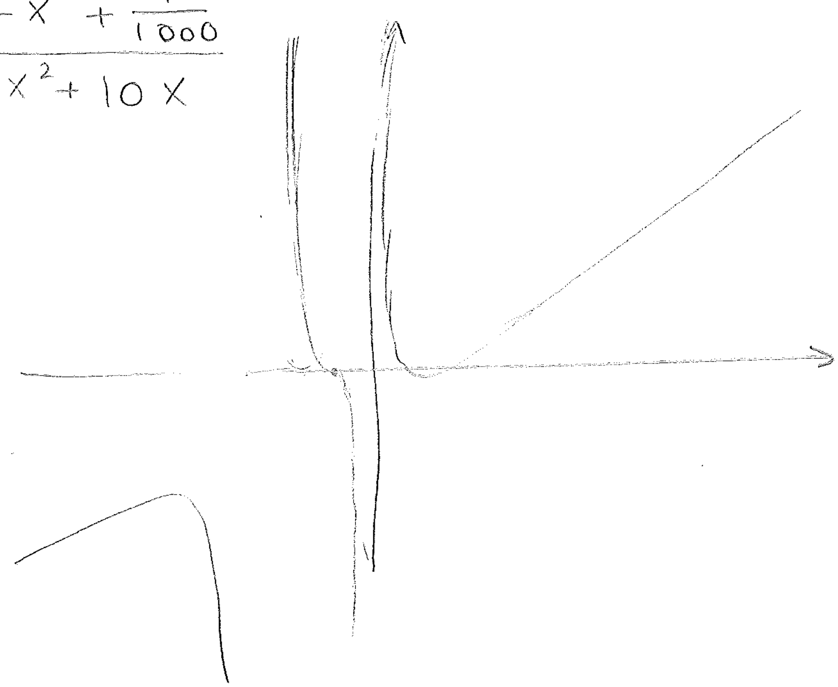
WE CAN APPROXIMATE LIMITS NUMERICALLY OR PLOT THEM, BUT WHAT WE REALLY WANT TO DO IS FINDING THEM ANALITICALLY, THAT IS, OBTAIN EXACT RESULTS

EXAMPLE OF WHY APPROXIMATING IS NOT ENOUGH

$$f(x) = \frac{3x^3 - x^2 + \frac{1}{1000}}{5x^2 + 10x}$$



Note: Try printing This in Wolfram Alpha!

$f(1) = \frac{2 + \frac{1}{1000}}{15} \simeq 0.1334$ $f(0.1) = -0.0057$ $f(0.01) = 0.0089$

$f(0.001) = 0.0998$ $f(0.0001) = 0.9994$ $f(10^{-8}) = 10{,}000$!!

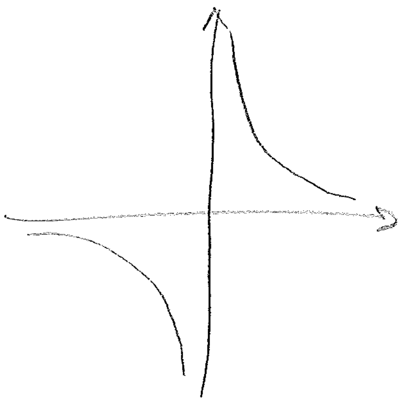
WHAT'S HAPPENING?

$$f(0) = \frac{3 \cdot 0^3 - 0^2 + \frac{1}{1000}}{5 \cdot 0^2 + 10 \cdot 0} = \frac{\frac{1}{1000}}{0} = \text{"}\infty\text{"}?!$$

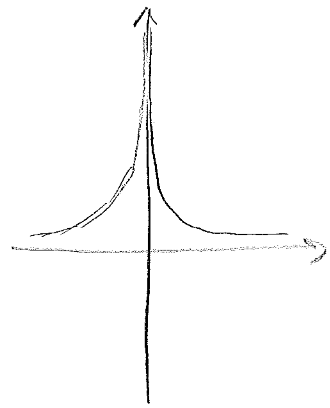
SO IT MAY HAPPEN THAT $f(x)$ DOES NOT HAVE A LIMIT FOR $x \to \bar{x}$?

YES! THREE THINGS CAN HAPPEN:

1) $f(x)$ IS NOT BOUNDED FROM ABOVE/BELOW



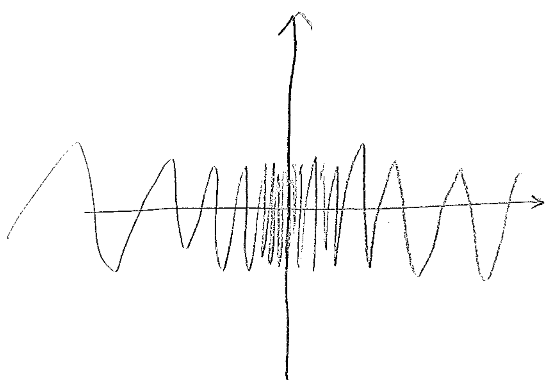
$f(x)=\frac{1}{x}$



$f(x)=\frac{1}{x^2}$

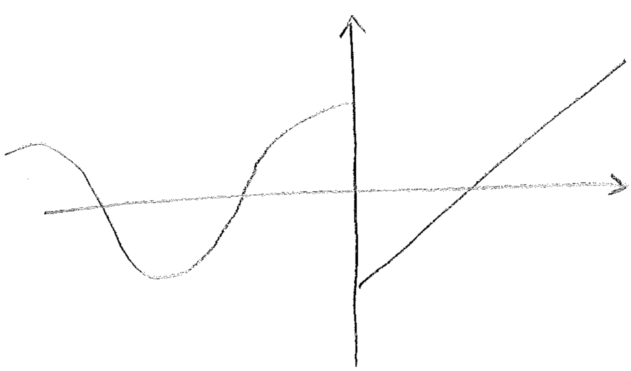
* Note: for now we will only consider <u>finite</u> limits.

2) $f(x)$ OSCILLATES TOO MUCH



$f(x)=\sin\left(\frac{1}{x}\right)$

3) $f(x)$ "JUMPS"



$$f(x)=\begin{cases}\cos(x) & x<0\\ x-1 & x>0\end{cases}$$

WHICH FUNCTIONS <u>BEHAVE WELL</u>? HERE ARE <u>SOME</u>

- FOR ALL OF THE FOLLOWING FUNCTIONS WE HAVE $\lim_{x \to T} f(x) = f(T)$ WHENEVER $f(T)$ IS DEFINED:

$f(x) =$

c - constants
x - identity
a^x - powers
$\log_a x$ - logarithms
$\sin x, \cos x$ - Trigonometric functions

As we will see, we are saying that these functions are <u>continuous</u>.

- SUPPOSE $\lim_{x \to T} f(x) = L$, $\lim_{x \to T} g(x) = K$. THEN

(1) $\lim_{x \to T} f(x) \pm g(x) = \lim_{x \to T} f(x) \pm \lim_{x \to T} g(x)$

(2) $\lim_{x \to T} b \cdot f(x) = b \cdot \lim_{x \to T} f(x)$ <u>for a number b</u>

(3) $\lim_{x \to T} f(x) \cdot g(x) = \lim_{x \to T} f(x) \lim_{x \to T} g(x)$

(4) IF $K \neq 0$ $\lim_{x \to T} \frac{f(x)}{g(x)} = \frac{L}{K}$

(5) $$\lim_{x \to T} f(x)^n = L^n$$

(6) $$\lim_{x \to T} \sqrt[n]{f(x)} = \sqrt[n]{L}$$

SUPPOSE NOW THAT

$$\lim_{x \to T} f(x) = L \quad \text{AND} \quad \lim_{x \to L} g(x) = K$$

THEN $\lim_{x \to T} g(f(x)) = K$.

HOW DO WE USE THESE RULES TO COMPUTE (simple) LIMITS?

EXAMPLE:

FIND $\lim_{x \to 3} x^3 + 5x - 4$.

A: BY (1) $\lim_{x \to 3} x^3 + 5x - 4 = \lim_{x \to 3} x^3 + \lim_{x \to 3} 5x - \lim_{x \to 3} 4$

$$= \underbrace{\left(\lim_{x \to 3} x\right)^3}_{(5)} + \underbrace{5 \lim_{x \to 3} x}_{(2)} - \lim_{x \to 3} 4 = 3^3 + 5 \cdot 3 - 4 = 38$$

IN GENERAL WE CAN DO THE SAME TO SHOW THAT IF $f(x) = a_0 x^m + a_1 x^{m-1} + \dots + a_m$ IS A POLYNOMIAL THEN

$$\lim_{x \to T} f(x) = f(T)$$

IN GENERAL WHEN LOOKING FOR A LIMIT

$\lim_{x \to \tau} f(x)$ WE WANT TO WRITE $f(x)$ AS A

COMBINATION OF FUNCTION OF WHICH WE KNOW

THE LIMIT AND THEN USE THE RULES ABOVE

(IMPORTANT: IF $f(x) = g(x)$ FOR ALL x CLOSE BUT NOT EQUAL τ, $\lim_{x \to \tau} f(x) = \lim_{x \to \tau} g(x)$)

EXAMPLES:

1) $\lim_{x \to -2} \frac{x^2 - x - 6}{x^2 + 2x}$ WE CAN'T APPLY RULE (4) AS $\lim_{x \to 2} x^2 + 2x = 0$

BUT, WRITE $P(x) = x^2 - x - 6$, $Q(x) = x^2 + 2x$.

WE HAVE $P(-2) = 0$, SO $(x+2)$ DIVIDES $P(x)$.

SOLVING WE GET $P(x) = (x+2)(x-3)$. ALSO

$Q(x) = x(x+2)$. THEN

$$\frac{P(x)}{Q(x)} = \frac{(x+2)(x-3)}{x(x+2)} = \begin{cases} \frac{(x-3)}{x} & x \neq -2 \\ \text{UNDEFINED} & x = -2 \end{cases}$$ SO $\lim_{x \to 2} \frac{x^2 - x - 6}{x^2 + 2x} =$

$$\lim_{x \to -2} \frac{x-3}{x} = \frac{-5}{-2} = \frac{5}{2}$$

2) $\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x}$ AGAIN WE CAN'T APPLY

(4) IMMEDIATELY. BUT!

$$\lim_{x \to 0} \frac{\sqrt{x-4} - 2}{x} = \lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x} \cdot \left(\frac{\sqrt{x+4} + 2}{\sqrt{x+4} + 2} \right) =$$

$$\lim_{x \to 0} \frac{(\sqrt{x+4} - 2)(\sqrt{x+4} - 2)}{x(\sqrt{x+4} + 2)} = \lim_{x \to 0} \frac{x + 4 - 4}{x(\sqrt{x+4} + 2)} = \lim_{x \to 0} \frac{1}{\sqrt{x+4} + 2}$$

$$= \frac{1}{4}$$

3) $$\lim_{x\to 2} \frac{\frac{3}{2x+1} - \frac{3}{5}}{x-2} = \lim_{x\to 2} \frac{\frac{3\cdot 5 - 3(2x+1)}{5(2x+1)}}{x-2}$$

$$= \lim_{x\to 2} \frac{\frac{3(5-2x-1)}{5(2x+1)}}{x-2} = \lim_{x\to 2} \frac{3}{5} \frac{\frac{(-2x+4)}{(2x+1)}}{(x-2)}$$

$$= \frac{3}{5}\lim_{x\to 2} \frac{-2(x-2)}{(x-2)(2x+1)} = \frac{-6}{5} \lim_{x\to 2} \frac{1}{2x+1} =$$

$$\frac{-6}{25}$$

A CASE WE SHOULD ALWAYS BE ABLE TO SOLVE IS RATIONAL FUNCTIONS, THAT IS, FUNCTIONS THAT CAN BE OBTAINED BY ADDING/MULTIPLYING POLYNOMIALS AND INVERSES OF POLYNOMIALS.

EX: COMPUTE

$$\lim_{x\to -3} \frac{\frac{30}{x^3-19x} - 1}{x+3}$$

BOTH = 0 FOR $x=-3$

a) REDUCE TO A FRACTION IN THE FORM $\frac{P(x)}{q(x)}$

$$\lim_{x\to -3} \frac{\frac{30}{x^3-19x} - 1}{x+3} = \lim_{x\to -3} \frac{30 - x^3 + 19x}{(x^3-19x)(x+3)}$$

b) WE CAN NOW APPLY THE FOLLOWING:

THM: LET $f(x)$ BE A POLYNOMIAL. IF $f(\tau)=0$ THEN $f(x)=g(x)(x-\tau)$

- $30-x^3+19x$ IS 0 FOR $x=-3$: $30-(-3)^3+19(-3)$ $= 30+27-57 = 0$
- POLYNOMIAL DIVISION:

$$
\begin{array}{l|l}
-x^3+19x+30 & x-(-3)=x+3 \\
\hline
-x^3-3x^2 & -x^2+3x+10 \\
\quad 3x^2+19x+30 & \\
\quad 3x^2+9x & \\
\qquad 10x+30 & \\
\qquad 10x+30 &
\end{array}
$$

c) USING THE FACT THAT $x-\tau$ IS NOT 0 OUTSIDE OF τ, SIMPLIFY

$$\lim_{x\to-3}\frac{30-x^3+19x}{(x+3)(x^3-19x)} = \lim_{x\to-3}\frac{(x+3)(-x^2+3x+10)}{(x+3)(x^3-19x)}$$

$$= \lim_{x\to-3}\frac{-x^2+3x+10}{x^3-19x}$$

d) IF THE DENOMINATOR IS STILL 0 AT τ, REPEAT a), b) (OR IF YOU CAN'T, LIMIT DOES NOT EXIST!)
IF NOT, FIND LIMIT USING RULE 4

$$\lim_{x\to-3}\frac{-x^2+3x+10}{x^3-19x} = \frac{\lim_{x\to-3} -x^2+3x+10}{\lim_{x\to-3} x^3-19x} = \frac{-8}{30} = \frac{-4}{15}$$

USING LIMITS IN REAL LIFE:
THE VELOCITY / TANGENT PROBLEM

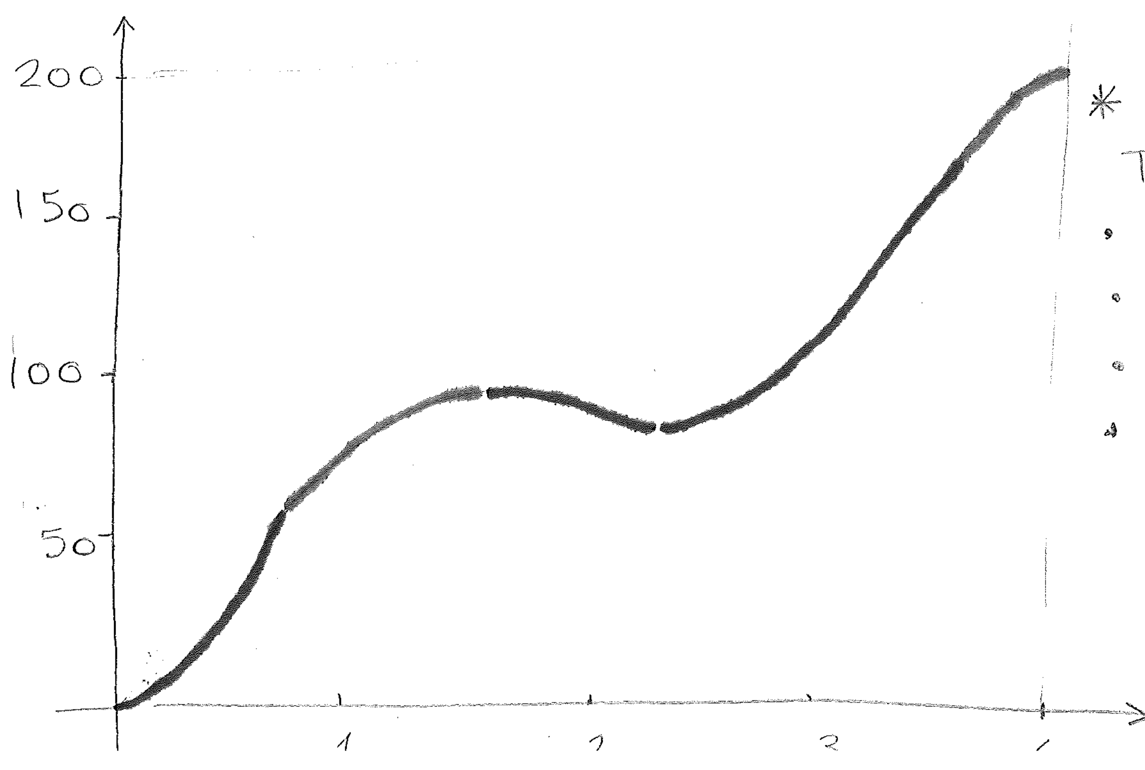
- Q: WHAT IS VELOCITY? WHAT DOES IT MEAN IF YOUR CAR SPEEDOMETER SAYS 100 KPH?
- A: $\text{VELOCITY} = \dfrac{\text{CHANGE IN DISTANCE}}{\text{CHANGE IN TIME}} = \dfrac{\Delta s}{\Delta t}$

WHERE $s(t)$ IS "DISTANCE AT TIME t"

EXAMPLE: IF WE TRAVEL 200 KM IN 4 HOURS WE HAVE $\dfrac{\Delta s}{\Delta t} = \dfrac{200\ (\text{KM})}{4\ (\text{H})} = 50\ \text{KM/H}$, 50 KILOMETERS PER HOUR.

BUT! OUR SPEED DURING THE 4 HOURS WAS DEFINITELY NOT ALWAYS 50 KM/H! THAT WAS JUST OUR AVERAGE SPEED. AND WE STILL HAVE NOT ANSWERED WHAT IS THAT OUR SPEEDOMETER IS SHOWING US.

LET'S DRAW A GRAPHIC FOR THESE 4 HOURS.



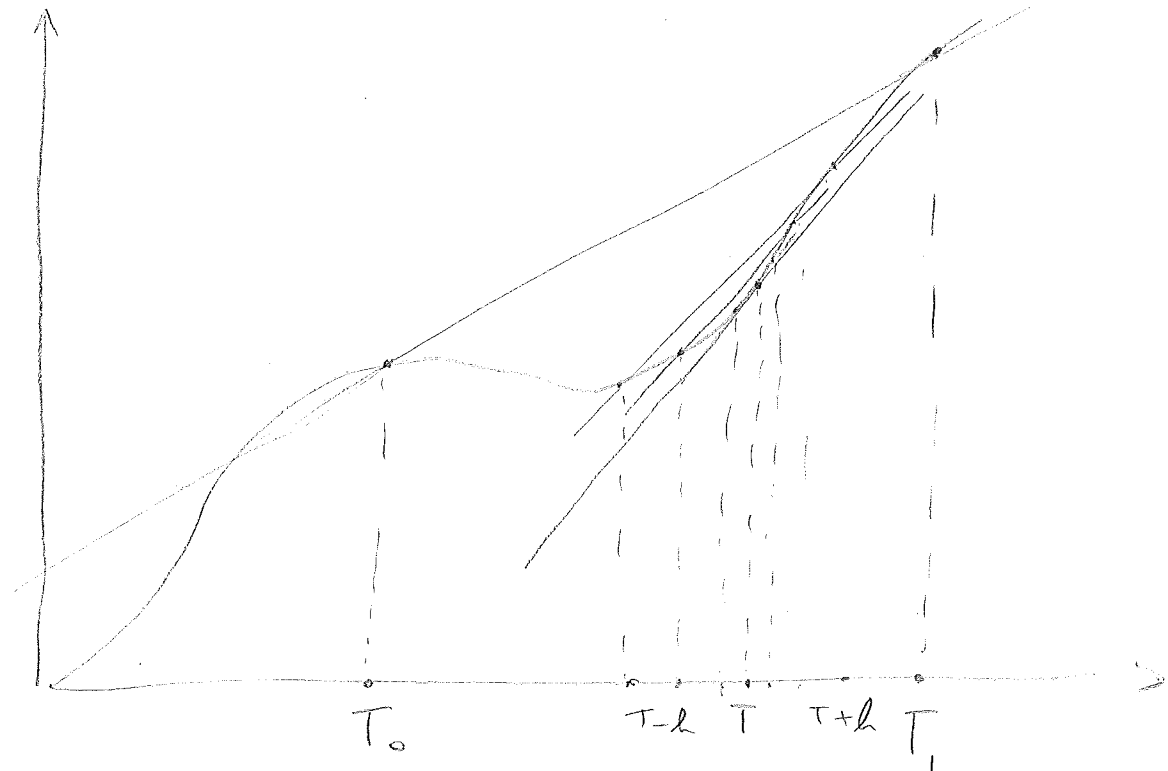
* Where is the car
- going reverse
- accelerating
- decelerating
- still

FIRST, HOW DO WE "DRAW" THE AVERAGE SPEED FROM T_0 TO T_1 ON THIS GRAPH?

AVERAGE SPEED BETWEEN T_0 AND T_1 =

$$\frac{\Delta S}{\Delta T} = \frac{S(T_1) - S(T_0)}{T_1 - T_0}$$

PICK THE POINTS $(T_0, S(T_0))$ AND $(T_1, S(T_1))$ ON THE GRAPH. TAKE THE LINE PASSING THROUGH THEM



WHAT'S THE EQUATION FOR THIS LINE?

$$y = \frac{S(T_1) - S(T_0)}{T_1 - T_0} x + \frac{S(T_1)T_0 - S(T_0)T_1}{T_1 - T_0}$$

THE AVERAGE VELOCITY IS THE SLOPE!

BUT WHAT HAPPENS IF WE SHRINK THE INTERVAL ΔT, THAT IS, WE TAKE T_0, T_1 VERY CLOSE TO EACH OTHER?

WE GET THE SLOPE OF THE TANGENT LINE!

Q1) SELECT ALL OF THE ANSWERS THAT YOU THINK ARE TRUE

A) THE FUNCTION $f(x) = \begin{cases} 3 & x < -1 \\ 2 & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$ DOES NOT HAVE A LIMIT FOR $x \to 0$

B) THE FUNCTION $f(x)$ IN POINT A HAS A LIMIT FOR $x \to 0$, AND THE LIMIT IS 2

C) THE FUNCTION $f(x) = \log(x^2)$ DOES NOT HAVE A LIMIT FOR $x \to 0$

D) THE FUNCTION $f(x) = \dfrac{x^2 - 1}{x - 1}$ IS EQUAL TO 2 WHEN $x = 1$

E) THE FUNCTION $f(x) = \dfrac{x^2 - 1}{x - 1}$ HAS A LIMIT FOR $x \to 0$, AND THE LIMIT IS 2

Q2) THE LIMIT $\displaystyle\lim_{x \to 0} \frac{\sqrt{x^2 + 1} - 1}{x}$

A) IS EQUAL TO 0

B) IS EQUAL TO 1

C) IS EQUAL TO $\frac{1}{2}$

D) DOES NOT EXIST

E) NONE OF THE ABOVE

Q3) LOOK AT THE PICTURE ON THE BLACKBOARD AND SELECT ALL ANSWERS YOU THINK ARE TRUE

A) THE CAR IS GOING FORWARD IN THE BLUE AND GREEN PARTS, REVERSE ON RED

B) THE CAR IS DECELERATING IN THE BLUE PARTS ACCELERATING ON GREEN

C) THE CAR IS NEVER GOING REVERSE

D) THERE ARE PRECISELY FOUR MOMENTS WHEN THE CAR'S SPEDOMETER SHOWED 0, INCLUDING START AND FINISH

E) THE CAR IS ACCELERATING IN THE BLUE PARTS, DECELERATING IN THE GREEN PARTS

Q4) OF THE FOLLOWING FORMULAS, SELECT THE ONES THAT CORRECTLY EXPRESS AVERAGE SPEED FROM T_0 TO T_1.

A) $\frac{S(T_1)}{T_1} - \frac{S(T_0)}{T_0}$

B) $\frac{S(T_1) - S(T_0)}{T_1 - T_0}$

C) $S(T_1)T_0 - S(T_0)T_1$

D) $\frac{S(T_1)}{T_1 - T_0} - \frac{S(T_0)}{T_1 - T_0}$

E) $\frac{S(T_1) - S(T_0)}{T_0 - T_1}$