WARM UP 1

1. $f(x)$ DEFINED ON $(-\infty, 0) \cup (0, +\infty)$

2. $\lim_{x \to 1^+} f(x) = 1$, $\lim_{x \to 1^-} f(x) = 3$

3. $\lim_{x \to 0} f(x) = +\infty$

4. $\lim_{x \to 2} f(x) = 1$

5. SLOPE OF TANGENT AT $(3,2)$ IS 1

---

1 + 3 TELL US THAT $f(x)$ MUST GO TO $+\infty$ AT $x=0$ FROM BOTH SIDES

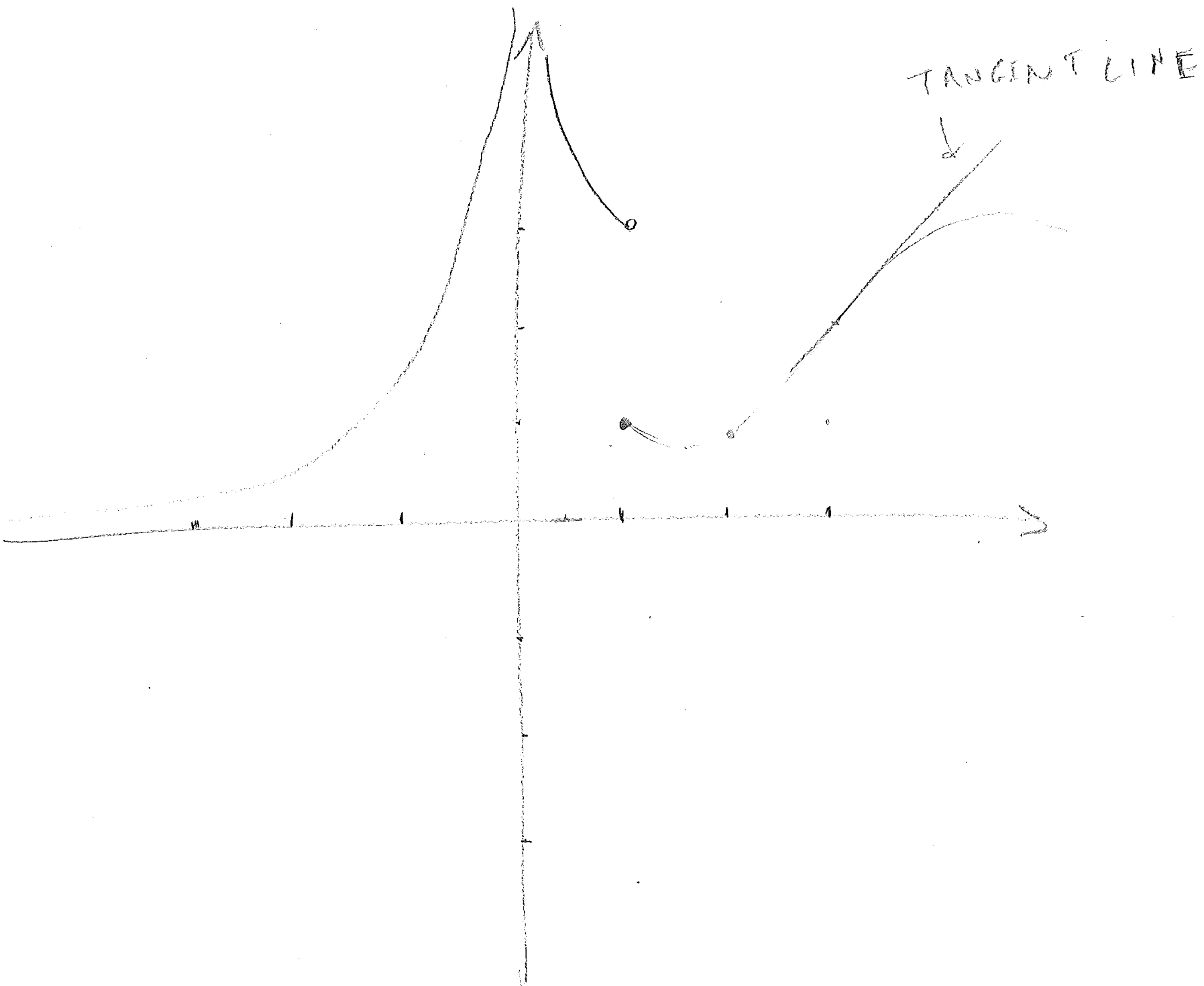
2 TELLS US THAT $f(x)$ IS DISCONTINUOUS AT $x=1$; IT GOES TO 1 FROM THE RIGHT AND TO 3 FROM THE LEFT

4 JUST TELLS US THAT $f(x)$ GOES TO 1 AT 2

5 TELLS US THAT

- $f(3) = 2$
- $f'(3) = 1$

A POSSIBLE SKETCH:



NOTE: NO VALUE IS PRESCRIBED AT $X=1$, BUT WE HAVE TO PICK ONE AS $f(x)$ MUST BE DEFINED THERE

- WARM UP 2

$$f(x) = \frac{x^3 + x^2 - 2x - 3}{x^2 - 3}$$

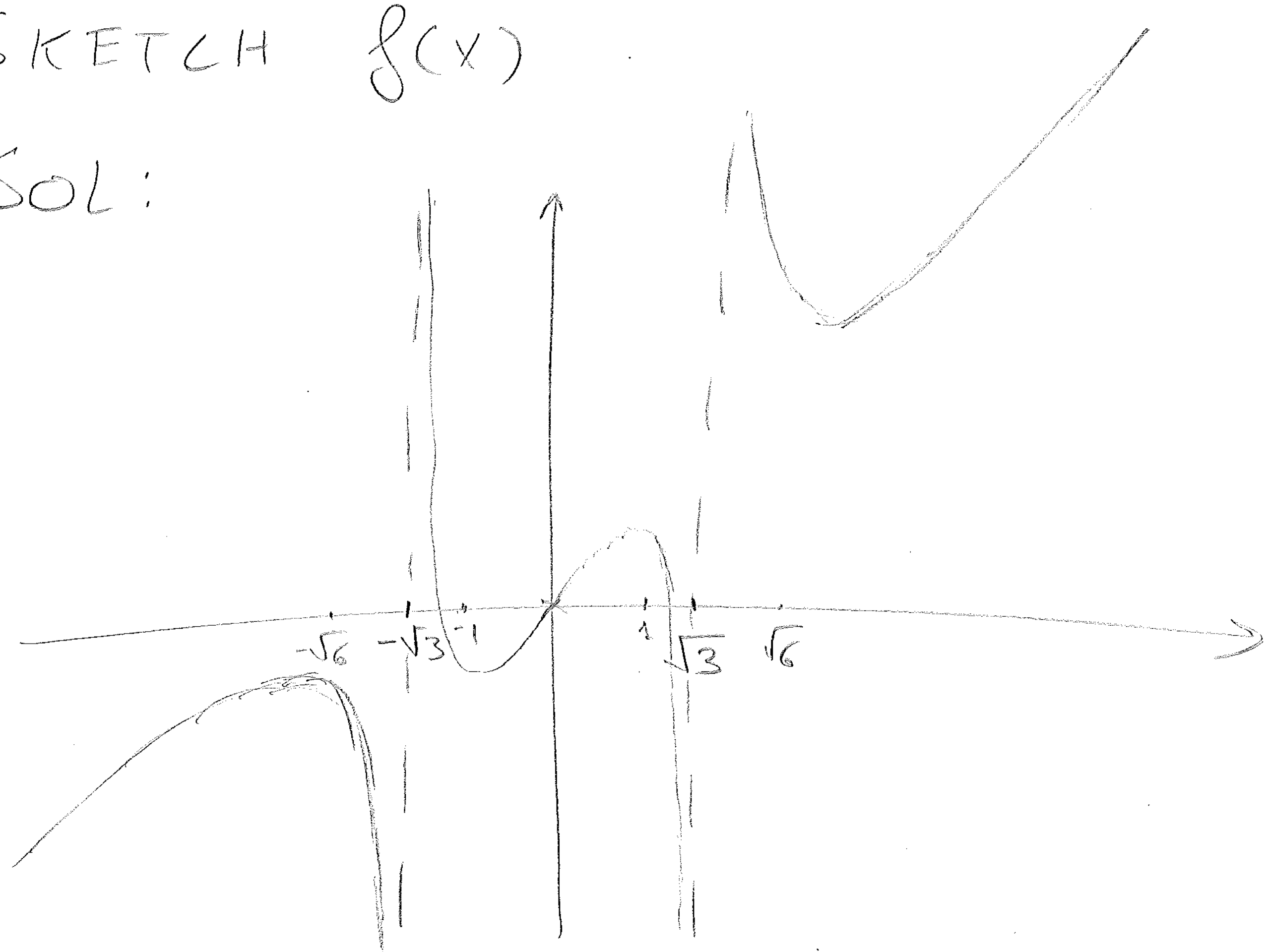
KNOWING THAT

$$f'(x) = \frac{(x^2-1)(x^2-6)}{(x^2-3)^2},$$

$$f''(x) = \frac{2x(x^2+9)}{(x^2-3)^3}$$

SKETCH $f(x)$

- SOL:



OPTIMIZATION PROBLEMS

(1) AT A CERTAIN MOMENT, THE STOCK OF A COMPANY LISTED ON THE VANCOUVER STOCK EXCHANGE IS SELLING AT $0.25 / SHARE. THE CONTROLLING SHARE-HOLDERS SPREAD A (QUICKLY DISPELLED) RUMOR THAT THE COMPANY HAS DISCOVERED A CURE FOR DEATH. THIS CAUSES A SURGE OF DEMAND FOR THE STOCK. AS A RESULT, WHEN $T \geq 0$ THE PRICE OF A SHARE $P(T)$ IS (IN DOLLARS)

$$P(T) = 0.25 + 30Te^{-\frac{T}{5}}$$

T IS IN DAYS AN $T=0$ IS THE INSTANT THE RUMOR STARTS.

AT WHAT TIME T IS THE PRICE OF THE STOCK INCREASING MOST RAPIDLY?

(2) A SMALL MANIFACTURER WHOLE SALES LEATHER JACKETS. THE MONTHLY DEMAND IS DESCRIBED BY

$$P = 400 - 50q$$

P = PRICE IN DOLLARS

Q = DEMAND IN 1000 UNITS / MONTH

THE MANIFACTURER'S MARGINAL COST IS

$$\frac{dC}{dq} = \frac{800}{q+5}.$$ DETERMINE Q TO MAXIMISE <u>PROFIT</u>.

3) "ABC STEEL" MANIFACTURES NUTS AND BOLTS. WHEN x NUTS ARE PRODUCED, THEY CAN BE SOLD FOR $-3x+500$ DOLLARS EACH

WHEN y BOLTS ARE PRODUCED, THEY CAN BE SOLD FOR $-y+300$ DOLLARS EACH

EACH WEIGH 0.5 KG. HOW MANY NUTS AND BOLTS CAN WE PRODUCE FROM 100 KG OF STEEL TO MAXIMISE REVENUE?

4) A RECTANGULAR CONTAINER WITH OPEN TOP IS TO HAVE A VOLUME OF $8\ m^3$. THE LENGTH OF THE BASE IS TWICE THE WIDTH. MATERIAL FOR THE BASE COST $\$4.50/m^2$, MATERIAL FOR FOR THE SIDES COST $\$6/m^2$. WHAT'S THE CHEAPEST CONTAINER WE CAN BUILD?

5) YOU ARE PLANNING A CITY TOUR FOR A GROUP OF 100 TOURISTS. IF YOU CAN SELL x BUS TOUR TICKETS, YOU CAN OFFER THEM AT $\$(30-x/4)$. IF YOU CAN SELL y BOAT TOUR TICKETS, YOU CAN OFFER THEM AT $\$(70-y/2)$. HOW MANY OF EACH SHOULD YOU SELL TO MAXIMIZE REVENUE?

SOL:

(1) $P(T) = 0.25 + 30Te^{-\frac{T}{5}}$

WE WANT TO MAXIMISE

$P'(T) = 30e^{-\frac{T}{5}} - 6Te^{-\frac{T}{5}} = 6e^{-\frac{T}{5}}(5-T)$

$P''(T) = -6e^{-\frac{T}{5}} - \frac{6}{5}e^{-\frac{T}{5}}(5-T) = -6e^{-\frac{T}{5}} - 6e^{-\frac{T}{5}} + \frac{6}{5}Te^{-\frac{T}{5}}$

$= \frac{6}{5}Te^{-\frac{T}{5}} - 12e^{-\frac{T}{5}} = e^{-\frac{T}{5}}\left(\frac{6}{5}T - 12\right)$

CRIT POINT $T = 10$

DERIVATIVE GOES FROM $-$ TO $+$ $\Rightarrow$ LOCAL MIN

GRAPHICAL ANALYSIS $\Rightarrow$ MAX IS AT 0.

(2) QUANTITIES: $P(q)$ PRICE
$C(q)$ COST

EQUATIONS: $p = 400 - 50q \quad p \geq 0$
$C' = \frac{800}{q+5} \quad q \geq 0$

OBJ FUNCTION: P PROFIT $= p \cdot q - C$

$= (400 - 50q) \cdot q - C$

$P' = 400 - 50q - 50q - C' = 400 - 100q - \frac{800}{q+5}$

$P' = 0$ WHEN $(q+5)(400-100q) = 800$

$\sim -100q^2 + \cancel{5}00q + 1200 = 0 \sim -q^2 - q + 12 = 0$

$q = 3, -4$

$q \geq 0$ SO 3 IS THE ONLY ADMISSIBLE CRIT POINT.

$$P'' = -100 + \frac{800}{(q+5)^2} \quad \text{FOR } q \geq 0 \quad P'' < -100 + 32 < 0$$

SO P IS CONC. DOWN ON $[0, +\infty)$

THEN $q = 3$ <u>GLOBAL MAX</u>.

(3) QUANTITIES: X : NUTS
y : BOLTS

CONSTRAINTS: $X \geq 0,\ y \geq 0,\ \frac{X}{2} + \frac{y}{2} = 100$

OBJ FUN (MAX): $(-3X + 500)X + (-y + 300)y$

$\frac{y}{2} + \frac{X}{2} = 100 \sim y = 200 - X$

$$f(x) = 500X - 3X^2 + (X + 100)(200 - X) =$$

$$500X - 3X^2 + 200X + 10000 - 100X - X^2 =$$

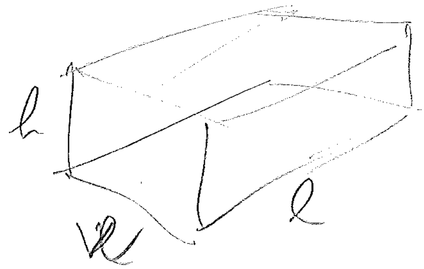
$$-4X^2 + 600X + 10000$$

$f'(x) = -8X + 600$ CRIT VAL $X = \frac{600}{8} = 75$

DERIVATIVE ONLY CHANGES SIGN ONCE FROM + TO − SO GLOBAL MAX.

SOL: $X = 75,\ Y = 125$

(4)



QUANTITIES: h, l, w

CONSTR: $l, h, w > 0$

$l = 2w$, $l \cdot h \cdot w = 8$

OBJ FUN (MIN):

$$2lh + 2wh + lw$$

- $w = \frac{l}{2}$ SO $2lh + 2wh + lw =$

$$2lh + lh + \frac{l^2}{2} = 3lh + \frac{l^2}{2}$$

- $\frac{l^2}{2} h = 8$ SO $h = \frac{16}{l^2}$,

$$f(l) = \frac{48}{l} + \frac{l^2}{2}.$$

- $f'(l) = -\frac{48}{l^2} + l$ $\quad f'(l) = 0$ WHEN

$l^3 = 48 \sim l = 2\sqrt[3]{6}$

- $f''(l) = \frac{96}{l^3} + 1 > 0$ FOR $l > 0$ SO

$f(l)$ CONC UP EVERYWHERE SO

$l = 2\sqrt[3]{6}$ IS <u>GLOBAL MIN</u>

SOL: $l = 2\sqrt[3]{6}$, $w = \sqrt[3]{6}$, $h = \frac{16}{4\sqrt[3]{36}} = \frac{4}{\sqrt[3]{36}}$

(5) QUANTITIES: x BUS TICKETS
y BOAT TICKETS

CONSTR: $x, y \geq 0$ $\quad x+y=100$

OBJ FUN (MAX):

$$x\left(30-\frac{x}{4}\right)+y\left(70-\frac{y}{2}\right)$$

- $y = 100 - x$ SO

$$f(x) = 30x - \frac{x^2}{4} + (100-x)\left(70-50+\frac{x}{2}\right)$$

$$= 30x - \frac{x^2}{4} + (100-x)\left(20+\frac{x}{2}\right) =$$

$$30x - \frac{x^2}{4} + 2000 + 50x - 20x - \frac{x^2}{2} = -\frac{3x^2}{4} + 60x + 2000$$

- $f'(x) = -\frac{3x}{2} + 60$

SO $x=40$ IS A LOCAL MAX.

- CLOSED INTERVAL METHOD ON $[0, 100]$

$f(0) = 2000 \quad f(40) = 3200 \quad f(100) = 500$

SO SOL: $x = 40$, $y = 60$