WARM-UP

i) USE LINEAR APPROX ON $f(x)=\sqrt[3]{x}=x^{\frac{1}{3}}$ AT $a=8$ TO APPROXIMATE $\sqrt[3]{7}$

ii) USE THE FORMULA $|\text{ERROR}|\le\frac{M}{2}(x-a)^2$ WHERE $M>0$, TO ESTIMATE AN ERROR BOUND FOR YOUR APPROX.

iii) IS YOUR APPROXIMATION OF $\sqrt[3]{7}$ TOO LARGE OR TOO SMALL? USE THIS AND POINT (ii), (i) TO CONSTRUCT THE <u>SMALLEST INTERVAL</u> YOU CAN GUARANTEE CONTAINS $\sqrt[3]{7}$

SOL:

$$f(8)=2,\quad f'(8)=\frac{1}{3}\cdot 8^{-\frac{2}{3}}=\frac{1}{12}$$

$$L_8(X)=2+\frac{X-8}{12}\qquad L_8(7)=2-\frac{1}{12}=\frac{23}{12}$$

$$|\text{ERROR}|\le\frac{M}{2}\cdot 1=\max_{[7,8]}\left|-\frac{2}{9}X^{-\frac{5}{3}}\right|=\frac{2}{9}7^{-\frac{5}{3}}<\frac{1}{50}$$

$f''<0$ SO WE ARE APPROX FROM ABOVE

$$\sqrt[3]{7} \text{ IS IN } \left[\frac{23}{12},\ \frac{23}{12}+\frac{2}{9}7^{-\frac{5}{3}}\right]$$

QUADRATIC APPROXIMATION

TAYLOR'S IDEA:

BY WRITING A POLYNOMIAL AS

$$P(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \dots$$

WE HAVE A GREAT DEAL OF IMMEDIATE CONTROL OVER THE DERIVATIVES OF $P(x)$ AT a

$$P(a) = b_0 \;,\; P'(a) = b_1 \;,\; P''(a) = 2b_2$$

$$\frac{d^3P}{dx^3}(a) = 6b_3 \;, \dots \;, \frac{d^n P(a)}{dx^n} = n!\,b_n$$

EXAMPLE:

$P(x) = 3 + 5(x-2) + 6(x-2)^2$

THEN

$P(2) = 3 + 5(0) + 6(0)^2 = 3$

$P'(x) = 5 + 12(x-2)$ SO $P'(2) = 5 + 12 \cdot 0 = 5$

$P''(x) = 12$ SO $P''(2) = 12$

OUR LINEAR APPROXIMATION IS ALREADY IN THIS FORM: GIVEN $f(x)$ AND a

$$L_a(x) = f(a) + (x-a)f'(a)$$

SO IF WE WANT A NEW APPROXIMATION SUCH THAT ALSO THE SECOND DERIVATIVE AGREES WITH THAT OF $f(x)$ AT a WE JUST NEED TO ADD A NEW TERM OF DEGREE 2

$$P_{2,a}(x) = f(a) + (x-a)f'(a) + (x-a)^2 \frac{f''(a)}{2}$$

DEF: THE ABOVE IS THE QUADRATIC APPROXIMATION OF $f(x)$ AT a

THEOREM:

LET $f(x)$ BE A FUNCTION THAT'S THREE TIMES DIFFERENTIABLE ON $[a, x]$ (OR $[x, a]$), THEN:

- $f(a) = P_{2,a}(a)$, $f'(a) = P'_{2,a}(a)$, $f''(a) = P''_{2,a}(a)$
- $|f(x) - P_{2,a}(x)| \leq |a-x|^3 \cdot \frac{M}{6}$ WHERE $M = \mathrm{MAX}_{[a,x]} \left| \frac{d^3 f}{dx^3} \right|$

EXAMPLE:

$f(x) = \cos(x)$. WE WANT TO ESTIMATE $\cos(110°)$

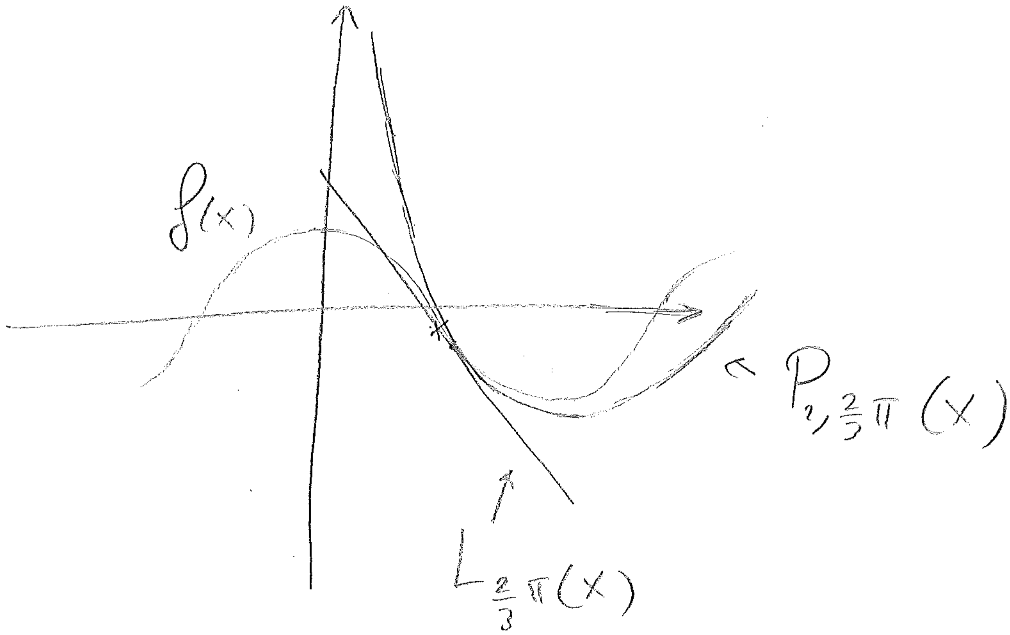
$a = 120° = \frac{2}{3}\pi$.

$$L_{\frac{2}{3}\pi}(x) = -\frac{1}{2} + \left(x - \frac{2}{3}\pi\right)\left(-\sin\left(\frac{2}{3}\pi\right)\right) = -\frac{1}{2} - \left(x - \frac{2}{3}\pi\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$P_{2,\frac{2}{3}\pi}(x) = L_{\frac{2}{3}\pi}(x) + \left(x - \frac{2}{3}\pi\right)^2 \frac{\left(-\cos\left(\frac{2}{3}\pi\right)\right)}{2} = -\frac{1}{2} - \left(x - \frac{2}{3}\pi\right)\frac{\sqrt{3}}{2} + \left(x - \frac{2}{3}\pi\right)^2 \cdot \frac{1}{4}$$

$$L_{\frac{2}{3}\pi}\left(110° = \frac{11}{18}\pi\right) = \frac{1}{2} + \frac{\pi}{18}\left(\frac{\sqrt{3}}{2}\right)$$

$$P_{2,\frac{2}{3}\pi}\left(\frac{11}{18}\pi\right) = \frac{1}{2} + \frac{\pi}{18}\frac{\sqrt{3}}{2} + \frac{\pi^2}{(18)^2 \cdot 4}$$



WORST CASE ERRORS:

FOR $L_{\frac{2}{3}\pi}(x)$ ERROR $\leq \left(x-\frac{2}{3}\pi\right)^2 = \left(\frac{\pi}{18}\right)^2 \frac{1}{2} \sim 0.016$

$\uparrow |f''(x)| \leq 1$

FOR $P_{2,\frac{2}{3}\pi}(x)$ ERROR $\leq \dfrac{\left|\left(x-\frac{2}{3}\pi\right)^3\right|}{2} = \left(\frac{\pi}{18}\right)^3 \cdot \frac{1}{6} \sim 0.001$

$\uparrow$

$|f'''(x)| \leq 1$

IT'S ~ 10× BETTER!

NOTE: SUPPOSE THAT $\dfrac{d^3 f}{dx^3}$ DOES NOT CHANGE SIGN BETWEEN a AND x

THEN:

IF $x > a$ $\quad$ SIGN $\left(f(x) - P_{2,a}(x)\right) =$ SIGN f'''

IF $x < a$ $\quad$ SIGN $\left(f(x) - P_{2,a}(x)\right) = -$SIGN f'''

EXAMPLE: IN THE QUADR. APPROX ABOVE,

$\cos x - P_{2,\frac{2}{3}\pi}(x) > 0$ FOR $x < \frac{2}{3}\pi$,

$\cos x - P_{2,\frac{2}{3}\pi}(x) < 0$ FOR $x > \frac{2}{3}\pi$ (CLOSE TO $\frac{2}{3}\pi$)

EXAMPLE:

$f(x) = \sqrt[4]{x}$ APPROXIMATE $\sqrt[4]{90}$

$$P_{2,81}(x) = f(81) + (x-81)f'(81) + (x-81)^2 \frac{f''(81)}{2}$$

$$= 3 + \frac{(x-81)}{4\cdot 27} - \frac{(x-81)^2}{2\cdot 4^2\cdot 3^6}$$

$$P_{2,81}(90) = 3 + \frac{9}{4\cdot 27} - \frac{9^2}{2\cdot 4^2\cdot 3^6} = 3 + \frac{1}{12} - \frac{1}{2\cdot(12)^2} = \frac{887}{288}$$

EXAMPLE:

$f(x)$ IMPLICITLY DEFINED BY $yx^2 + x = 0$ $f(-1) = 1$
$\overset{\shortparallel}{y}$ USE A QUADRATIC APPROXIMATION TO ESTIMATE y AT $-\frac{1}{2}$

$yx^2 + x = 0 \sim y'x^2 + 2xy + 1 = 0$

$y' = \frac{-2xy-1}{x^2}$ AT $(-1, 1)$ $y' = 1$

$$y'' = \frac{(-2y - 2xy')x^2 - 2x(-2xy-1)}{x^4} \quad \text{AT } (-1,1)$$

$$y'' = \frac{0 - 2}{1} = -2$$

$P_{2,-1}(x) = 1 + (x+1) - (x+1)^2$ $P_{2,-1}(-\frac{1}{2}) = \frac{3}{2} - \frac{1}{4} = \frac{5}{4}$

WRAP-UP :

1. WRITE DOWN THE LINEAR AND QUADRATIC APPROXIMATION OF $\tan^{-1}(x^2)$ WITH CENTER $a = 1$.

2. USE A QUADRATIC APPROXIMATION TO ESTIMATE $\sqrt{10}$. GIVE AN ERROR BOUND.

3. WRITE DOWN THE LINEAR AND QUADRATIC APPROXIMATION OF THE FUNCTION IMPLICITLY DEFINED BY $x^2y^4=1$ AT $(4, \frac{1}{2})$

4. USE LINEAR AND QUADRATIC APPROX TO ESTIMATE $\ln(0.8)$. GIVE A BOUND FOR MAXIMUM ERROR.

SOL:

1. $$\left(\tan^{-1}(x^2)\right)' = \frac{2x}{1+x^4}, \quad \left(\tan^{-1}(x^2)\right)'' = \frac{2(1+x^4)-2x(1+4x^3)}{(1+x^4)^2}$$

AT $a=1$ $\quad f(1) = \frac{\pi}{4}, \quad f'(1) = 1,$

$$f''(1) = -\frac{6}{4} = -\frac{3}{2}$$

SO

$$L_1(x) = \frac{\pi}{4} + (x-1)$$

$$P_{2,1}(x) = \frac{\pi}{4} + (x-1) - (x-1)^2\frac{3}{4}$$

(2) $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$, $f''(x) = \frac{-1}{4\sqrt{x^3}}$

$a = 9$

$L_9(x) = 3 + \frac{1}{6}(x-9)$ $L_9(10) = 3 + \frac{1}{6}$

$P_{2,9}(x) = 3 + \frac{1}{6}(x-9) - \frac{(x-9)^2}{4 \cdot 27}$ $P_{2,9}(10) = 3 + \frac{1}{6} - \frac{1}{108}$

$|\text{LINEAR ERROR}| \leq \frac{M}{2}(10-9)^2 = \frac{M}{2} = \frac{1}{216}$

$M = \max_{[9,10]} \left| \frac{-1}{4\sqrt{x^3}} \right| = \frac{1}{4 \cdot 27} = \frac{1}{108}$

$|\text{QUADR. ERROR}| \leq \frac{M}{6}(10-9)^3 = \frac{M}{6} = \frac{3}{6 \cdot 648} = \frac{1}{1296}$

$M = \max_{[9,10]} \left| \frac{3}{8} x^{-\frac{5}{2}} \right| = \frac{3}{8} \cdot \frac{1}{3^5} = \frac{3}{8 \cdot 81}$

(3)

$x^2 y^4 = 1$ $2xy^4 + 4y'y^3x^2 = 0$

$y' = \frac{-2xy^4}{4y^3x^2} = -\frac{y}{2x}$ $y'' = \frac{-y'2x + 2y}{4x^2}$

AT $(4, \frac{1}{2})$ $y' = -\frac{1}{16}$ $y'' = \frac{\frac{1}{2} + 1}{64} = \frac{3}{128}$

$L_4(x) = \frac{1}{2} - \frac{(x-4)}{16}$ $P_{2,4} = \frac{1}{2} - \frac{(x-4)}{16} + \frac{(x-4)^2}{128} \cdot 3$

4 $f'(x) = \frac{1}{x}$ $\quad f''(x) = -\frac{1}{x^2}$

WE PICK $a = 1$

$f'(1) = 1 \quad f''(1) = -1$

$L_a(x) = 0 + (x-1) \quad P_{1,a}(x) = 0 + (x-1) - \frac{(x-1)^2}{2}$

$L_a(0.8) = -0.2 \quad P_{2,a}(0.8) = -0.2 - \frac{0.04}{2}$

ERROR

$|\text{LINEAR ERR}| \le \frac{M}{2}(1-0.8)^2 = \frac{0.04}{2\cdot(0.64)} = \frac{1}{32}$

$|\text{QUAD ERR}| \le \frac{M}{6}(|1-0.8|)^3 = \frac{0.008}{6} \cdot \frac{2}{(0.8)^3}$

$= \frac{1}{100} \cdot \frac{1}{3\cdot(0.64)} = \frac{1}{3\cdot 64} = \frac{1}{192}$