WARM-UP

APPROXIMATE $\sin(40°)$ WITH A LINEAR APPROXIMATION CENTERED AT $a = \frac{\pi}{4}$

$$40° = \frac{\pi}{4} - \frac{\pi}{36}$$

- $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $\quad (\sin)'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

SO $L_{\frac{\pi}{4}}(x) = \left(x - \frac{\pi}{4}\right) \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$

AT $x = \frac{\pi}{4} - \frac{\pi}{36}$ $(= 40°)$

$$L_{\frac{\pi}{4}}\left(\frac{\pi}{4} - \frac{\pi}{36}\right) = \frac{\pi}{36} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}\left(1 + \frac{\pi}{36}\right)$$

MAX ERROR: $\leq \left(\frac{\pi}{36}\right)^2 \cdot \frac{M}{2}$ $\quad M = \max_{\left[\frac{\pi}{4} - \frac{\pi}{36}, \frac{\pi}{4}\right]} \left(|\sin(x)|\right)$

SOMETIMES IT'S EASIER TO USE A "STUPID" BOUND. $M \leq 1$ FOR SURE!

SO MAX ERROR $\leq \frac{\overset{\leq 10}{\pi^2}}{\underset{\geq 10^3}{(36)^2 \cdot 2}} \leq \frac{1}{200}$

EXAMPLE:

APPROXIMATE $\log_2 10$

WE KNOW $\log_2 8 = 3$, SO WE PICK $a = 8$

$$(\log_2)'(8) = \left(\frac{\ln}{\ln 2}\right)'(8) = \frac{1}{8\ln 2}$$

SO $L_8(x) = \frac{1}{8\ln 2}(x-8) + 3$, $L_8(10) = \frac{2}{8\ln 2} + 3$

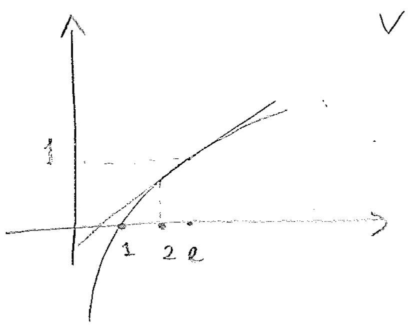
$$= \frac{1}{4} \cdot \frac{1}{\ln 2} + 3$$

$$|(\log_2)''(x)| = \left|\left(\frac{\ln}{\ln 2}\right)''(x)\right| = \left|-\frac{1}{x^2 \ln 2}\right|$$ MAX AT $x = 8$ $\frac{1}{64\ln 2}$

SO $L_8(10) = \frac{1}{4\ln 2} + 3$ WITH WORST CASE ERROR

$$(10-8)^2 \cdot \frac{1}{128\ln 2} = \frac{1}{32\ln 2}$$

BUT HOW ABOUT $\ln 2$?



VALUES WE KNOW: $\ln(1) = 0$, $\ln(e) = 1$

LESS "BENT" AT e

$$(\ln)'(e) = \frac{1}{e} \qquad (\ln)'' = -\frac{1}{x^2}$$

$$L_e(x) = \frac{(x-e)}{e} + 1, \quad L_e(2) = \frac{2}{e}$$

MAX ERROR: $(e-2)^2 \cdot \frac{1}{2 \cdot 2^2} \sim \frac{0.51}{8} \sim 0.06$

NOTE: THIS IS STILL A PRETTY POOR APPROXIMATION! $\frac{2}{e} \cong 0.74$, $\ln 2 \cong 0.69$ ...

CLOSE TO MAX ERROR! ALSO, WHAT ABOUT e ITSELF?

HIGHER APPROXIMATION

LINEAR APPROXIMATION IS GOOD BUT NOT SO GREAT! IT CAN FAIL PRETTY BADLY FOR A VERY BENT FUNCTION. UNLESS WE CAN PICK A CENTER THAT'S EXTREMELY CLOSE TO THE VALUE WE WANT TO APPROXIMATE.

EXAMPLE:

WE TRY TO APPROXIMATE THE VALUE OF e WITH A LINEAR APPROX.

THE ONLY VALUE WE REALLY KNOW IS $e^0 = 1$, SO WE APPROXIMATE WITH $a = 0$.

$e^0 = 1$, $(e^x)' = e^x$, SO $L_0(x) = x + 1$

THEN $L_0(1) = 2 \ldots$ THAT'S A TERRIBLE APPROXIMATION!

THE PROBLEM HERE IS THAT $f(x) = e^x$ IS VERY BENT FOR $x \geq 0$.

IDEA:

WHAT IF WE TAKE INTO ACCOUNT BOTH SLOPE AND BEND AT OUR CENTER?

WE LOOK FOR AN APPROXIMATION IN THE FORM $g_a(x) = b_0 + b_1 x + b_2 x^2$. IN OUR CASE $a = 0$.

$g_a(0) = b_0$ SO $b_0 = e^0 = 1$

$g_a'(0) = b_1$ SO $b_1 = e^0 = 1$

$g_a''(0) = 2 b_2$ SO $b_2 = \frac{e^0}{2} = \frac{1}{2}$

SO $g_a(x) = 1 + x + \frac{x^2}{2}$, $g_a(1) = 2.5$. THAT'S MUCH BETTER!

LET'S TRY APPROXIMATING $\sqrt[4]{90}$. WE KNOW THAT $\sqrt[4]{81} = 3$ SO, WE TAKE $\underline{a = 81}$.

AGAIN WE WANT

- $g_a(81) = f(81) = 3$
- $g_a'(81) = f'(81) = \dfrac{1}{4\sqrt[4]{(81)^3}} = \dfrac{1}{4 \cdot 3^3}$
- $g_a''(81) = f''(81) = \dfrac{-3}{16\sqrt[4]{(81)^7}} = \dfrac{-1}{16 \cdot 3^6}$

$$g_a(x) = b_0 + b_1 x + b_2 x^2$$

$$3 = g_a(81) = b_0 + 81 b_1 + 81^2 b_2$$
$$\frac{1}{4 \cdot 3^3} = g_a'(81) = b_1 + 2 \cdot 81 b_2$$
$$\frac{-1}{16 \cdot 3^3} = g_a''(81) = 2 b_2$$

THREE LINEAR EQUATIONS IN THREE VARIABLES...

WE $\underline{\text{CAN}}$ SOLVE THIS TO GET

$$g_a(x) = \frac{-x^2}{23{,}328} + \frac{7x}{432} + \frac{63}{32}$$

THIS LOOKS HARD! ALSO

$$L_a(x) = \frac{x - 81}{4 \cdot 3^3} + 3 = \frac{x}{108} + \frac{9}{4}$$

THEY LOOK UNRELATED, SO WE CANNOT GO BY STEPS... THIS LOOKS <u>AWFUL</u>!

<u>BUT</u>! (SEE NEXT PAGE)

BROOK TAYLOR'S GREAT IDEA (1712)

WHAT IF WE SHIFTED EVERYTING TO $x=a$?
MORE PRECISELY; IT'S EASY TO APPROXIMATE FUNCTIONS AT $a=0$, BECAUSE IF WE HAVE

$$P(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_m x^m$$

THEN

$$P(0) = b_0, \quad P'(0) = b_1, \quad P''(0) = 2b_2, \dots$$

$$\dots, \quad \frac{d^m P(0)}{dx^m} = 2 \cdot 3 \cdot \dots \cdot m \cdot b_m$$

NOTATION: WE WRITE $m!$ FOR THE NUMBER $1 \cdot 2 \cdot \dots \cdot m$, AND WE CALL IT <u>M FACTORIAL</u>

NOW, WRITE A POLYNOMIAL $P_a(x)$ AS $P_a(x) = b_0 + (x-a) b_1 + (x-a)^2 b_2 + \dots + (x-a)^m b_m$. THEN IT'S NOT HARD TO SEE THAT

$$P_a(a) = b_0, \quad P_a'(a) = b_1, \quad P_a''(a) = 2b_1, \quad \frac{d^m P_a(a)}{dx^m} = m! \, b_m$$

THIS LOOKS MUCH EASIER TO HANDLE!

DEFINITION:

LET $f(x)$ BE A FUNCTION SUCH THAT $\frac{df}{dx}, \dots, \frac{d^m f}{dx^m}$ EXIST. WE DEFINE ITS <u>TAYLOR POLYNOMIAL</u> OF DEGREE m WITH

CENTER a AS:

$$P_{m,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{1}{m!}\frac{d^m}{dx^m}f(a)(x-a)^m$$

WHEN $m=2$, WE CALL IT THE <u>QUADRATIC APPROXIMATION</u> OF $f(x)$ AT a

$$P_{2,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

<u>THEOREM</u>

LET $f(x)$ BE A FUNCTION DIFFERENTIABLE $m+1$ TIMES ON $[a,x]$ (RESP $[x,a]$), THEN

$$|P_{m,a}(x) - f(x)| \leq \frac{M_m}{m+1!}(x-a)^m$$

WHERE $M_m = \max_{[a,x]}\left|\frac{d^m f}{dx^m}\right|$

IN PARTICULAR, FOR THE QUADRATIC APPROXIMATION

$$|P_{2,a}(x) - f(x)| \leq \frac{M_2}{6}(x-a)^3$$

WHERE $M_2 = \max_{[a,x]}\left|\frac{d^3 f}{dx^3}\right|$

MOREOVER, IF $\frac{d^{m+1}f}{dx^{m+1}} \geq 0$ ON $[a,x]$ THEN $P_{m,a}(x) \leq f(x)$ AND IF $\frac{d^{m+1}f}{dx^{m+1}} \leq 0$ ON $[a,x]$ THEN $P_{m,a}(x) \geq f(x)$

SO

- THE TAYLOR POLYNOMIALS APPROXIMATE $f(x)$ WITH INCREASING PRECISION AT a
- ONCE WE KNOW THE TAYLOR POLYNOMIAL OF DEGRE m AT a WE KNOW THE FIRST m COEFFICIENTS OF ALL THE FOLLOWING ONES!