APPROXIMATION

WHICH FUNCTIONS CAN WE COMPUTE <u>EXACTLY</u>?

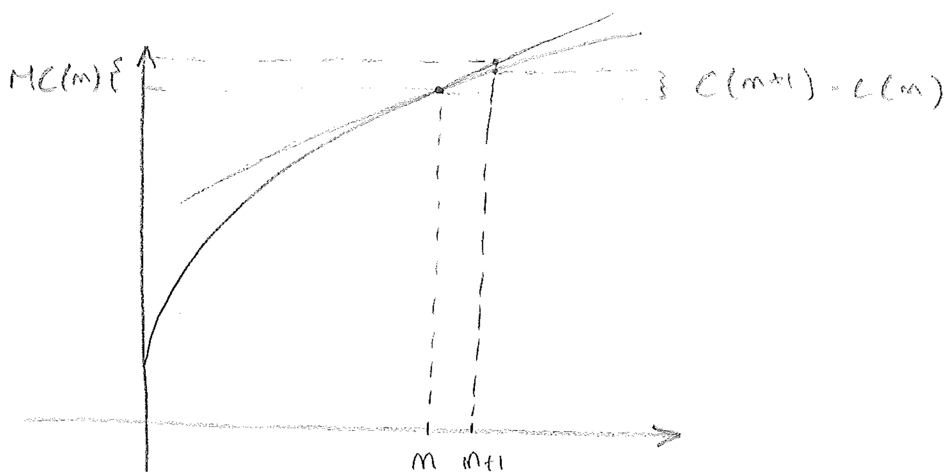
- POLYNOMIALS
- RATIONAL FUNCTIONS
- THAT'S IT.

THIS ALSO APPLIES TO COMPUTERS. SO HOW DO WE <u>NUMERICALLY</u> UNDERSTAND <u>ANYTHING</u>? HOW DOES A SCIENTIFIC CALCULATOR WORK?

BY USING <u>APPROXIMATION</u>.

<u>EXAMPLE</u>:

WHEN WE USE THE MARGINAL COST $MC(n)$ TO APPROXIMATE THE COST OF THE $n+1$-TH UNIT, WE'RE APPROXIMATING $C(q)$ WITH ITS TANGENT AT $q=n$



BUT <u>WHY</u> SHOULD THIS WORK?

WELL, LET'S PICK A VALUE x VERY CLOSE TO a.

WE HAVE $\lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a} = f'(a)$, SO

$$f'(a) \cong \frac{f(x)-f(a)}{x-a} \Rightarrow \underbrace{f(a)+f'(a)(x-a)}_{\text{TANGENT LINE AT } x\,!} \cong f(x)$$

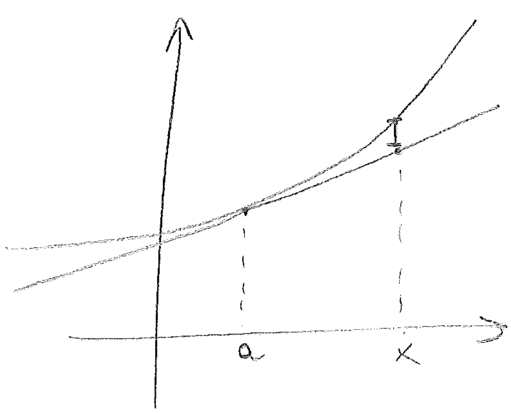
SO IF WE HAVE $f(a)$, $f'(a)$ WE CAN USE THE TANGENT TO ESTIMATE $f(x)$, FOR x CLOSE TO a.

<u>BUT:</u>

- IS THIS ESTIMATE ANY GOOD?
- ARE WE OVERSHOOTING OR UNDERSHOOTING? THAT IS, ARE WE APPROXIMATING FROM ABOVE OR BELOW?

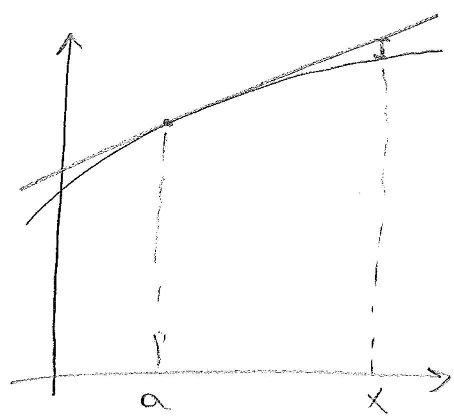
WELL, WE DO HAVE A PROPERTY OF f TELLING US WETHER THE TANGENT LIES <u>ABOVE</u> OR <u>BELOW</u> OUR FUNCTION!

f CONCAVE UP AT a



WE'RE UNDERESTIMATING

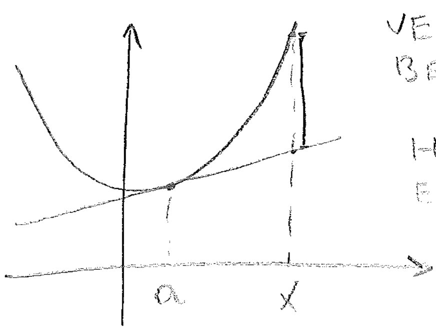
f CONCAVE DOWN AT a



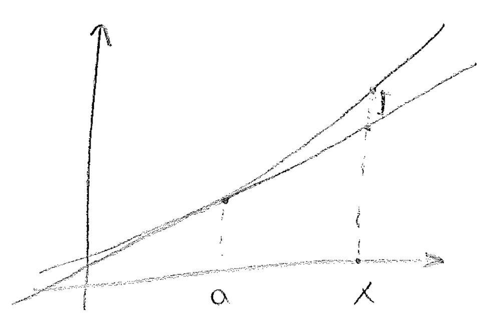
WE'RE OVERESTIMATING

SO IT LOOKS LIKE THE DIFFERENCE BETWEEN $f(x)$ AND OUR APPROXIMATION HAS SIGN OPPISITE TO $f''(x)$

AND HOW BIG WILL THE ERROR BE? IT DEPENDS ON "HOW BENT" $f(x)$ IS!



VERY BENT = HIGH ERROR



NOT VERY BENT = LOW ERROR

LET'S TRY TO GET THINGS IN THE RIGHT ORDER.

AN APPROXIMATION IS <u>USELESS</u> IF WE CANNOT SAY ANYTHING ABOUT <u>HOW WELL</u> IT APPROXIMATES OUR FUNCTION $f(x)$ (CLOSE TO A GIVEN POINT a)

SO WE'LL NEED:

- AN APPROXIMATING FUNCTION (AT a) $g_a(x)$ SATISFYING SOME CONDITIONS (EXAMPLE: $g_a(a) = f(a)$, $g_a'(a) = f'(a)$; THE TANGENT AT a IS THE "SIMPLEST" FUNCTION SATISFYING THESE TWO)
- A BOUND FOR THE ERROR (OR "WORST CASE ERROR") CLOSE TO a, THAT IS, AN ESTIMATE OF

$$|f(x) - g_a(x)| \text{ IN TERMS OF } |x-a|$$

IDEALLY WE WANT $|f(x) - g_a(x)|$ TO BE <u>MUCH SMALLER</u> THAN $|x-a|$ WHEN x IS CLOSE TO a.

LINEAR APPROXIMATION

LET $f(x)$ BE A FUNCTION, DIFFERENTIABLE AT a. THE <u>LINEAR APPROXIMATION</u> OF $f(x)$ AT a

IS THE FUNCTION $L_a(x) = f'(a)(x-a) + f(a)$

IT SATISFIES

- $L_a(a) = f(a)$
- $L_a'(a) = f'(a)$

, MOREOVER:

THEOREM: SUPPOSE $f(x)$ IS DIFFERENTIABLE TWICE ON $[a,x]$ (RESP. $[x,a]$ IF $x<a$). THEN THE FOLLOWING BOUND FOR THE ERROR OF LINEAR APPROXIMATION HOLDS

$$|f(x)-L_a(x)| \le \frac{M}{2}|x-a|^2$$

WHERE $M =$ MAX OF $|f''|$ ON $[a,x]$ (RESP. $[x,a]$).

MOREOVER, IF $f''>0$ ON $[a,x]$ THEN

$$f(x)-L_a(x) \ge 0$$

IF $f''<0$ ON $[a,x]$ THEN

$$f(x)-L_a(x) \le 0.$$

EXAMPLE:

APPROXIMATE $\sqrt[5]{31}$ USING A LINEAR APPROXIMATION.

STEP 1: WE START FROM A VALUE WE KNOW

$$32 = 2^5 \Longrightarrow \sqrt[5]{32} = 2$$

WE WANT TO USE LINEAR APPROXIMATION AT 32

$a = 32$, $f(x) = \sqrt[5]{x}$

$$f(a) = 2, \quad f'(a) = \frac{1}{5} \cdot a^{\frac{1}{5}-1} = \frac{1}{5} \cdot 32^{-\frac{4}{5}} = \frac{1}{5} \cdot \frac{1}{(\sqrt[5]{32})^4}$$

$$= \frac{1}{5} \cdot \frac{1}{16} = \frac{1}{80}$$

SO $L_{32}(x) = \dfrac{x-32}{80} + 2$, $L_{32}(31) = 2 - \dfrac{1}{80}$

THE WORST CASE ERROR IS:

$$\left|\sqrt[5]{32} - \left(2-\frac{1}{80}\right)\right| \le \frac{M}{2}(32-31)^2 = \frac{M}{2} \qquad M = \mathrm{MAX}_{[31,32]}(|f''|)$$

$f''(x) = -\dfrac{4}{25}x^{-\frac{9}{5}}$ MAX OF $|f''|$ AT 31 $\left|\dfrac{-4}{25} 31^{-\frac{9}{5}}\right| < \dfrac{1}{200}$

EXAMPLE:

APPROXIMATE $\log_2 10$

WE KNOW $\log_2 8 = 3$, SO WE PICK $a = 8$

$$(\log_2)'(8) = \left(\frac{\ln}{\ln 2}\right)'(8) = \frac{1}{8\ln 2}$$

SO $L_8(x) = \frac{1}{8\ln 2}(x-8) + 3$, $L_8(10) = \frac{2}{8\ln 2} + 3$

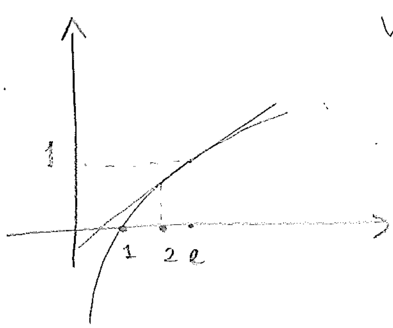
$$= \frac{1}{4} \cdot \frac{1}{\ln 2} + 3$$

$$|(\log_2)''(x)| = \left|\left(\frac{\ln}{\ln 2}\right)''(x)\right| = \left|-\frac{1}{x^2 \ln 2}\right| \quad \text{MAX AT } x = 8 \quad \frac{1}{64\ln 2}$$

SO $L_8(10) = \frac{1}{4\ln 2} + 3$ WITH WORST CASE ERROR

$$(10-8)^2 \cdot \frac{1}{128\ln 2} = \frac{1}{32\ln 2}$$

BUT HOW ABOUT $\ln 2$?



VALUES WE KNOW: $\ln(1) = 0$, $\ln(e) = 1$

LESS "BENT" AT e

$$(\ln)'(e) = \frac{1}{e} \qquad (\ln)'' = -\frac{1}{x^2}$$

$$L_e(x) = \frac{(x-e)}{e} + 1, \quad L_e(2) = \frac{2}{e}$$

MAX ERROR: $(e-2)^2 \cdot \frac{1}{2 \cdot 2^2} \sim \frac{0.51}{8} \sim 0.06$

NOTE: THIS IS STILL A PRETTY POOR APPROXIMATION! $\frac{2}{e} \cong 0.74$, $\ln 2 \cong 0.69 \ldots$

CLOSE TO MAX ERROR! ALSO, WHAT ABOUT e ITSELF?

WRAP-UP:

1) USE LINEAR APPROXIMATION TO ESTIMATE $(7.5)^{\frac{1}{3}}$ APPROXIMATING $x^{\frac{1}{3}}$ AT $a=8$. GIVE A BOUND FOR THE ERROR. OVER OR UNDER?

SOL: $f(8)=2$, $f'(8)=\frac{1}{3\,8^{\frac{2}{3}}}=\frac{1}{12}$

$L_8(x)=\frac{x-8}{12}+3 \qquad L_8(7.5)=3-\frac{1}{24}=\frac{71}{24}$

$$\text{ERROR} \le \max_{[7.5,8]} |f''(x)| \cdot \frac{0.5^2}{2}$$

$f''(x)=-\frac{2}{9}\cdot\frac{1}{x^{\frac{5}{3}}}$ ← OVER   MAX IS AT 7.5 $\quad |f''(x)|=\frac{2}{9\cdot(7.5)^{\frac{5}{3}}}$

SO MAX ERROR $\le \frac{0.25}{9\cdot(7.5)^{\frac{5}{3}}}=\frac{1}{36}\cdot\frac{1}{7.5}\cdot\frac{1}{\sqrt[3]{(7.5)^2}}<\frac{1}{750}$

2) APPROXIMATE $\ln(1.25)$ BY USING LINEAR APPROXIMATION AT $a=1$. GIVE A BOUND FOR ERROR, OVER OR UNDER?

SOL: $\ln(1)=0$, $\ln'(1)=1$

$L_1(x)=x-1 \qquad L_1(1.25)=0.25 \qquad f''=-\frac{1}{x^2}$ ← OVER

$$\text{ERROR} \le \max_{[1,1.25]}\left|-\frac{1}{x^2}\right|\cdot\frac{(0.25)^2}{2}$$

MAX AT $x=1$ IS 1 SO ERROR $\le \frac{(0.25)^2}{2}=\frac{1}{32}$

SOME EXERCISES FROM THE MIDTERM:

4a) $(p+10)(q+20) = p^2 q$. FIND ϵ

$q+20+q'(p+10) = 2pq + p^2 q'$

$q'(-p^2+p+10) = 2pq - q - 20 \qquad p=10,\ q=5$

$q'(-80) = 75 \qquad q' = -\frac{75}{80} = -\frac{15}{16}$

$\epsilon = q' \cdot \frac{p}{q} = 2q' = -\frac{15}{8}$

4b) $\lim_{x \to -\infty} \frac{\sqrt{4x^6 - 2x^4 + 5x^2 - 1}}{5x^3 - x^2 + 3x + 25}$

NOTE: $\sqrt{p(x)}$ IS ALWAYS ≥ 0

$\sqrt{x^2} \neq x$, $= |x|$; SO $\sqrt{x^6} = |x^3| = \begin{cases} x^3 & x > 0 \\ -x^3 & x < 0 \end{cases}$

SO: $\frac{\sqrt{4x^6 - 2x^4 + 5x^2 - 1}}{5x^3 - x^2 + 3x + 25} = \frac{-2x^3}{5x^3} \cdot \frac{\sqrt{1 - \frac{1}{2x^2} + \frac{5}{4x^4} - \frac{1}{4x^6}}}{1 - \frac{1}{5x} + \frac{3}{5x^2} + \frac{25}{5x^3}}$

FOR $x < 0$!

$\lim_{x \to -\infty} \frac{\sqrt{4x^6 - 2x^4 + 5x - 1}}{5x^3 - x^2 + 3x + 25} = \underbrace{\lim_{-\infty} \frac{-2x^3}{5x^3}}_{= -\frac{2}{5}} \cdot \frac{\sqrt{1 - \frac{1}{2x^2} + \frac{5}{4x^4} - \frac{1}{4x^6}}}{1 - \frac{1}{5x} + \frac{3}{5x^2} + \frac{25}{5x^3}}$ GOES TO 1

$= -\frac{2}{5}$

5a) $qe^{2P} = 3$ $\quad \frac{dP}{dT} = -3 \quad \frac{dq}{dT} = ? \quad P = 10$

$$q = 3e^{-2P} \qquad \frac{dq}{dT} = \frac{d\,3e^{-2P}}{dT} = -6P'e^{-2P} = \frac{18}{e^{20}}$$

5b) $f(x) = x^5 + x \quad (f^{-1})'(2)$? TANGENT AT 2.?

$$f^{-1}(x) = y \qquad f(y) = x \qquad y^5 + y = x$$

AT $x = 2$ $\quad y^5 + y = 2$, $y = 1$ IS SOL

$$y' = \frac{1}{f'(y)} = \frac{1}{5y^4 + 1} = \frac{1}{6}$$

TANGENT $\quad y = \frac{1}{6}(x-2) + 1$

6) $f(x) = \frac{x^3}{e^x}$, $f'(x) = \frac{x^2(3-x)}{e^x}$, $f''(x) = \frac{x(x^2-6x+6)}{e^x}$

- INCR. WHEN $f' > 0$, DEC WHEN $f' < 0$

$e^x > 0$ SO WE CHECK $x^2(3-x)$

$+ \quad 0 \quad + \quad 3 \quad -$

$3-x$: $+ \quad 3 \quad -$

x^2: $+ \quad 0 \quad +$

INCR ON $(-\infty, 3)$, DECR ON $(3, +\infty)$,

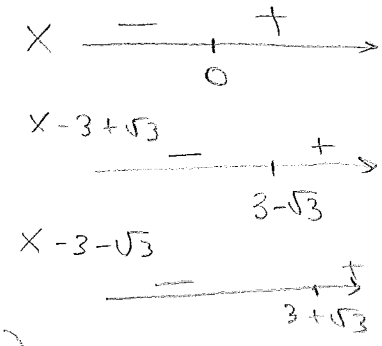
MAX AT $x = 3$

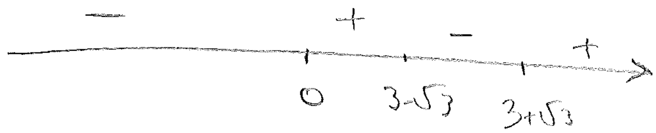
- C. UP WHEN $f'' > 0$
  C. DOWN WHEN $f'' < 0$

  WE NEED SIGN OF $x(x^2-6x+6)$

  $x(x^2-6x+6) = x(x-3+\sqrt{3})(x-3-\sqrt{3})$

  INFL PTS $0,\ 3+\sqrt{3},\ 3-\sqrt{3}$





  C. UP ON $(-\infty, 0) \cup (3-\sqrt{3}, 3+\sqrt{3})$

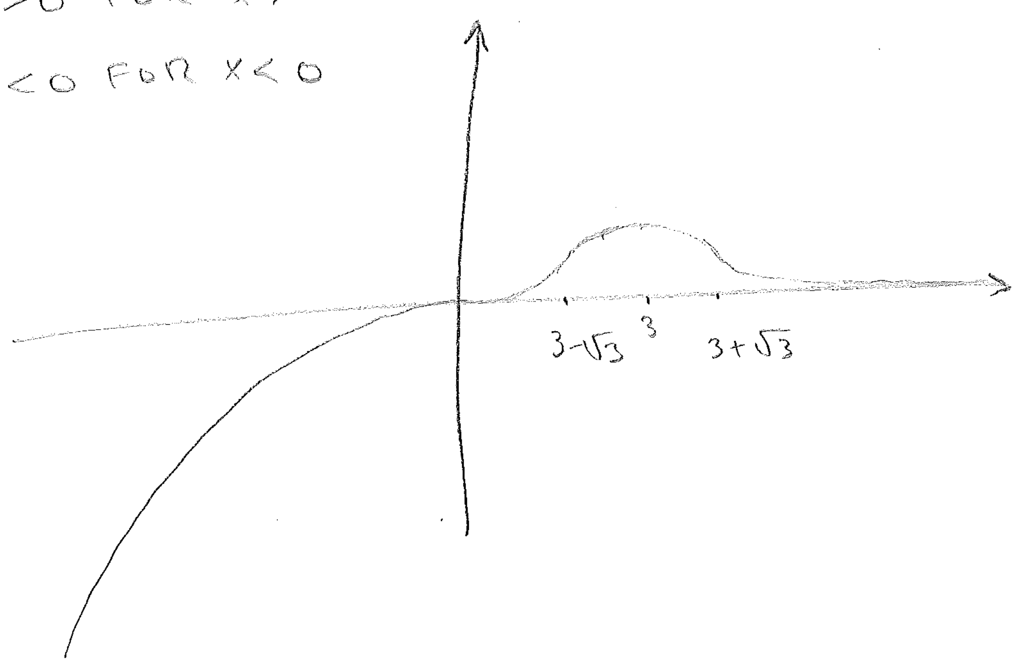
  C. DOWN ON $(0, 3-\sqrt{3}) \cup (3+\sqrt{3}, +\infty)$

- SKETCH:

  $\lim_{x\to+\infty} \frac{x^3}{e^x} = 0$ $\qquad$ $\lim_{x\to-\infty} \frac{x^3}{e^x} = \lim_{x\to+\infty} -x^3 e^x = -\infty$

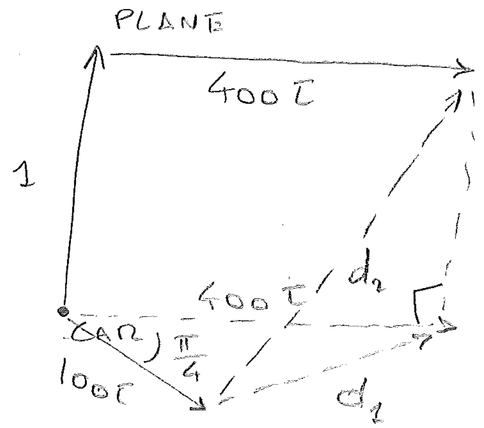
  $f > 0$ FOR $x > 0$

  $f < 0$ FOR $x < 0$



THE NASTY ONE: ALTERNATIVE SOL)

$$\tau = \frac{1}{100}$$



COSINE LAW:

$$d_1^2 = (100\tau)^2 + (400\tau)^2 - 2\cdot(400\cdot 100)\tau^2\cdot\frac{\sqrt{2}}{2} = 100\tau^2(1+16-2\sqrt{2})$$

$$d_1 = 100\tau\sqrt{(17-4\sqrt{2})}$$

$$d_2 \underset{\uparrow}{=} \sqrt{1+d_1^2} = \sqrt{1+10^4\tau^2(17-4\sqrt{2})}$$

PITHAGORAS

$$\frac{d}{d\tau}d_2 = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}\cdot 10^4\tau(17-4\sqrt{2})}{\sqrt{1+10^4\tau^2(17-4\sqrt{2})}} \qquad \frac{d(d_2)}{d\tau}\left(\frac{1}{100}\right) =$$

$$\frac{100(17-4\sqrt{2})}{\sqrt{1+(17-4\sqrt{2})}} = \frac{1700-400\sqrt{2}}{\sqrt{18-4\sqrt{2}}} \ \text{Km/h}$$