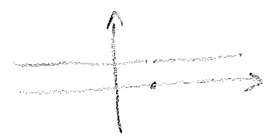
SOME MORE EXAMPLES ON CONTINUITY

- CHOCOLATE QUESTION! SEE SECTION PAGE.
- IT'S IMPORTANT NOT TO CONFUSE THE STATEMENTS

  "$f(x)$ ADMITS A LIMIT AT τ" AND

  "$f(x)$ IS CONTINUOUS AT τ". THE FUNCTION

$$f(x) = \begin{cases} 1 & x \neq 1 \\ 0 & x = 0 \end{cases}$$



HAS THE LIMIT $L = 1$ FOR $\tau = 1$ BUT IS NOT CONTINUOUS AT 1.

- SOME FUNCTIONS ARE NATURALLY DEFINED ON A CLOSED INTERVAL $[a, b]$ OR AN OPEN INTERVAL (a, b). FOR EXAMPLE

  $f(x) = \sqrt{1 - x^2}$ IS DEFINED ON $[-1, 1]$

  $f(x) = \tan(x)$ IS DEFINED ON $(-\frac{\pi}{2}, \frac{\pi}{2})$

  OR EVEN ON AN INTERVAL $(a, b]$:

  $f(x) = \tan\left(\frac{\pi}{2}\sqrt{1-x}\right)$ IS DEFINED ON $(0, 1]$

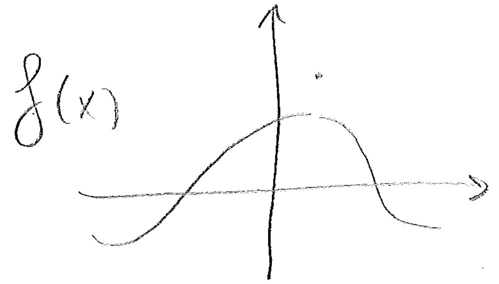
  OR EVEN ON $[a, b)(b, c]$

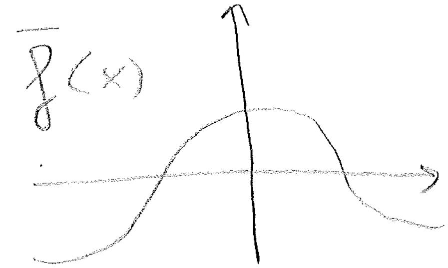
  $f(x) = \tan\left(\frac{\pi}{2}\sqrt{1-x^2}\right)$ IS DEFINED ON $[-1, 0)(0, 1]$

- IF $f(x)$ IS NOT DEFINED OR NOT CONTINUOUS AT τ BUT $\lim_{x \to \tau} f(x)$ EXISTS (RESP. $\lim_{x \to \tau^+}$ OR $\lim_{x \to \tau^-}$ IF IT IS AN EXTREME) WE SAY f HAS A <u>REMOVABLE DISCONTINUITY</u>. THE FUNCTION

$$\bar{f}(x) = \begin{cases} f(x) & x \neq \tau \\ \lim_{x \to \tau} f(x) & x = \tau \end{cases}$$

IS CONTINUOUS AT τ.





EXERCISE:

LET $f(x)$ BE THE FUNCTION:

$$f(x) = \begin{cases} \dfrac{x^2 - c_1}{x - 3} & x < 3 \\ e^{c_2 x} & x \geqslant 3 \end{cases} \qquad c_1, c_2 \in \mathbb{R}$$

FIND c_1, c_2 SUCH THAT $f(x)$ IS CONTINUOUS EVERYWHERE.

SOLUTION:

LET US LOOK AT $\dfrac{x^2 - c_1}{x - 3}$ FIRST. IT MUST ADMIT A LH LIMIT AT $I = 3$ FOR $f(x)$ TO BE CONTINUOUS.

THIS MEANS THAT $x^2 - c_1$ MUST BE 0 AT 3, SO.

$c_1 = 9$.

NOW, $e^{c_2 x}$ IS CONTINUOUS EVERYWHERE, SO WE HAVE

$$\lim_{x \to 3^+} e^{c_2 x} = e^{c_2 3} = (e^{c_2})^3$$

WE NEED

$$(e^{c_2})^3 = \lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3^-} x + 3 = 6$$

SO WE SET $c_2 = \ln(\sqrt[3]{6}) = \frac{1}{3} \ln(6)$.

HOW DO WE DECIDE IF A FUNCTION f IS CONTINUOUS? WE KNOW THAT POLYNOMIALS, ROOTS, TRIG FUNCTIONS AND EXPONENTIALS/LOGARITHMS ARE, BUT WHAT ABOUT, SAY

$$f(x) = \frac{\log(e^{3x} + x^2 - 5)}{x^2 + 4} + \sqrt{x^6 + 12} \ ?$$

WE CAN USE THE LIMIT PROPERTIES:

<u>THEOREM</u>:

- SUPPOSE $f(x), g(x)$ ARE CONTINUOS AT $\bar{x}$. THEN

$$f(x) + g(x), \quad f(x)g(x), \quad \frac{f(x)}{g(x)} \text{ (IF DEFINED AT } \bar{x}\text{)}$$

  ARE CONTINUOUS AT $\bar{x}$

- SUPPOSE $g(x)$ IS CONTINUOUS AT $\bar{x}$, $f(x)$ IS CONTINUOUS AT $g(\bar{x})$. THEN $f(g(x))$ IS CONTINUOUS AT $\bar{x}$.

- SUPPOSE $f(x), g(x)$ ARE CONTINUOUS (ON THEIR RESPECTIVE DOMAINS), THEN

$$(f+g)(x), \ f \cdot g(x), \ f(g(x)), \ \frac{f}{g}(x) \text{ ARE}$$

  CONTINUOUS WHERE THEY ARE DEFINED.

WHAT DOES CONTINUITY MEAN "MORALLY"?

A CONTINUOUS FUNCTION IS ONE WE CAN SKETCH WITHOUT LIFTING OUR PEN FROM THE SHEET.

IMPORTANT: SUMS, PRODUCT, FRACTIONS, ROOTS, COMPOSITIONS o CONTINUOUS FUNCTIONS ARE CONTINUOUS WHERE THEY ARE DEFINED (FOLLOWS FROM LIMIT PROPERTIES)

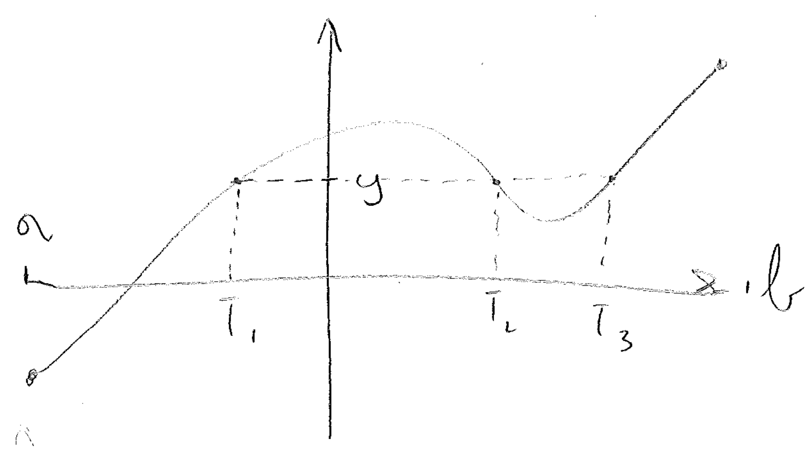
WHY ARE CONTINUOUS FUNCTIONS USEFUL?

EXHIBIT A:

INTERMEDIATE VALUE THEOREM:

SUPPOSE $f(x)$ IS CONTINUOUS ON A CLOSED INTERVAL $[a, b]$. WE CAN ASSUME WITHOUT LOSS OF GENERALITY THAT $f(a) \leq f(b)$ (Or we Take $-f$)

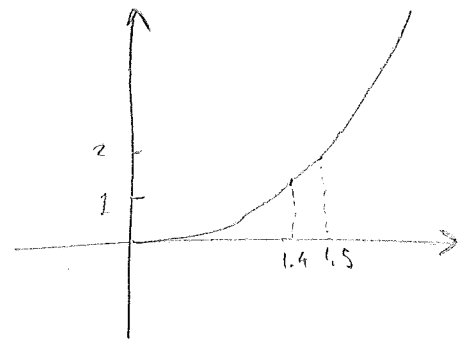
THEN FOR EVERY $f(a) \leq y \leq f(b)$ THERE IS T IN $[a, b]$ SUCH THAT $f(T) = y$



WHY IS THE I.V.T USEFUL?

EXAMPLE: SHOW THAT $1.4 < \sqrt{2} < 1.5$

CONSIDER $f(x) = x^2$. WE KNOW THAT $f(x)$ IS CONTINUOUS.



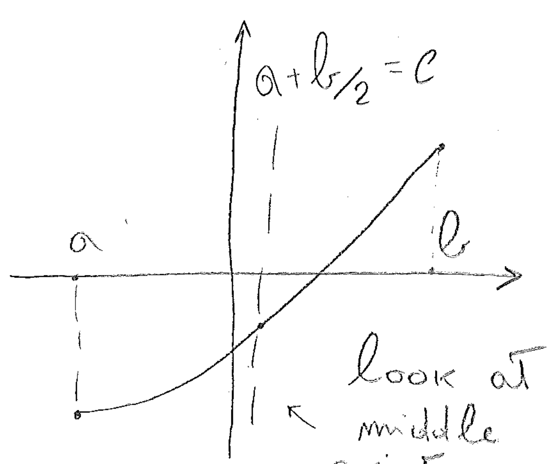
WE HAVE $f(1.4) = 1.96$, $f(1.5) = 2.25$.

BY THE I.V.T. ON $[1.4, 1.5]$ WE KNOW THERE IS $1.4 < \tau < 1.5$ WITH $f(\tau) = 2$, THAT IS, $\tau^2 = 2$.

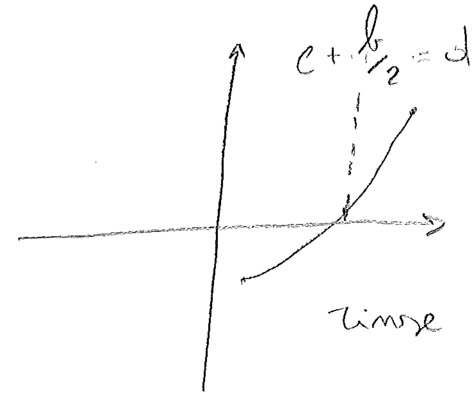
EXAMPLE: <u>FINDING ZEROES</u>

WE HAVE A CONTINUOUS FUNCTION $f(x)$ ON $[a, b]$. WE KNOW $f(a) < 0$, $f(b) > 0$. THEN f HAS A ZERO IN $[a, b]$! HOW DO WE FIND IT?

1



2 $f(c) < 0$ so there is a 0 in $[c, b]$



$f(d) > 0$ zero is in $[c, d]$ ...

rinse and repeat!

AN EXPLICIT EXAMPLE OF ROOT-SEARCHING

$$f(x) = 17x^3 - 3x^2 + 17x - 3$$

WE WANT TO FIND A 0. DOES IT HAVE ONE?

$$f(1) = 28 \qquad f(0) = -3$$

SO IT MUST HAVE A ROOT IN $[0, 1]$ BY I.V.T

$$f\left(\tfrac{1}{2}\right) = \frac{17}{8} - \frac{3}{4} + \frac{17}{2} - 3 = \frac{85 - 30}{8} = \frac{55}{8} > 0$$

SO IT MUST HAVE A ROOT IN $[0, \tfrac{1}{2}]$ BY I.V.T.

$$f\left(\tfrac{1}{4}\right) = \frac{17}{64} - \frac{3}{16} + \frac{17}{4} - 3 = \frac{17 \cdot 17 - 3 \cdot 68}{64} = \frac{85}{64}$$

" " " " " " " $[0, \tfrac{1}{4}]$ BY I.V.T.

$f\left(\tfrac{1}{8}\right) = -\frac{455}{512}$, ROOT IN $[\tfrac{1}{8}, \tfrac{1}{4}]$

$f\left(\tfrac{3}{16}\right) \approx 0.19$, " " $[\tfrac{1}{8}, \tfrac{3}{16}]$

$f\left(\tfrac{5}{32}\right) \approx -0.35$, " " $[\tfrac{5}{32}, \tfrac{3}{16}]$

$f\left(\tfrac{11}{64}\right) \approx +0.08$, " " $[\tfrac{11}{64}, \tfrac{3}{16}]$

$f\left(\tfrac{23}{128}\right) \approx 0.05$, " " $[\tfrac{11}{64}, \tfrac{23}{128}]$ PRECISION OVER $1/100$!

CHECK: THE ROOT IS $\frac{3}{17}$, $\left|\frac{3}{17} - \frac{45}{256}\right| = 0.0006$