}{X^2}$$
- $$\left((X^2-5X+1)X^{-1}\right)' = (X^2-5X+1)' \cdot X^{-1} + (X^{-1})'(X^2-5X+1) = \frac{X^2+X-1}{X^2}$$

GRAPHICALLY RECOGNIZING DERIVATIVES AND TANGENTS

WE SHOULD BE ABLE, GIVEN A DIFFERENTIABLE FUNCTION

- TO SKETCH THE FUNCTION
- TO FIND THE DERIVATIVE AND SKETCH IT
- TO FIND THE TANGENT AT A POINT AND SKETCH IT

FOR NOW, SUPPOSE WE CAN ACTUALLY DO THE FIRST TWO THINGS WELL (WHICH WE DON'T AT THIS POINT OF THE COURSE)

WE HAVE A FUNCTION, SAY

- $f(x) = \dfrac{1-\cos x}{x^2}$

USING THE QUOTIENT RULE WE GET

- $$f'(x) = \frac{(1-\cos x)'x^2 - (x^2)'(1-\cos x)}{x^4} = \frac{(\sin x)x^2 - 2x + 2x\cos x}{x^4}$$

$$= \frac{(\sin x)x - 2 + 2\cos x}{x^3}$$

WHAT'S THE EQUATION FOR THE TANGENT TO $f(x)$ AT $x = \frac{\pi}{4}$

$$\text{SLOPE} = \frac{\sin\left(\frac{\pi}{4}\right)\frac{\pi}{4} - 2 + 2\cos\frac{\pi}{4}}{\left(\frac{\pi}{4}\right)^3} = \frac{\frac{\sqrt{2}}{2}\cdot\frac{\pi}{4} - 2 + 2\frac{\sqrt{2}}{2}}{\frac{\pi^3}{64}}$$

EQUATION FOR THE LINE THROUGH (s_0, s_1) WITH SLOPE a?

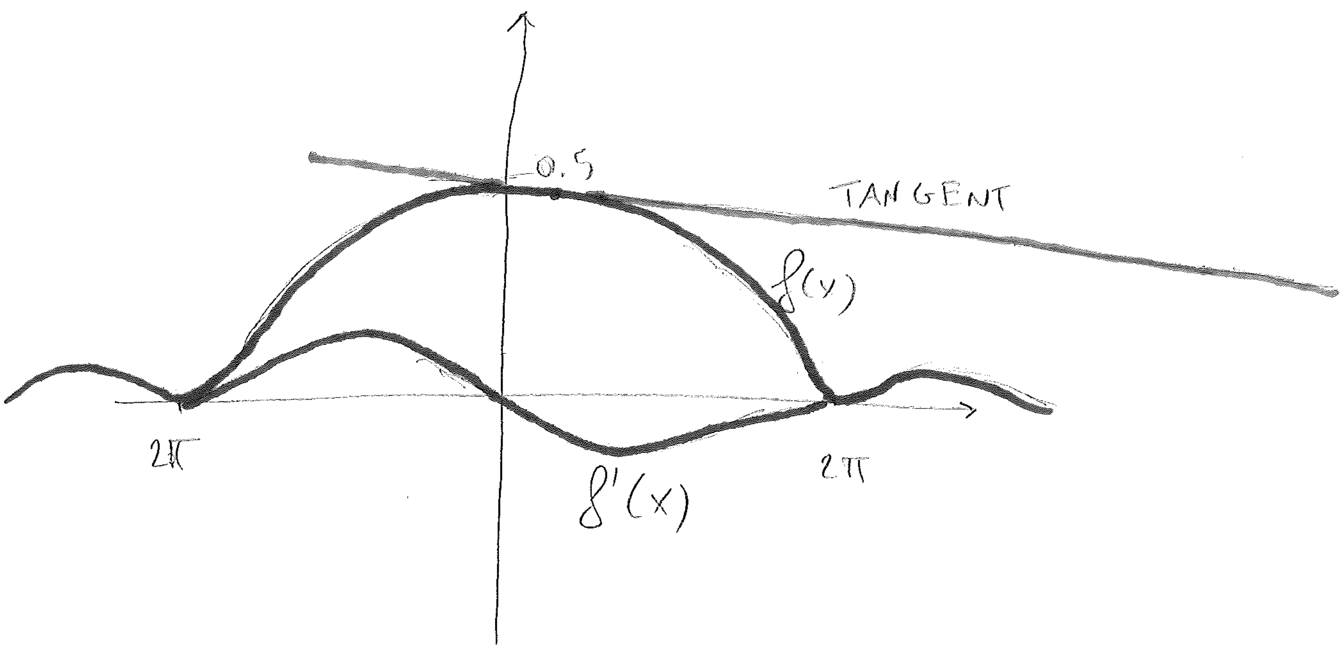
- $y = ax + b$ SO $s_1 = as_0 + b$ SO $b = s_1 - as_0$
- $s_0 = \frac{\pi}{4}$, $s_1 = f\left(\frac{\pi}{4}\right)$

THEN OUR LINE WILL BE

$$y = f'\left(\frac{\pi}{4}\right) x + \left(f\left(\frac{\pi}{4}\right) - f'\left(\frac{\pi}{4}\right)\frac{\pi}{4}\right) \quad \text{THAT IS}$$

$$y = \frac{\sqrt{2}\left(\frac{\pi}{8} - \sqrt{2} + 1\right)}{\frac{\pi^3}{64}} x + \left(\frac{1 - \frac{\sqrt{2}}{2}}{\frac{\pi}{4}} - \frac{\sqrt{2}\left(\frac{\pi}{8} - \sqrt{2} + 1\right)}{\frac{\pi^2}{16}}\right)$$

$$\simeq \quad y = -0.06\, x + 0.42$$



ONE MORE!

$$- f(x) = \frac{x^3 + 12x + 3}{e^x}$$

$$- f'(x) = \frac{(x^3 + 12x + 3)' e^x - (e^x)'(x^3 + 12x + 3)}{e^{2x}} = \frac{(3x^2 + 12)e^x - e^x(x^3 + 12x + 3)}{e^{2x}}$$

$$= \frac{-x^3 + 3x^2 - 12x + 9}{e^x}$$

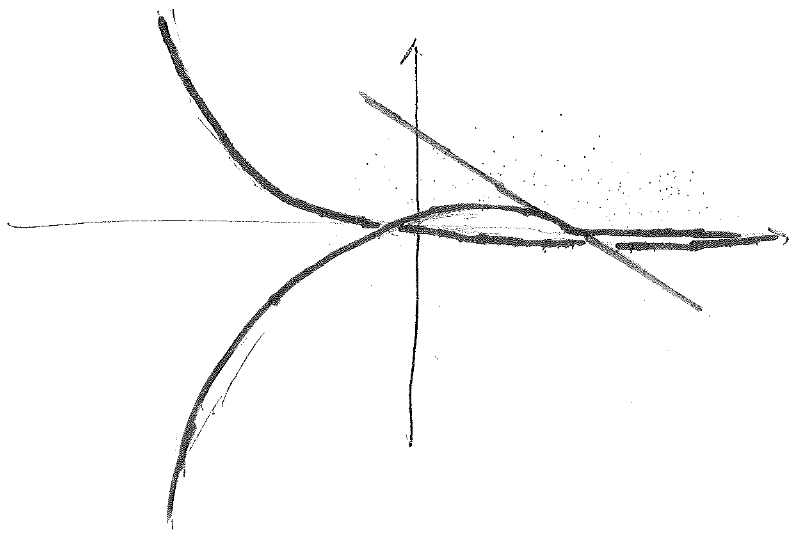
FIND THE TANGENT LINE AT $x=2$

- $f(2) = \frac{35}{e^2} \approx 4.73$

- $f'(2) = \frac{-11}{e^2} \approx -1.5$

TANGENT LINE

$$y = \frac{-11}{e^2}x + \frac{35}{e^2} + \frac{22}{e^2}.$$



HIGHER ORDER DERIVATIVES: RATES OF CHANGE OF RATES OF CHANGE OF RATES OF CHANGE...

ONE THING WE CAN DO IS TAKE THE DERIVATIVE OF THE DERIVATIVE OF A FUNCTION. IT LOOKS LIKE THIS:

- $f(x) = e^x x^2$
- $f'(x) = e^x(2x + x^2)$
- $(f'(x))' = e^x(x^2 + 4x + 2)$
- $((f'(x))')' = e^x(x^2 + 6x + 6)$

AS YOU CAN SEE, WE DID NOT ACCIDENTALLY DESTROY THE UNIVERSE.

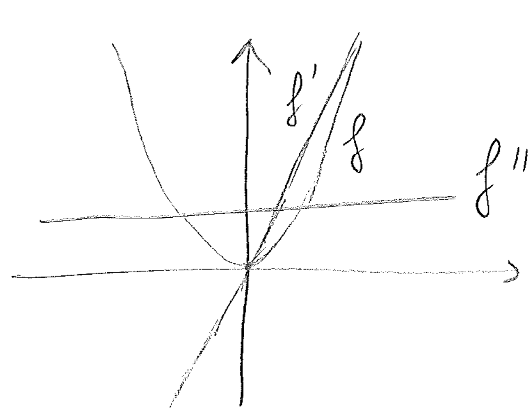
FOR NOTATIONAL SIMPLICITY, WE KEEP USING SIMILAR SYMBOLS WHEN WE TAKE A DERIVATIVE MULTIPLE TIMES.

FUNCTION	$f(x)$		
DERIVATIVE	$f'(x)$	OR	$\frac{df}{dx}(x)$
SECOND "	$f''(x)$	OR	$\frac{d^2f}{dx^2}(x)$
THIRD "	$f'''(x)$	OR	$\frac{d^3f}{dx^3}(x)$
n-TH "	$f^n(x)$	OR	$\frac{d^nf}{dx^n}(x)$

BUT WHAT DO THESE <u>HIGHER ORDER DERIVATIVES MEAN</u>?

WELL, LET'S JUST MAKE SURE WE UNDERSTAND WHAT THE SECOND DERIVATIVE MEANS.

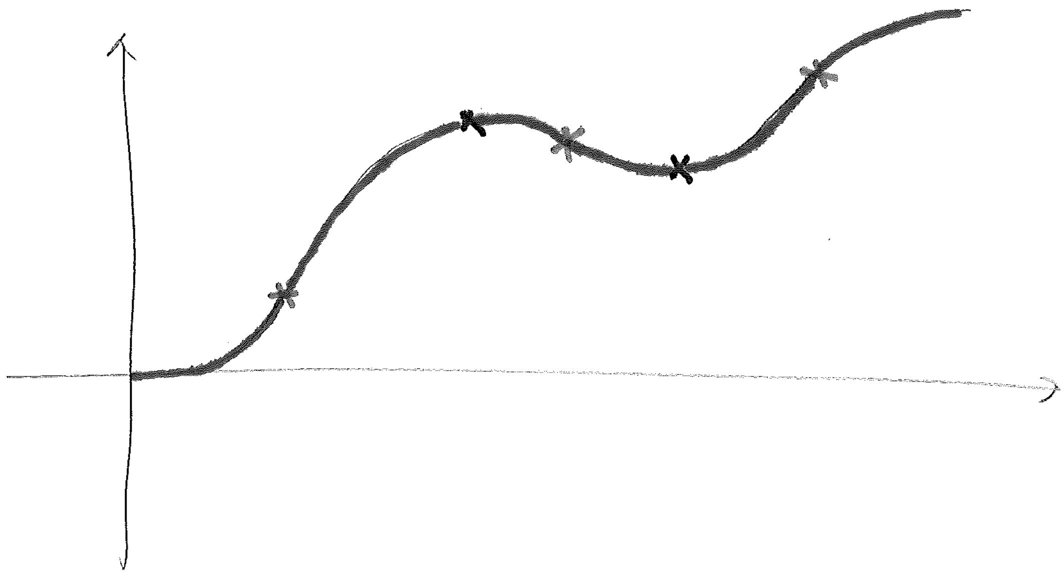
- $f(x) = 4.9x^2$ • $f'(x) = 9.8x$ • $f''(x) = 9.8$



SAY WE ARE DESCRIBING A POINT MOVING. IF $f'(x)$ IS THE VELOCITY, THEN $f''(x)$ IS THE

RATE OF CHANGE OF VELOCITY. THAT IS, $f''(x)$ EXPRESSES THE INSTANTANEOUS ACCELERATION.

BACK TO THE CAR EXAMPLE:



GIVEN THE FUNCTION ABOVE, IDENTIFY WHEN

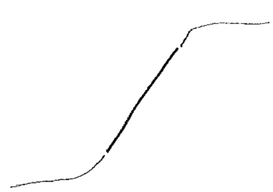
- $f''(x) > 0$ $\qquad f'(x) > 0$
- $f''(x) < 0$ $\qquad f'(x) < 0$
- $f''(x) = 0$ $\qquad f'(x) = 0$

SO $f'' > 0$ THE FUNCTION IS GETTING STEEPER

$f'' < 0$ " " " " LESS STEEP

$f'' = 0$ TRANSITION BEETWEEN THE TWO OR

f IS LOCALLY A LINE



WHY IS THE SECOND DERIVATIVE VERY IMPORTANT?

$$F = ma$$