OPTIMIZATION

HOW DO WE APPLY THINGS SUCH AS EXTREME VALUES TO REAL LIFE SITUATIONS?

EXAMPLE

WE HAVE A BIG YARD AND 100 m OF FENCING. WE WANT TO CREATE A RECTANGULAR ENCLOSURE AS WIDE AS POSSIBLE FOR OUR DOG.

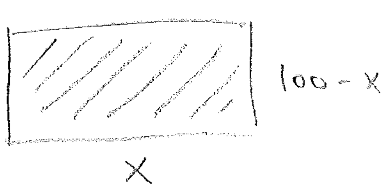
AREA OF ENCLOSURE xy WHERE x, y ARE THE SIDES.

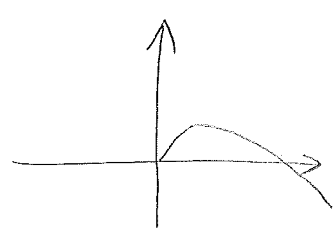
CONSTRAINTS $x \geq 0$, $y \geq 0$, $x + y = 100$

WE CAN REWRITE $y = 100 - x$

SO AREA $= x(100 - x)$ WHERE $0 \leq x \leq 100$
(BECAUSE x, $100 - x$ MUST BE ≥ 0)

FINDING THE BEST SOLUTION





$A(0) = A(100) = 0$

$$\frac{dA}{dx} = -2x + 100$$

CRITICAL POINT $x = 50$ $\quad A(50) = 2500 \text{ m}^2$

MUST BE A MAX BY E.V.T. + MIN/MAX CRITERIA.

SOLVING OPTIMIZATION PROBLEMS

1. UNDERSTAND THE PROBLEM. IDENTIFY THE QUANTITY WE NEED MINIMIZED OR MAXIMISE. A SKETCH MIGHT HELP.

2. WRITE DOWN EQUATIONS RELEVANT TO THE CONTEXT. ONE OF THEM SHOULD DESCRIBY THE QTY TO MIN/MAX. WE CALL IT THE <u>FUNDAMENTAL EQUATION</u>

3. IF THE FUNDAMENTAL EQUATION HAS <u>MORE THAN ONE VARIABLE</u>, USE THE OTHER EQUATIONS TO REDUCE IT TO <u>ONE VARIABLE</u>

$$Q = f(x)$$

4. IDENTIFY THE DOMAIN OF $f(x)$, KEEPING CONTEXT IN MIND.

5. FIND EXTREME VALUES ON OUR DOMAIN

6. IDENTIFY THE VALUES OF ALL RELEVANT QTYs OF THE PROBLEM.

NOTE:

STEP 5 IS WHERE MOST OF THE MATH IS DONE.

YOU NEED TO SHOW THAT YOU HAVE AN ABSOLUTE MAX/MIN.

ON A CLOSED INTERVAL, WE CAN USE THE CLOSED INTERVAL METHOD.
IF YOU ARE NOT ON A CLOSED INTERVAL, YOU NEED TO STUDY THE FUNCTION.

- SOME REASONS WHY YOU COULD SAY A GIVEN VALUE IS A MAX/MIN (GLOBAL)
  - $f(x)$ HAS UNIFORM CONCAVITY.
  - $f'(x)$ ONLY CHANGES SIGN ONCE
  - GRAPHICAL JUSTIFICATION
- REMEMBER TO CHECK SINGULARITIES AND ASYMPTOTES.
- REMEMBER TO CHECK ALL CONSTRAINTS, FOR EXAMPLE, $x \geq 0$ IN THE PREVIOUS PROBLEM.

Guidelines for Solving Optimization Problems:

(a) **Understand the problem:** Read the problem carefully until it is clearly understood. What is the unknown? What are the given quantities? What are the given conditions?

(b) **Draw a diagram:** If possible draw a diagram and identify the given and required quantities.

(c) **Introduce notation:** Identify the objective function (the function to be optimized). Write it in terms of the variables of the problem.

(d) **Reduce to a single variable:** Use the constraints to eliminate all but one variable in the objective function. What is the domain?

(e) **Solve the problem:** Do Calculus. Use techniques from the course to find the absolute maximum or minimum as required.

- Closed Interval Method;
- Absolute Extremum Theorem;
- Concavity Argument;
- Graphical Solution.

Be warned though, when the domain is not a closed interval, it is possible that there might not be an absolute extreme value!

(f) **Reflect:** Take some time to think about your answer. Is it reasonable? Write a concluding statement.

Complete tasks (a) - (d) for each of the following problems before next class.

Question 1.
A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He does not need a fence along the river. What are the dimensions of the field that has the largest area?

Question 2.
Find two numbers whose difference is 100 and whose product is a minimum.

Question 3.
Find the point of the line $6x + y = 9$ that is closest to the point $(-3, 0)$.

Question 4.
A cylindrical can is being made to contain 1 L of oil. Find the dimensions that will minimize the amount of metal needed to make the can.

Question 5.
If 1200 cm^2 of material is available to make a box with a square base and open top, find the largest possible volume of the box.

Question 6.
The manager of a 100-unit apartment complex knows from experience that all the units will be occupied if the is $800 per month. A market survey indicates that one additional unit will remain vacant for each $10 increase in rent. What rent should the manager charge to maximize revenue?

Question 7. You stand on a cliff at point $(0, 0)$ overlooking a river. You see a boat due north at point $(0, 2)$. The boat is traveling down the river along the curve $y = \sqrt{x+4}$ towards the harbour at $(-4, 0)$. You want to wave to the boat at the point where it is closest to you. Find the coordinates of this point.

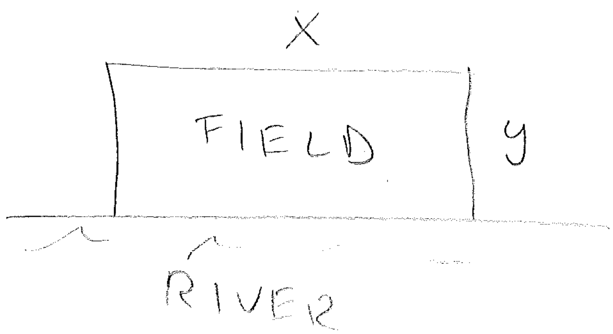
Question 8. A tutoring company is offering a workshop for the upcoming exam in December. Market research suggests that setting the price of the solution at p in dollars will yield

$$q(p) = 20(18 - 2\sqrt{p})$$

students registering. What price should they set in order to maximize revenue?

800 m

Question 1.
A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He does not need a fence along the river. What are the dimensions of the field that has the largest area?



QUANTITIES:

RIVER SIDE $X \geq 0$

PERP. SIDE $y \geq 0$

CONSTRAINTS:

PERIMETER OF FENCE = 800 m

$X + 2y = 800$ m $\quad (x, 2y \leq 800)$

OBJECTIVE FUNCTION (MAXIMIZE):

$X \cdot y = A(x, y)$

- $X + 2y = 800 \qquad y = 400 - \frac{X}{2}$

$A(x,y) = A(x, 400 - \frac{x}{2}) = 400X - \frac{X^2}{2} \quad (= f(x))$

- $f' = 400 - X, \quad f'' = -1$
- LOCAL MAX $X = 400, \quad f(x) = \frac{(400)^2}{2} = 8000\ m^2$
- f CONCAVE DOWN EVERYWHERE SO IT IS GLOBAL. SOLUTION $X = 400, y = 200$.
- ALTERNATIVELY: CLOSED INTERVAL METHOD ON $[0, 800]$.

Question 2.
Find two numbers whose difference is 100 and whose product is a minimum.

QUANTITIES: X, y NUMBERS

CONSTRAINTS: $X - y = 100$

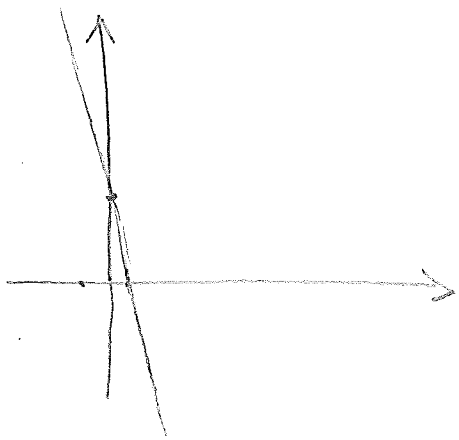
OBJECTIVE FUNCTION (MINIMIZE):

$X \cdot y$

- $X - y = 100$ SO $y = X - 100$
- $X \cdot y = X^2 - 100x \; (= f(x))$
- $f'(x) = 2X - 100$
- CRITICAL POINT $X = 50$; $f'(x) < 0$ FOR $X < 50$, $f'(x) > 0$ FOR $X > 50$

  SO IT'S A GLOBAL MINIMUM.
- SOLUTION $X = 50$, $y = -50$

Question 3.
Find the point of the line $6x + y = 9$ that is closest to the point $(-3, 0)$.



QUANTITIES:
X, Y COORDINATES

CONSTRAINTS:
$6X + Y = 9$

OBJECTIVE FUNCTION (MINIMIZE):

$\sqrt{(X+3)^2 + Y^2}$ (PITHAGORAS)
$\quad \parallel$
$d(X, Y)$

- $6X + Y = 9 \qquad Y = 9 - 6X$
- SO $d(X, Y) = \sqrt{X^2 + 6X + 9 + 81 + 36X^2 - 54X} =$

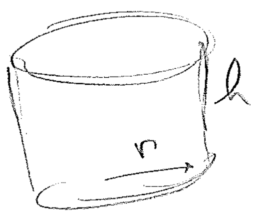
$\sqrt{37X^2 - 48X + 90} \qquad (= f(X))$

- $f'(X) = \dfrac{74X - 48}{2\sqrt{37X^2 - 48X + 90}} \leftarrow > 0 \qquad X = \dfrac{48}{74}$ CRIT VALUE

- $f'(X)$ ONLY CHANGES SIGN ONCE, SO IT'S A GLOBAL MIN/MAX. GRAPHICALLY WE SEE IT'S A MIN.

- SOLUTION: $X = \dfrac{48}{74} \qquad Y = 9 - 6 \cdot \dfrac{48}{74}$

Question 4.
A cylindrical can is being made to contain 1 L of oil. Find the dimensions that will minimize the amount of metal needed to make the can.



QUANTITIES: RADIUS $r \geq 0$
HEIGHT $h \geq 0$
CONSTRAINTS: $\pi r^2 h = 1$

OBJECTIVE FUNCTION (MINIMIZE):
$2\pi r h + 2\pi r^2 = A(r,h)$

- $h = \dfrac{1}{\pi r^2}$ $\quad A(r,h) = \dfrac{2}{r} + 2\pi r^2 = f(r)$
- $f'(r) = -\dfrac{2}{r^2} + 4\pi r$

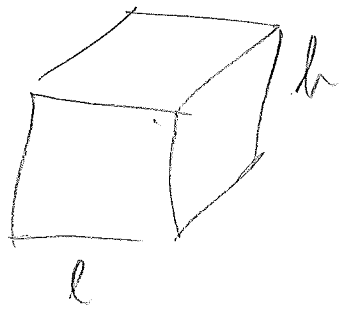
  $f'(r) = 0 \sim 4\pi r = \dfrac{2}{r} \sim 2\pi r^2 = 1 \quad r = \dfrac{1}{\sqrt{2\pi}}$
- $f''(r) = \dfrac{\cancel{4}}{r^3} + 4\pi > 0$ FOR $r \geq 0$

SO FUNCTION IS CONCAVE UP ON DOMAIN

- THEN $r = \dfrac{1}{\sqrt{2\pi}}$ CORRESPONDS TO GLOBAL MIN
- SOLUTION $r = \dfrac{1}{\sqrt{2\pi}}$, $h = 2$. AREA

IS $2\sqrt{2\pi} + 1$.

Question 5.
If 1200 cm^2 of material is available to make a box with a square base and open top, find the largest possible volume of the box.



QUANTITIES:
SIDE $l \geq 0$
HEIGHT $h \geq 0$

CONSTRAINTS:
$$l^2 + 4hl = 1200$$
OBJECTIVE FUNCTION (MAXIMISE):
$$V(h,l) = hl^2$$

- $l^2 + 4hl = 1200 \qquad h = \frac{300}{l} - \frac{l}{4}$
- So $V(h,l) = 300l - \frac{l^3}{4} \quad (= f(l))$
- $f'(l) = 300 - \frac{3}{4}l^2 \qquad l = \sqrt{400} = 20$ CRIT VAL.
- $f'(l)$ ONLY CHANGES SIGN ONCE FOR $l > 0$ AND GOES TO $-\infty$ SO IT MUST BE A MAXIMUM
- SOLUTION $l = 20$, $h = 10$, VOLUME IS 4000 cm^3.

Question 6.
The manager of a 100-unit apartment complex knows from experience that all the units will be occupied if the is \$800 per month. A market survey indicates that one additional unit will remain vacant for each \$10 increase in rent. What rent should the manager charge to maximize revenue?

Question 7.
You stand on a cliff at point $(0,0)$ overlooking a river. You see a boat due north at point $(0,2)$. The boat is traveling down the river along the curve $y = \sqrt{x+4}$ towards the harbour at $(-4,0)$. You want to wave to the boat at the point where it is closest to you. Find the coordinates of this point.

Question 8.
A tutoring company is offering a workshop for the upcoming exam in December. Market research suggests that setting the price of the solution at p in dollars will yield

$$q(p) = 20(18 - 2\sqrt{p})$$

students registering. What price should they set in order to maximize revenue?