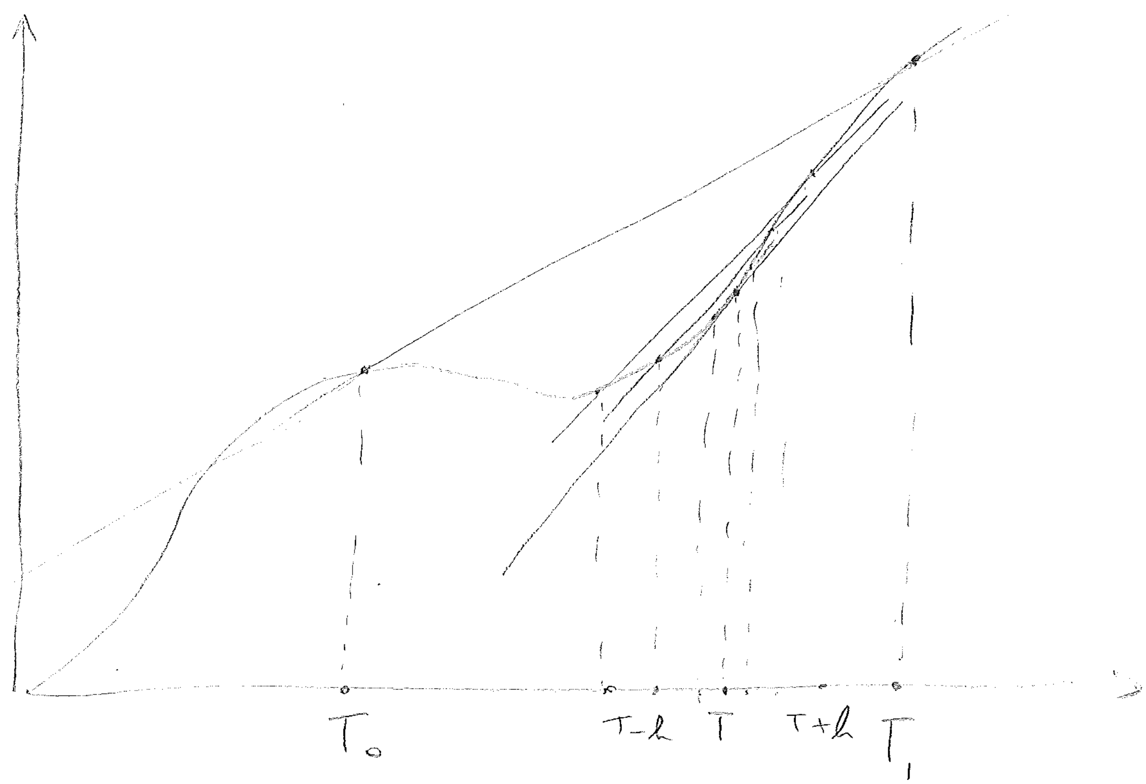
FIRST, HOW DO WE "DRAW" THE AVERAGE SPEED FROM T_0 TO T_1 ON THIS GRAPH?

AVERAGE SPEED BETWEEN T_0 AND T_1 =

$$\frac{\Delta S}{\Delta T} = \frac{S(T_1) - S(T_0)}{T_1 - T_0}$$

PICK THE POINTS $(T_0, S(T_0))$ AND $(T_1, S(T_1))$ ON THE GRAPH. TAKE THE LINE PASSING THROUGH THEM



WHAT'S THE EQUATION FOR THIS LINE?

$$y = \frac{S(T_1) - S(T_0)}{T_1 - T_0} x + \frac{S(T_1)T_0 - S(T_0)T_1}{T_1 - T_0}$$

THE AVERAGE VELOCITY IS THE SLOPE!

BUT WHAT HAPPENS IF WE SHRINK THE INTERVAL ΔT, THAT IS, WE TAKE T_0, T_1 VERY CLOSE TO EACH OTHER?

WE GET THE SLOPE OF THE TANGENT LINE!

WE CAN NOW ANSWER THE QUESTION ABOUT INSTANTANEOUS VELOCITY!

Q: WHAT IS THE INSTANTANEOUS VELOCITY AT A TIME T?

A: IT IS EQUAL TO THE LIMIT

$$v(T) = \lim_{\Delta T \to 0} \frac{s(T+\Delta T) - s(T)}{\Delta T}$$

WHICH IS <u>ALSO</u> THE SLOPE OF THE TANGENT LINE TO THE GRAPH OF s AT T!

BUT CAN WE ACTUALLY COMPUTE IT?

EX: WE DROP A BALL FROM A VERY TALL TREE. SINCE GALILEO WE KNOW THAT THE DISTANCE OUR (VERY AERODYNAMIC) BALL WILL FALL IN A TIME T IS

$$s(T) \simeq 4.9\,T^2 \text{ METERS.}$$

Q: HOW FAST IS THE BALL AFTER 1.5 SECONDS?

LET'S START BY COMPUTING THE AVERAGE SPEED CLOSE TO $T = 1.5$

- $$\frac{s(1.6) - s(1.5)}{1.6 - 1.5} = \frac{4.9(1.6)^2 - 4.9(1.5)^2}{0.1} = \frac{4.9(2.56) - 4.9(2.25)}{0.1}$$
  $= 15.19$

- $$\frac{s(1.49) - s(1.5)}{1.49 - 1.5} = \frac{-0.14651}{0.01} = 14.651$$

ΔT CAN BE NEGATIVE!

SO IT SHOULD BE AROUND 14.6-14.7, BUT CAN WE SOLVE THE LIMIT?

YES!
$$\lim_{\Delta T \to 0} \frac{4.9(1.5+\Delta T)^2 - 4.9(1.5)^2}{\Delta T} =$$

$$\lim_{\Delta T \to 0} \frac{4.9(2.25+3\Delta T+\Delta T^2 - 2.25)}{\Delta T} =$$

$$\lim_{\Delta T \to 0} \frac{14.7\,\Delta T + 4.9\,\Delta T^2}{\Delta T} = \lim_{\Delta T \to 0} 14.7 + \Delta T \cdot 4.9$$

$$= 14.7 + 0$$

THE SPEED AFTER 1.5 SECONDS IS 14.7 M/S!
BUT CAN WE COMPUTE THE SPEED AT ANY TIME, AS A FUNCTION $v(T)$? YES!

FIX T. WE WANT

$$\lim_{\Delta T \to 0} \frac{4.9(T+\Delta T)^2 - 4.9T^2}{\Delta T^2} = \lim_{\Delta T \to 0} \frac{4.9(T^2+2T\Delta T+\Delta T^2-T^2)}{\Delta T}$$

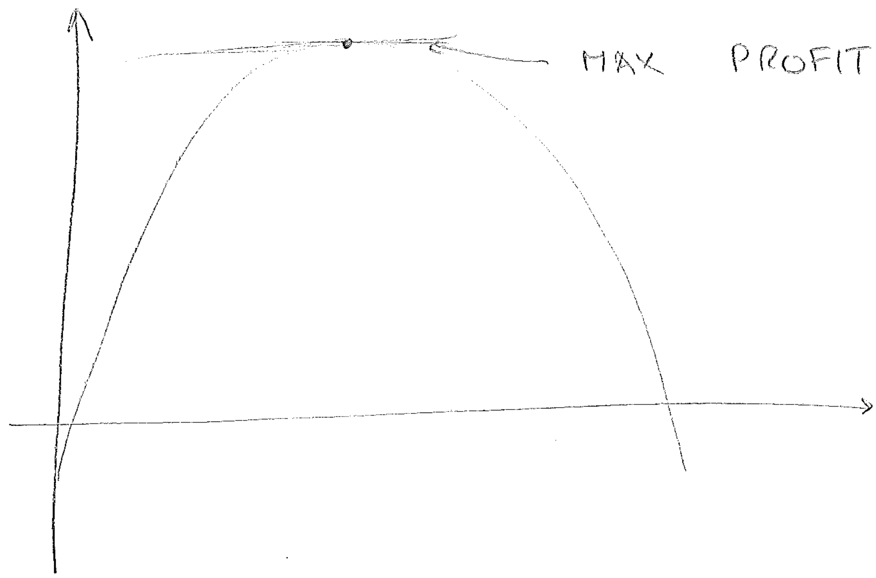
$$= \lim_{\Delta T \to 0} \frac{4.9(2T\Delta T + \Delta T^2)}{\Delta T} = \lim_{\Delta T \to 0} 4.9(2T+\Delta T)$$

$$= 9.8T \;!$$

WE CAN ALSO USE THIS TO FINISH THE BUISNESS PROBLEM FROM DAY 1!

THE PROFIT FUNCTION FOR THE PAYSTATION WAS

$$P(q) = 225q - \frac{q^2}{50} - 10\,000$$



GRAPHICALLY, WE DETERMINED THAT THE POINT OF MAX PROFIT IS THE PEAK OF THE PARABOLA. WHAT'S THE TANGENT AT THIS PEAK? A HORIZONTAL LINE! NOW WE KNOW HOW TO FIND THE SLOPE AT A POINT q!

$$\text{SLOPE}(q) = \lim_{\Delta q \to 0} \frac{P(q+\Delta q) - P(q)}{\Delta q} =$$

$$= \lim_{\Delta q \to 0} \frac{225(q+\Delta q) - \frac{(q+\Delta q)^2}{50} - 10000 - 225q + \frac{q^2}{50} + 10.000}{\Delta q}$$

$$= \lim_{\Delta q \to 0} \frac{225\Delta q - \frac{2q\Delta q}{50} - \frac{\Delta q^2}{50}}{\Delta q} = \lim_{\Delta q \to 0} 225 - \frac{2}{50}q$$

$$= 225 - \frac{2}{50}q$$

THE SLOPE OF A HORIZONTAL LINE IS 0,
SO WE SOLVE

$$225 - \frac{2}{50} q = 0 \sim q = \frac{225 \cdot 50}{2} = 5625$$

MAX PROFIT IS AT $q = 5625$

SOME QUESTIONS:

- WHAT DOES $\frac{\Delta P}{\Delta q}$ REPRESENT HERE?

- WHY IS THE SLOPE 0 AT THE POINT THAT MAXIMISES $P(q)$? IS THIS A GENERAL THING?

CONTINUITY (1.4)

WE JUST DISCOVERED HOW TO COMPUTE THE <u>RATE OF CHANGE</u> OF A FUNCTION. <u>BUT!</u>

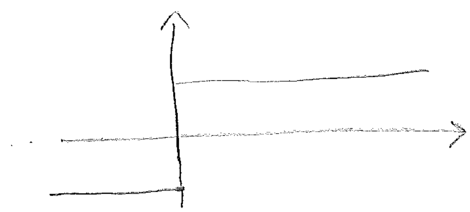
IT INVOLVES TAKING A LIMIT

$$\text{RATE OF CHANGE AT } T = \lim_{\Delta_T \to 0} \frac{f(T+\Delta_T) - f(T)}{\Delta_T}$$

WE'VE SEEN THAT LIMITS MAY WELL <u>NOT EXIST</u>.

AN EXAMPLE:

$$f(x) = \begin{cases} 1 & x > 0 \\ -1 & x \leq 0 \end{cases}$$



WHAT'S THE RATE OF CHANGE OF f AT 0?

$$\lim_{\Delta_x \to 0} \frac{f(\Delta_x) - f(0)}{\Delta_x} = \lim_{\Delta_x \to 0} \frac{f(\Delta_x) + 1}{\Delta_x} = \lim_{\Delta_x \to 0} \begin{cases} \frac{2}{\Delta_x} & \Delta_x > 0 \\ 0 & \Delta_x \leq 0 \end{cases}$$

DOES NOT EXIST! THERE ARE TWO RELEVANT THINGS AT PLAY HERE

A) <u>ONE SIDED LIMITS</u>

IN THE FUNCTION ABOVE EVERYTHING SEEM OK AS LONG AS WE APPROACH 0 "FROM THE LEFT". WHAT DOES THIS MEAN?

WE WRITE: "THE LIMIT OF $f(x)$ AS x APPROACHES T FROM THE LEFT IS L" OR "THE LEFT-HAND LIMIT OF f AT T IS L", DENOTED

$$\lim_{x \to T^-} f(x) = L$$

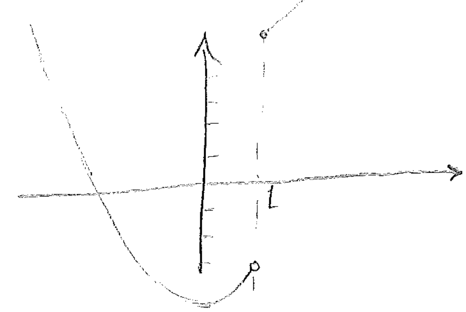
IF AS $x < \tau$ GETS ARBITRARILY CLOSE TO τ $f(x)$ GETS ARBITRARILY CLOSE TO L.

SIMILARLY WE SAY "THE LIMIT OF $f(x)$ AS x APPROACHES τ FROM THE RIGHT IS L" OR "THE RIGHT-HAND LIMIT OF $f(x)$ AT τ IS L", DENOTED

$\lim_{x \to \tau^+} f(x) = L$ IF AS $x > \tau$ GETS ARBITRARILY CLOSE TO τ $f(x)$ GETS ARBITRARILY CLOSE TO L.

EXAMPLE:

$$f(x) = \begin{cases} x+5 & x \geq 1 \\ x^2-4 & x < 1 \end{cases}$$



WE HAVE $\lim_{x \to 1^-} f(x) = -3$,

$\lim_{x \to 1^+} f(x) = 6$.

IMPORTANT:

- ALL PROPERTIES OF LIMITS HOLD FOR ONE-SIDED LIMITS, WHEN APPROPRIATELY ADJUSTED.
- TO COMPUTE A ONE-SIDED LIMIT, WE JUST DO THE SAME WE WOULD DO FOR A NORMAL LIMIT, KEEPING IN MIND $x < \tau$ OR $x > \tau$.

VERY IMPORTANT:

THE LIMIT $\lim_{x \to \tau} f(x)$ EXIST IF AND ONLY IF $\lim_{x \to \tau^-} f(x) = \lim_{x \to \tau^+} f(x)$ IN WHICH CASE $\lim_{x \to \tau} f(x)$ IS THE SAME.

B) CONTINUITY!

LET'S LOOK AGAIN AT THE RATE OF CHANGE OF $f(x)$ AT T

$$\lim_{\Delta_x \to 0} \frac{f(T+\Delta_x) - f(T)}{\Delta_x}$$

must go to 0!

goes to 0

WE NEED THE NUMERATOR OF THIS FRACTION TO GO TO 0! WHAT DOES THIS MEAN?

WE WANT

$$\lim_{\Delta_x \to 0} f(T+\Delta_x) - f(T) = 0$$

AS Δ_x APPROACHES 0, $T+\Delta_x$ APPROACHES T, SO

$$\lim_{\Delta_x \to 0} f(T+\Delta_x) - f(T) = \lim_{x \to T} f(x) - f(T)$$

SO WE WANT $\lim_{x \to T} f(x) = f(T)$!!

DEFINITION:

LET T BE A POINT S.T. $f(x)$ IS <u>DEFINED</u> AT T. WE SAY THAT $f(x)$ IS <u>CONTINUOUS AT T</u> IF

$$\lim_{x \to T} f(x) = f(T)$$

WE SAY THAT $f(x)$ IS <u>CONTINUOUS</u> IF THE ABOVE HOLDS FOR ALL T WHERE $f(T)$ IS DEFINED.

EQUIVALENTLY, $f(x)$ IS CONTINUOUS AT T IF $f(T)$ EXISTS AND

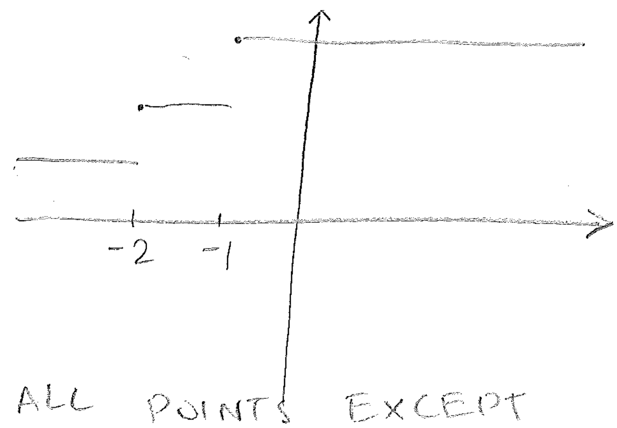
$$\lim_{x\to T^-} f(x) = \lim_{x\to T^+} f(x) = f(T).$$

WE CALL THE SET OF POINTS WHERE $f(x)$ IS CONTINUOUS THE <u>DOMAIN OF CONTINUITY</u> OF $f(x)$.

EXAMPLES:

1) POLYNOMIALS, EXPONENTIALS, SIN, COS ARE CONTINUOUS EVERYWHERE

2) $$f(x) = \begin{cases} 1 & x < -2 \\ 2 & -2 \le x < -1 \\ 3 & -1 \le x \end{cases}$$



IS CONTINUOUS FOR ALL POINTS EXCEPT $x = -2, -1$

3) $$f(x) = \begin{cases} x^2 - 1 & x < 2 \\ c(x+2) & x \ge 2 \end{cases}$$

IS CONTINUOUS AT $x = 2$ IF AND ONLY IF $c = \frac{3}{4}$.

4) HOW ABOUT $f(x) = \sqrt{x}$? IT IS NOT DEFINED FOR $x < 0$. IF A FUNCTION IS DEFINED ON A SET $[a, b]$ WE CONSIDER

$$\lim_{x\to a^+} f(x) \quad \text{AND} \quad \lim_{x\to b^-} f(x).$$

Q1: WHAT IS THE CORRECT FORMULA FOR AVERAGE VELOCITY? (CHOOSE ONE OPTION)

A) $\frac{S(T_1)}{T_1} - \frac{S(T_0)}{T_0}$

B) $-\frac{S(T_0)}{T_1-T_0} + \frac{S(T_1)}{T_1-T_0}$

C) $S(T_0)T_1 - S(T_1)T_0$

D) $\frac{S(T_1)-S(T_0)}{-T_0+T_1}$

E) BOTH B AND D ARE CORRECT

Q2: CONSIDER THE FUNCTION $f(x) = \begin{cases} 1 & x < -2 \\ 2 & -2 \leq x < -1 \\ 3 & -1 \leq x \end{cases}$
WHICH OF THE FOLLOWING IS TRUE:

A) $\lim_{x \to -2^-} f(x) = 2$

B) $f(x)$ NEVER HAS A LH LIMIT

C) $f(x)$ HAS A RH AND LH LIMIT EVERYWHRE

D) $f(x)$ HAS A LIMIT EVERYWHERE

E) $f(x)$ HAS A LIMT EVERYWHERE EXCEPT $T = -2$

Q3: CONSIDER THE FUNCTION $f(x) = \begin{cases} -\sqrt{10+x} & -10 \leq x < 6 \\ \sqrt{x-6} - 4 & 6 \leq x \leq 10 \\ \frac{x+2}{x^2-20x+100} & 10 < x \end{cases}$
WHAT IS THE DOMAIN OF CONTINUITY OF $f(x)$?

A) $[-10, 10) \cup (10, +\infty)$

B) $[-10, 6) \cup (6, 10] \cup (10, +\infty)$

C) $[-10, 6) \cup (6, 10) \cup (10, +\infty)$

D) $[-\infty, 10) \cup (10, +\infty)$

E) $(-10, 10) \cup (10, +\infty)$