1.2 EVEN MORE MOTIVATING EXAMPLE:

COMPOUND INTEREST

PURPOSE: INTRODUCE EXP, LOG, LIMIT

Does anyone here have a credit card? Or a saving account? Or works as a loan shark?
What are the rates? How often is interest applied?

EXERCISE:

WE HAVE AN INVESTMENT OF 1000 \$ FOR 1 YEAR, AT 10%
WHAT IS OUR FUTURE BALANCE (FV) IF:

a) INTEREST IS COMPOUNDED ANNUALLY?	$FV = 1000 \cdot \left(1+\frac{1}{10}\right) = 1100$
b) " " " QUARTERLY?	$FV = 1000 \cdot \left(1+\frac{1}{40}\right)^4 = 1103.8$
c) " " " MONTHLY?	$FV = 1000 \cdot \left(1+\frac{1}{120}\right)^{12} = 1104.7$
d) " " " DAILY?	$FV = 1000 \cdot \left(1+\frac{1}{3650}\right)^{365} = 1105.16$

e) INTEREST IS COMPOUNDED <u>CONTINUOUSLY</u>?

WAIT, WHAT? LET'S PRETEND THAT THERE ARE NO LIMITS TO HOW FAST WE CAN DO THINGS, AND WE COMPOUND AN IMMENSE NUMBER OF TIMES PER SECOND, AND EVEN MORE.

- SO IF WE COMPOUND n TIMES, THE FV IS

$$FV = 1000 \cdot \left(1 + \frac{1}{10n}\right)^n$$

- THIS FUNCTION INCREASES WITH n. WHY? CONVINCE YOURSELF OF IT!
- WILL THIS FUNCTION "EXPLODE"? CAN WE USE IT TO GET INFINITELY RICH?

WELL, $FV = \lim_{n \to \infty} 1000\left(1 + \frac{1}{10n}\right)^n = 1000 \cdot e^{0.1} =$

$$\approx \$\, 1105.17$$

WE GAINED 1 CENT FROM $n = 365$ TO INFINITY.

- BUT! IT WAS AN IMPORTANT QUESTION, WHICH LED TO THE DISCOVERY OF e IN 1683 (JACOB BERNOULLI)

DEFINITION: THE EXPONENTIAL FUNCTION e^x OR $\exp(x)$ CAN BE DEFINED IN MANY WAYS. ONE IS

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

COMPOUND INTEREST:

KEY TERMS/FORMULAS

- PV PRESENT VALUE
- FV FUTURE VALUE
- J EFFECTIVE INTEREST RATE
- i NOMINAL " "
- r CONTINUOUS " "
- T TIME UNIT (IN YEARS)

- SIMPLE INTEREST RATE We only get interest on our initial value

$$FV = (1 + T \cdot J)\, PV$$

- EFFECTIVE I. R. compounds every year

$$FV = (1 + J)^T\, PV$$

- NOMINAL I. R. WITH A FREQUENCY m, EITHER VERBAL (MONTHLY) OR NUMERICAL (12)

$$FV = \left(1 + \frac{i}{m}\right)^{m \cdot T} PV$$

- CONTINUOUS I. R.

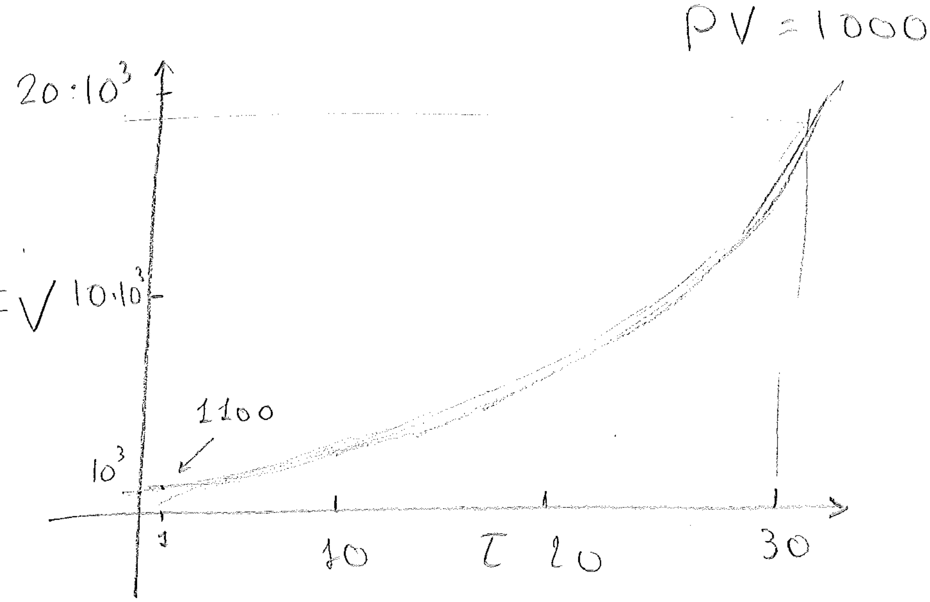
$$FV = PV e^{rT} \quad \left(= PV \cdot \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^T\right)$$

- ALL INTERESTS* CAN BE "CONVERTED" TO EFFECTIVE BY TAKING YEARLY GAIN.

MANY COUNTRIES REQUIRE BANKS TO TELL YOU EFFECTIVE INTEREST.

* EXCEPT SIMPLE

THEY ARE ALL THE SAME! BUT THE RATES ARE <u>NOT</u>



$$FV = e^{\ln(1.1)\tau} PV \approx e^{0.09531\tau} \cdot PV \quad \text{or} \quad FV = (1.10)^{\tau} PV$$

$$\text{EFFECTIVE RATE} = e^{\text{CONTINUOUS RATE}} - 1$$

$$\text{CONTINUOUS RATE} = \ln(\text{EFFECTIVE RATE})$$

WHY USE CONTINUOUS RATE? MUCH BETTER FOR COMPUTATIONS!

Q: MONTHLY RATE IS A) $e^{12 \text{ CONTINUOUS RATE}} - 1$?

B) $e^{\frac{\text{CONTINUOUS RATE}}{12}} - 1$

<u>EXERCISES</u>

a) YOU BORROW 20.000 DOLLARS FROM SHARK-FACE JACK. HE CHARGES A CONTINUOUS INTEREST RATE r. AFTER 5 YEARS YOU PAY HIM \$100.000. WHAT WAS THE CONTINUOUS RATE r? WHAT IS THE EFFECTIVE RATE?

b) HOW MANY YEARS DOES IT TAKE FOR 10.000 DOLLARS TO GROW TO 15000 AT A NOMINAL 15% INTEREST, COMPOUNDED QUARTERLY?

c) WHAT CONTINUOUS INTEREST RATE IS EQUIVALENT TO 8% NOMINAL RATE, COMPOUNDED SEMI-MONTHLY?

SOLUTIONS

a) WE HAVE $e^{5r} \cdot 20.000 = 100.000$

$$\sim \quad e^{5r} = 5 \quad \sim \quad 5r = \ln 5 \quad \sim \quad r = \frac{\ln 5}{5}$$

EFFECTIVE RATE IS $e^{\frac{\ln 5}{5}} - 1 = \sqrt[5]{5} - 1$

b) WE HAVE

$$FV = \left(1 + \frac{0.15}{4}\right)^{4T} PV \quad \text{SO}$$

$$15.000 = \left(1 + \frac{0.15}{4}\right)^{4T} 10.000, \quad \text{SOLVE FOR } T$$

$$\frac{3}{2} = \left(1 + \frac{0.15}{4}\right)^{4T} \sim \ln\left(\frac{3}{2}\right) = 4T \ln\left(1 + \frac{0.15}{4}\right)$$

$$T = \frac{\ln\left(\frac{3}{2}\right)}{4 \ln\left(1 + \frac{0.15}{4}\right)} \approx \frac{0.405}{4 \cdot (0.036)} \approx 3.308 \text{ YRS}$$

NOTE: THIS IS A PERFECTLY COMPLETE ANSWER.

c) WE HAVE $FV = \left(1 + \frac{0.08}{24}\right)^{24T} PV$

COMPARE WITH CONTINUOUS $FV = e^{rT} PV$

$$PV\left(1 + \frac{0.08}{24}\right)^{24T} = e^{rT} PV \sim$$

$$\left(1 + \frac{0.08}{24}\right)^{24T} = e^{rT} \sim 24T \ln\left(1 + \frac{0.08}{24}\right) = rT$$

$$r = 24 \ln\left(1 + \frac{0.08}{24}\right)$$

A+ PROBLEM ON COMPOUND INTEREST

a) A YOUNG STUDENT NAMED SCROOGE MCDUCK HAS A TRUST CURRENTLY VALUED 1.000.000 \$. HE GETS AN ANNUAL INTEREST RATE OF 6%, COMPOUNDED CONTINUOUSLY. HE PLAS TO WITHDRAW AS SOON AS HE CAN WITHDRAW \$ 50,000 / MONTH INDEFINITELY. HOW LONG WILL IT TAKE FOR HIM TO RETIRE?

A: HE NEEDS TO GET TO THE POINT WHERE

$$\underbrace{PV \cdot e^{0.06 \cdot \frac{1}{12}} - PV}_{\text{MONTHLY INCREASE}} = 50.000$$

WE WANT TO SOLVE FOR PV.

$$PV \cdot e^{0.06 \cdot \frac{1}{12}} - PV = 50.000 \sim PV\left(e^{0.06 \cdot \frac{1}{12}} - 1\right) = 50.000$$

$$\sim PV = \frac{50,000}{e^{\frac{0.06}{12}} - 1} \left(\simeq 9.975.000 \text{ \$}\right)$$

TO GET TO THAT HE'LL NEED:

$$10^6 e^{0.06T} = \frac{50.000}{e^{0.06/12} - 1} \sim e^{0.06T} = \frac{1}{(e^{0.06/12} - 1) 20} \sim$$

$$(0.06) T = \ln \frac{1}{(e^{0.06/12} - 1) 20} \sim T = \frac{100}{6} \ln \frac{1}{(e^{0.06/12} - 1) 20}$$

$$\simeq \frac{100}{6} (2.3) \simeq 38.33 \text{ YEARS.}$$

(1.3) LIMITS: INTRODUCTION

CALCULUS: 17^{Th} CENTURY - PRESENT

CHILD OF ISAAC NEWTON, GOTTFRIED LEIBNIZ AND MANY OTHERS.

SOME THINGS WE DID NOT UNDERSTAND BEFORE CALCULUS:

- AREAS OF ARBITRARY SHAPES
- RATES OF CHANGE (e.g. if I move at a given non-costant speed how far do I get in a time t?)
- MAXIMISING/MINIMISING FUNCTIONS

CALCULUS SOLVED ALL THESE PROBLEMS, WHICH ARE RELATED. THE FOUNDATION OF CALCULUS IS THE NOTION OF LIMIT.

WHAT IS A LIMIT?

CONSIDER THE FUNCTION $f(x) = \dfrac{1 - \cos(x)}{x^2}$ *

* Note: we are not going to use a lot of trigonometry in this course. Do not be scared.

WHAT IS $f(1)$? WELL, $f(1) = \dfrac{1 - \cos(1)}{1^2}$

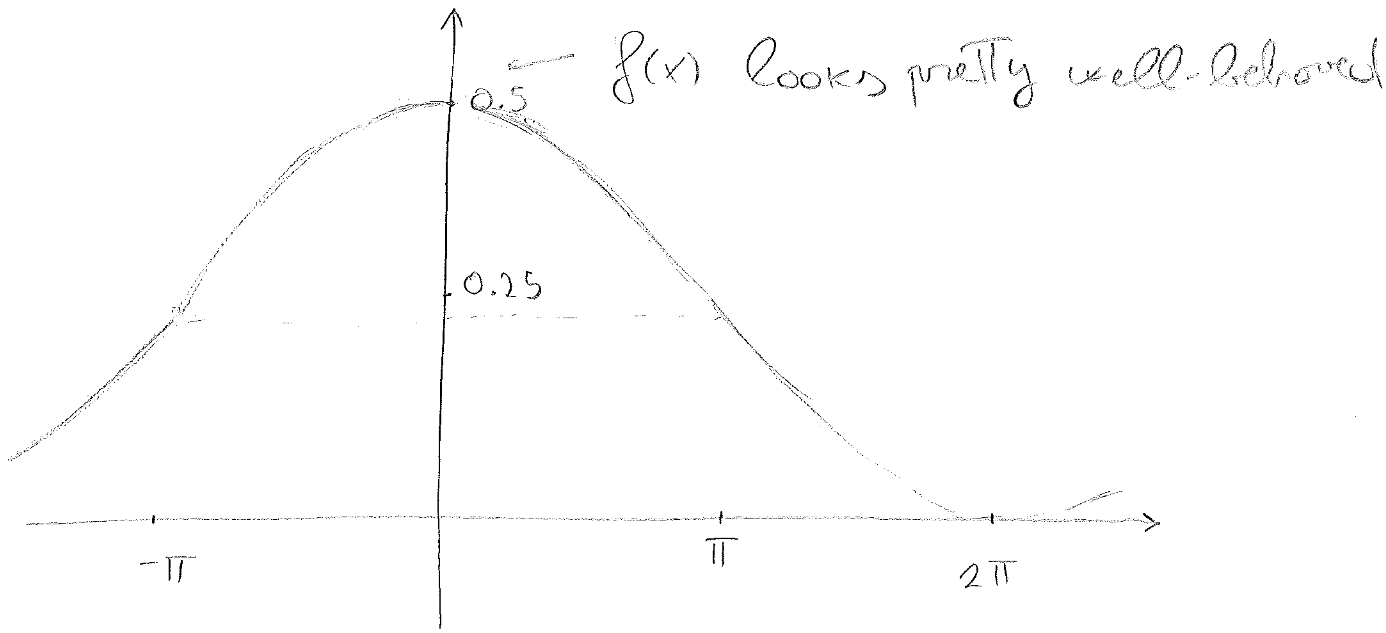
$= 1 - \cos^2(1) = 0.459\ldots$

BUT WHAT ABOUT $f(0)$?

WE CANNOT JUST PLUG 0 INTO $f(x)$:

$\dfrac{1-\cos'(0)}{(0)^2} = \dfrac{1-1}{0} = \dfrac{0}{0}$ AN EXPRESSION THAT HAS NO VALUE

DOES THIS MEAN THAT THERE CAN BE A MEANINGFUL VALUE FOR $f(0)$? LET'S PLOT $f(x)$ "AROUND" 0



LET'S COMPUTE SOME VALUES

$f(1) = 0.459\ldots \quad f(0.1) = 0.499\ldots \quad f(0.01) = 0.49999$

IT LOOKS LIKE WHEN X IS "NEAR" 0 $f(x)$ IS "NEAR" 0.5.

<u>FINDING A LIMIT</u> IS EXACTLY THIS; UNDERSTANDING HOW A FUNCTION BEHAVES <u>NEAR</u> (BUT NOT <u>AT</u>) A CERTAIN VALUE.

<u>INFORMAL DEFINITION OF LIMIT</u>

WE SAY "THE LIMIT OF $f(x)$ AS X APPROACHES $\bar{c}$ IS L" IF THE VALUE OF $f(x)$ GETS <u>ARBITRARILY CLOSE</u> TO L WHEN X IS <u>ARBITRARILY CLOSE</u> (BUT <u>NOT EQUAL</u>) TO $\bar{c}$.

WE WRITE IT $\lim_{x \to \bar{c}} f(x) = L$.

FOR EXAMPLE, IN THE CASE ABOVE THE LIMIT OF $\frac{1-\cos(x)}{x^2}$ AS X APPROACHES 0 IS $\frac{1}{2}$. OR SHORTLY $\lim_{x \to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$

IMPORTANT NOTE:

WE DO NOT CARE WHAT f DOES AT $\bar{c}$

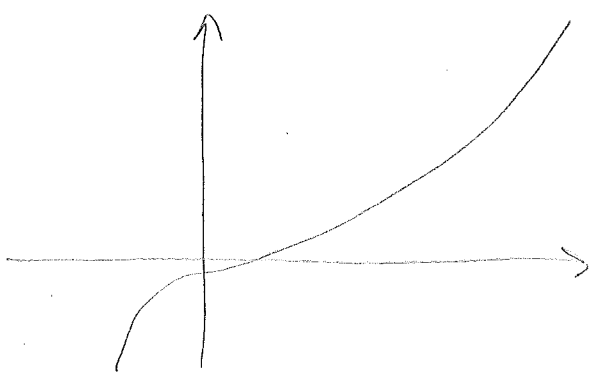
$f(x)$ CAN BE UNDEFINED AT $\bar{c}$

a) $$\lim_{x \to 2} \frac{(x-2)(x^3-4+3x)}{(x-2)(x+2)}$$ undefined at $x=2$

$f(x)$ CAN HAVE ANY VALUE AT $\bar{c}$

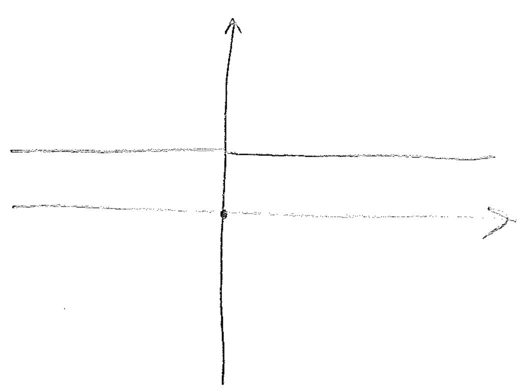
b) $\lim_{x \to 0} f(x)$ WHERE $f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$

a)



$$\lim_{x \to 2} \frac{(x-2)(x^3-4+3x)}{(x-2)(x+2)} = \frac{5}{2}$$

b)

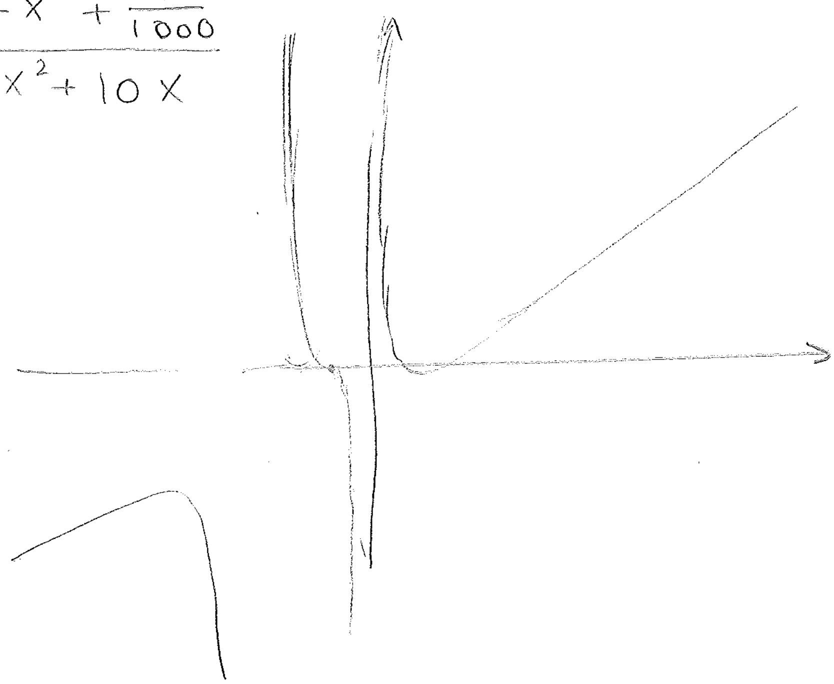


$$\lim_{x \to 0} f(x) = 1$$

WE CAN APPROXIMATE LIMITS NUMERICALLY OR PLOT THEM, BUT WHAT WE REALLY WANT TO DO IS FINDING THEM ANALITICALLY, THAT IS, OBTAIN EXACT RESULTS

EXAMPLE OF WHY APPROXIMATING IS NOT ENOUGH

$$f(x) = \frac{3x^3 - x^2 + \frac{1}{1000}}{5x^2 + 10x}$$



Note: Try printing This in Wolfram Alpha!

$f(1) = \dfrac{2 + \frac{1}{1000}}{15} \simeq 0.1334$ $\quad f(0.1) = -0.0057$ $\quad f(0.01) = 0.0089$

$f(0.001) = 0.0998$ $\quad f(0.0001) = 0.9994$ $\quad f(10^{-8}) = 10{,}000$!!

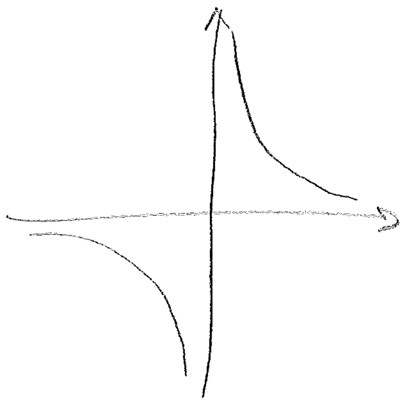
WHAT'S HAPPENING?

$$f(0) = \frac{3 \cdot 0^3 - 0^2 + \frac{1}{1000}}{5 \cdot 0^2 + 10 \cdot 0} = \frac{\frac{1}{1000}}{0} = \text{"}\infty\text{"}?!$$

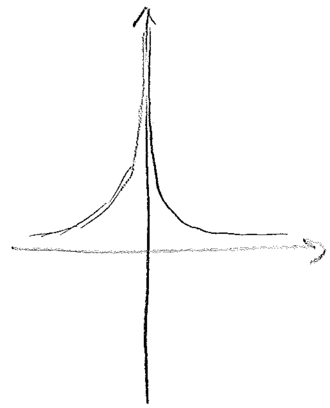
SO IT MAY HAPPEN THAT $f(x)$ DOES NOT HAVE A LIMIT FOR $x \to \tau$?

YES! THREE THINGS CAN HAPPEN:

1) $f(x)$ IS NOT BOUNDED FROM ABOVE/BELOW



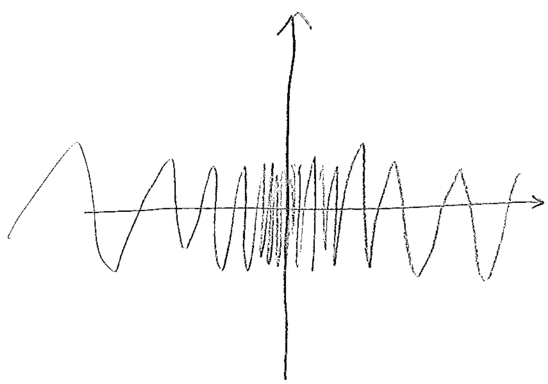
$f(x) = \frac{1}{x}$



$f(x) = \frac{1}{x^2}$

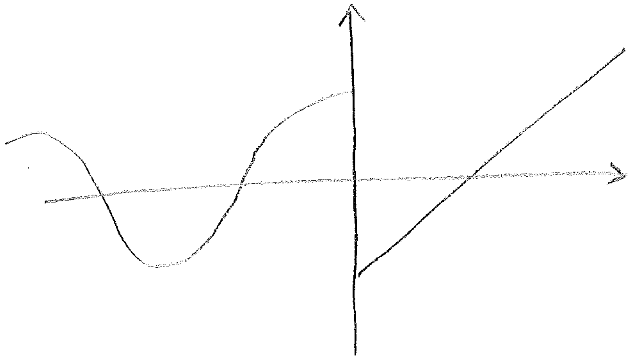
* Note: for now we will only consider finite limits.

2) $f(x)$ OSCILLATES TOO MUCH



$f(x) = \sin\left(\frac{1}{x}\right)$

3) $f(x)$ "JUMPS"



$$f(x) = \begin{cases} \cos(x) & x < 0 \\ x - 1 & x > 0 \end{cases}$$