THE TAYLOR POLYNOMIAL

GIVEN A FUNCTION $f(x)$ THAT IS DIFFERENTIABLE TWICE, WE DEFINED THE QUADRATIC APPROXIMATION WITH CENTER a:

$$P_{2,a}(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2}$$

SO, a IS THE CENTER, BUT WHY THE 2? YOU CAN GUESS – THERE ARE MORE!

DEF: LET $f(x)$ BE A FUNCTION SUCH THAT $\frac{d^m}{dx^m}f(x)$ EXISTS (ON A NEIGHBOURHOOD OF a) THE m-TH TAYLOR POLYNOMIAL OF $f(x)$ AT a IS

$$P_{m,a}(x) = f(a) + f'(a)(x-a) + \dots + f^{(m)}(a)\frac{(x-a)^m}{m!}$$

(SO: COEFFICIENT OF $(x-a)^r = \frac{d^r}{dx^r}f(a) \cdot \frac{1}{r!}$ FOR $r = 0, \dots, m$ $(0! = 1)$)

THE TAYLOR POLYNOMIALS GIVE US A SEQUENCE OF APPROXIMATIONS OF $f(x)$ OF INCREASING PRECISION

THM: (LAGRANGE'S REMAINDER – OPTIONAL)
SUPPOSE $f(x)$ IS DIFF $m+1$ TIMES ON $[a, x]$. THEN

$$f(x) - P_{m,a}(x) = \frac{d^{m+1}}{dx^{m+1}}f(\xi_x) \cdot \frac{(x-a)^{m+1}}{m+1!} \quad \text{FOR SOME } \xi_x$$

IN $[a, x]$.

THE THEOREM ABOVE (WHICH IS <u>OPTIONAL</u>, DON'T WORRY IF IT CONFUSES YOU) ALSO JUSTIFIES EVERYTHING WE SAID ABOUT THE ERROR

$$|f(x) - L_a(x)| \quad \text{AND} \quad |f(x) - P_{2,a}(x)|$$

(TRY TO UNDERSTAND WHY!)

EXAMPLE:

WE WANT A BETTER ESTIMATE OF e. WE CAN TAKE $a=0$, THEN

$\frac{d^m}{dx^m} e^x = e^x$ FOR ALL m, SO

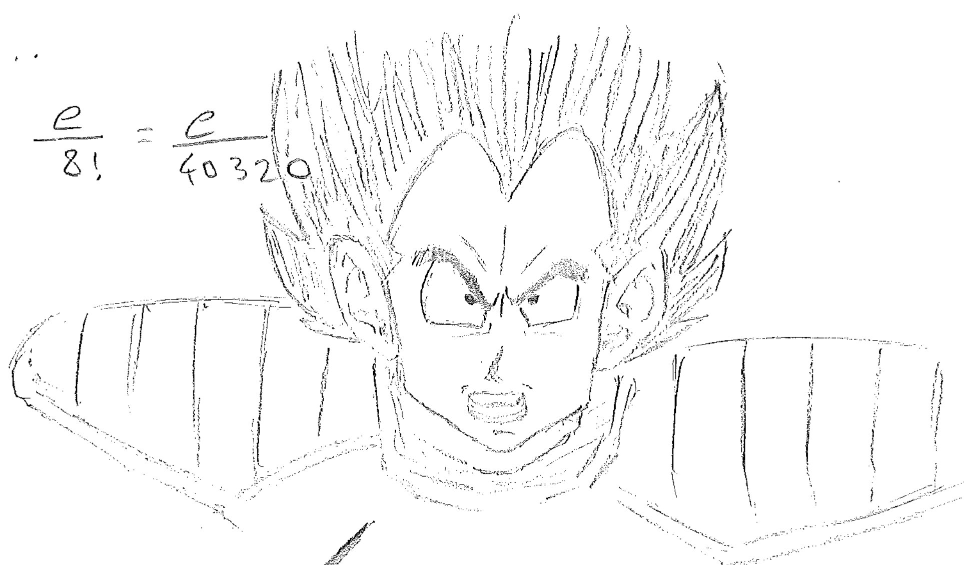
$$P_{m,0}(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots + \frac{x^m}{m!}$$

TAKE FOR EXAMPLE $m = 7$

$$P_{7,0}(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$$

$$= 2.71825\dots$$

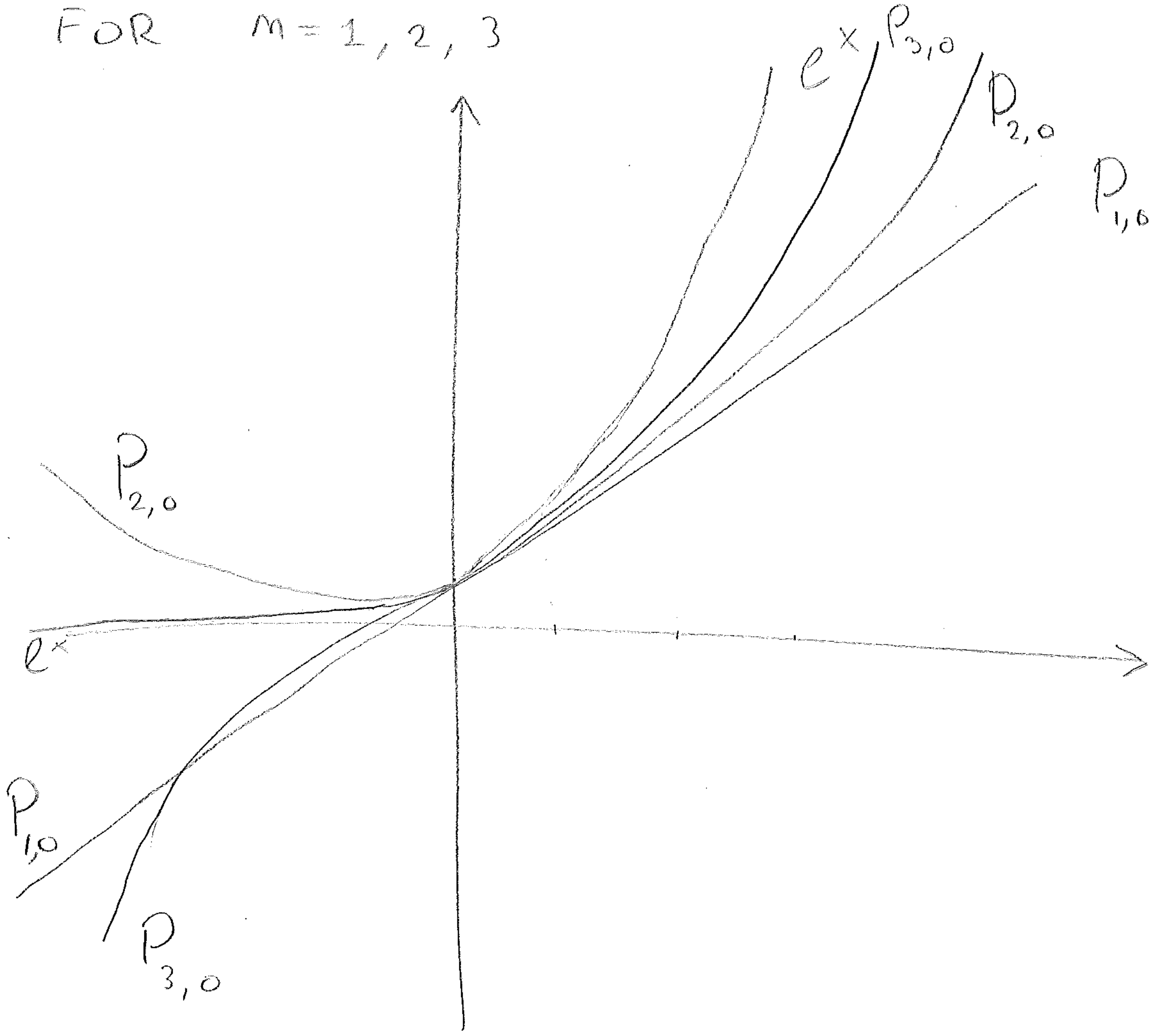
MAX ERROR: $\frac{e}{8!} = \frac{e}{40320}$



IT'S UNDER 1/9000!!

HERE IS A SKETCH OF $P_{m,0}(x)$

FOR $m = 1, 2, 3$



EXAMPLE:

SAY WE WANT TO COMPUTE π.

IDEA: $\tan^{-1}(1) = \frac{\pi}{4}$

SO WE CAN TRY TO APPROXIMATE $4\tan^{-1}(1)$.

WE KNOW THAT $\tan^{-1}(0) = 0$, SO WE TAKE $a = 0$

$f(x) = \tan^{-1}(x)$ $\quad f'(x) = \frac{1}{x^2+1}$ $\quad f''(x) = \frac{-2x}{(x^2+1)^2}$ $\quad f'''(x) = \frac{6x^2-2}{(x^2+1)^3}$

SO $P_{3,0}(x) = 0 + x + 0 - \frac{x^3}{3}$ $\qquad P_{3,0}(1) = \frac{2}{3}$

$4P_{3,0}(1) = \frac{8}{3}$ NOT A GREAT APPROX OF π...

TO GET A BETTER ONE WE NEED $P(1) = 3$?

EXAMPLE:

HOW ABOUT sin OF, SAY, 0.1 ~ 6°?

$a=0$ $\quad \sin'(x) = \cos(x),\ \sin''(x) = -\sin(x),$

$\sin^{(3)}(x) = -\cos(x), \ldots$

$$P_{3,0}(x) = \sin(0) + \cos(0)x - \sin(0)\frac{x^2}{2} - \cos(0)\frac{x^3}{\cancel{3}\,6}$$

$$= x - \frac{x^3}{6}$$

$$P_{3,0}(0.1) = 0.1 - \frac{0.001}{6} = \frac{0.599}{6} \sim 0.099983$$

sin(X) SEEMS PRETTY CLOSE TO X WHEN X IS CLOSE TO 0!

ACTUALLY, USING THE TAYLOR POLYNOMIAL IT'S <u>VERY EASY</u> TO SHOW THAT

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \ !$$

MCLAURIN POLYNOMIALS

WHEN $a=0$ THE TAYLOR POLYNOMIAL WITH CENTER 0 IS CALLED <u>MCLAURIN POLYNOMIAL</u>

THESE ARE USED EVERYWHERE AND YOU SHOULD REMEMBER <u>THE FOLLOWING</u>:

FOR e^x $\quad P_{0,m}(x) = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{m!}$ $\quad$ m-TH TERM $\frac{x^m}{m!}$

FOR $\sin x$ $\quad P_{0,m}(x) = 0 + x + 0 - \frac{x^3}{6} + 0 + \frac{x^5}{120} + \ldots$

m TH TERM: 0 m EVEN

$-\frac{x^m}{m!}$ IF $m = 4k-1$, $\frac{x^m}{m!}$ IF $m = 4k+1$

FOR $\cos(x)$ $P_{0,m}(x) = 1 + 0 - \frac{x^2}{2} + 0 + \frac{x^4}{24} + 0 + \dots$

m-TH TERM $\begin{cases} 0 & m \text{ ODD} \\ -\frac{x^m}{m!} \quad m = 4k+2, & \frac{x^m}{m!} \quad m = 4k \end{cases}$

EXAMPLE:

ESTIMATE $\cos(1)$ USING A 4-TH DEGREE EXPANSION

$$P_{0,4}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} \qquad P_{0,4}(1) = 1 - \frac{1}{2} + \frac{1}{24} = \frac{13}{24}$$

TIPS FOR THE FINAL

EXAM STRUCTURE

- 150 min (2.5 hrs) LONG
- 100 MARKS
- ~48 MARKS ARE SHORT ANSWER QUESTIONS
  - 3 PTS EACH
- ~52 MARKS ARE LONG ANSWER
- LAST YEAR'S EXAM IS A GOOD GUIDE FOR LAYOUT

WHAT, WHERE, WHEN

- CLOSED BOOK. NO NOTES, CALCULATORS, PHONES.
- BRING ID, THINGS TO WRITE.
- YOU ARE RESPONSIBLE FOR BEING THERE AT RIGHT TIME AND PLACE, YOU WON'T BE ADMITTED TO ANOTHER SECTION'S EXAM

HTTP://STUDENTS.UBC.CA/ENROLMENT/EXAMS/EXAM-SCHEDULE/

- DOUBLE, TRIPLE, QUADR. CHECK

TOPICS AND DIFFICULTY

- ALL TOPICS COVERED IN CLASS MAY BE ON EXAM
- NOT ALL QUESTIONS ARE EQUALLY DIFFICULT
- SKIM ALL QUESTIONS BEFORE STARTING AND START WITH THE EASY ONES.
- THERE WILL BE A COUPLE VERY HARD QUESTIONS. IDENTIFY THEM AND DO THEM LAST.
- REMEMBER THAT YOU DO NOT NEED TO ANSWER ALL QUESTIONS TO DO WELL ON THE EXAM.

PEN VS PENCIL

- USE A DARK PENCIL OR A BLUE OR BLACK PEN
- BRING AN APPROPRIATE ERASER / WHITEOUT

EXTRA PAPER

- THERE ARE 2 BLANK PAGES AT THE END IF YOU NEED EXTRA SPACE
- THEY WON'T BE GRADED, UNLESS YOU CLEARLY INDICATE THAT YOU CONTINUED A QUESTION THERE
  - ---> ON QUESTION PAGE WRITE "CONTINUED ON PAGE —"
  - ---> ON BLANK PAGE WRITE QUESTION NUMBER

WRAP UP

- WRITE DOWN THE THIRD DEGREE TAYLOR POLYNOMIAL OF $f(x) = \sin(x)\cos(x)$ AT $a=0$
- IF THE THIRD DEGREE TAYLOR POLY AT $a=1$ OF $f(x)$ IS x^3+3x AND THE THIRD DEGREE TAYLOR POLY AT $a=1$ OF $g(x)$ IS x^2+x+2 WHAT'S THE THIRD DEGREE TAYLOR POLY AT $a=1$ OF $f(x)g(x)$?
- IF $f(x)$ IS A POLYNOMIAL OF DEGREE m, WHAT'S ITS m-TH DEGREE TAYLOR POLYNOMIAL AT ANY POINT? HINT: USE THE REMAINDER THM; WHAT IS THE $m+1$ TH DERIVATIVE OF AN m-TH DEGREE POLY?