LIMITS GOING TO INFINITY AT A FINITE POINT

USING THE TERMINOLOGY WE DEVELOPED, WE CAN NOW ALSO CONSIDER THE SITUATION WHER A FUNCTION $f(x)$ "EXPLODES":

DEF:
WE SAY THE LIMIT OF $f(x)$ FOR $x \mapsto c$ IS $+\infty$ (RESP. $-\infty$) IF $f(x)$ GETS ARBITRARILY BIG AND POSITIVE (RESP. NEGATIVE) FOR x ARBITRARILY CLOSE TO c. WE WRITE

$$\lim_{x \to c} f(x) = +\infty \quad (\text{RESP. } -\infty)$$

DEF:
WE ANALOGOUSLY EXTEND THE NOTION OF ONE-SIDED LIMITS TO INCLUDE $\pm\infty$ AS POSSIBLE LIMITS. WE WRITE

$$\lim_{x \to c^+} f(x) = \pm\infty \quad \text{OR} \quad \lim_{x \to c^-} = \pm\infty$$

REMARK: ONE-SIDED LIMITS ARE NECESSARY HERE; A SIMPLE EXAMPLE IS $f(x) = \frac{1}{x}$.

THM: ALL LIMIT RULES EXTEND TO THIS CASE, ASSUMING WE HAVE A DEFINITE FORM.

IT WILL BE USEFUL TO ADD TWO NEW SYMBOLS TO OUR TERMINOLOGY:

0^+ : TO INDICATE THAT A FUNCTION GOES TO 0 BUT STAYS POSITIVE

0^- : TO INDICATE THAT A FUNCTION GOES TO 0 BUT STAYS NEGATIVE

WE DEFINE 1 NEW DEFINITE FORMS:

$$\frac{1}{0^+} = +\infty \quad , \quad \frac{1}{0^-} = -\infty$$

MOREOVER, ALL DEFINITE FORMS WITH 0 ARE DEFINITE WHEN WE SUBSTITUTE 0^+ OR 0^-

EXAMPLE / EXERCISE:

- DOES $\lim_{x \to 0} \frac{1}{x}$ EXIST?
- WHAT ABOUT $\lim_{x \to 0^+} \frac{1}{x}$ AND $\lim_{x \to 0^-} \frac{1}{x}$?
- FIND $\lim_{x \to 1^-} \frac{x^2+2}{x^2-1}$
- FIND $\lim_{x \to 0^+} \ln(x)$

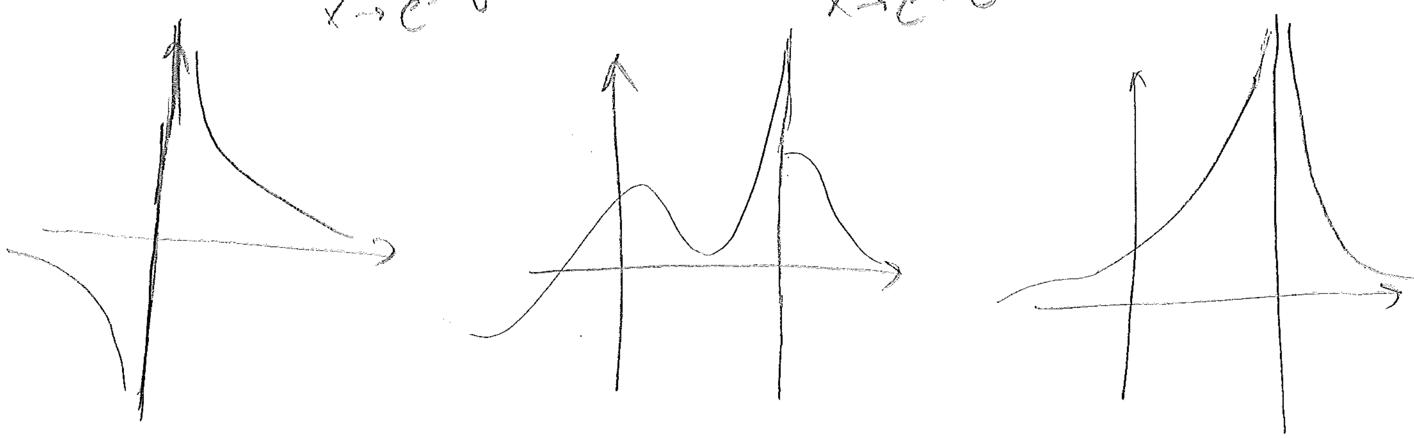
ASYMPTOTES:

SAYING $f(x)$ HAS AN ASYMPTOTE IS BASICALLY SAYING THAT THE "INFINITY" RELATED BEHAVOIR OF $f(x)$ CAN BE APPROXIMATED BY A LINE. WE WILL ONLY CONSIDER <u>HORIZONTAL</u> AND <u>VERTICAL</u> ASYMPTOTES.

DEF:

$f(x)$ HAS A <u>VERTICAL</u> ASYMPTOTE AT $x=c$ IF AT LEAST ONE OF THESE TWO STATEMENTS IS TRUE:

$$\lim_{x\to c^-} f(x) = \pm\infty \qquad \lim_{x\to c^+} f(x) = \pm\infty$$



$f(x)$ HAS A <u>HORIZONTAL</u> ASYMPTOTE AT $+\infty$ (RESP $-\infty$) IF

$$\lim_{x\to+\infty} f(x) = L \quad \left(\text{RESP } \lim_{x\to-\infty} f(x) = L\right)$$

FOR A FINITE NUMBER L.

SOME NOTEWORTHY LIMITS

WE'LL QUICKLY GO THROUGH SOME RELEVANT LIMITS:

THM:

- $$\lim_{x \to +\infty} \frac{x^\alpha}{e^{\lambda x}} = 0 \quad \text{FOR ALL } \alpha > 0, \lambda > 0$$
- $$\lim_{x \to +\infty} \frac{(\ln(x))^\beta}{x^\alpha} = 0 \quad \text{FOR ALL } \beta > 0, \alpha > 0$$

IN WORDS:

ANY EXPONENTIAL GROWS BIGGER THAN ANY POWER. ANY POWER GROWS BIGGER THAN ANY (POWER OF) LOGARITHM.

THIS CAN BE USED IN A LOT OF CONTEXTS!

EXAMPLES:

- $$\lim_{x \to 0^+} x \ln(x) = \lim_{y \to +\infty} \frac{1}{y} \ln\left(\frac{1}{y}\right) = \lim_{x \to +\infty} -\frac{\ln y}{y} = 0$$
- $$\lim_{x \to +\infty} \frac{x^{10} + 33 \ln(x)^2 + \ln(x) x^5}{e^x + 3\cos x \, e^{\frac{x}{2}}} = \lim_{x \to +\infty} \frac{x^{10}\left(1 + \frac{33 \ln x}{x^{10}} + \frac{\ln(x)}{x^5}\right)}{e^x\left(1 + 3\cos x \, e^{-\frac{x}{2}}\right)}$$

$$= \lim_{x \to +\infty} \frac{x^{10}}{e^x} \lim_{x \to +\infty} \frac{\left(1 + \frac{33 \ln x}{x^{10}} + \frac{\ln(x)}{x^5}\right)}{\left(1 + 3\cos x \, e^{-\frac{x}{2}}\right)} = 0 \cdot 1 = 0$$

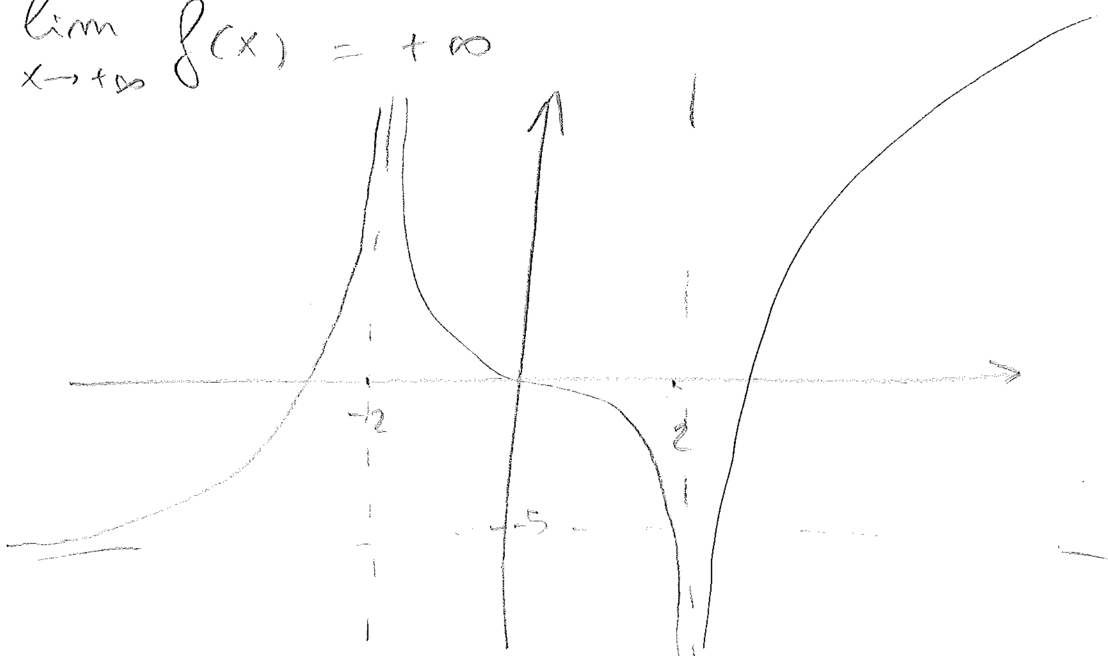
BACK TO CURVE SKETCHING

WE JUST LEARNED TO FIND THE ASYMPTOTES OF A FUNCTION. LET'S PRACTICE DRAWING THEM

EXERCISE

(1) DRAW A FUNCTION $f(x)$ WITH

- THE HORIZONTAL ASYMPTOTE $y = -5$ AT $-\infty$
- VERTICAL ASYMPTOTES AT $-2, 2$
- $\lim_{x \to +\infty} f(x) = +\infty$

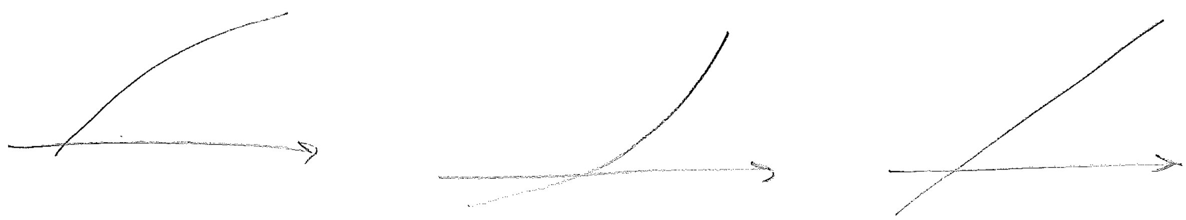


(2) CAN A FUNCTION $f(x)$ HAVE

- THE HORIZONTAL ASYMPTOTE $y = 3$ AT $+\infty$ AND $-\infty$
- NO LOCAL MAXIMA/MINIMA?

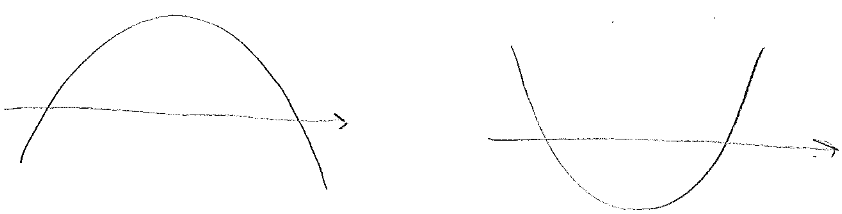
NOW THAT WE UNDERSTAND ASYMTOTES, LET'S MOVE ON TO THE <u>DATA</u> IN THE <u>DERIVATIVE</u>

- POSITIVE DERIVATIVE: $f(x)$ INCREASING



- NEGATIVE DERIVATIVE: $f(x)$ DECREASING

- DERIVATIVE <u>CHANGES</u> <u>SIGN</u>: LOCAL MIN/MAX



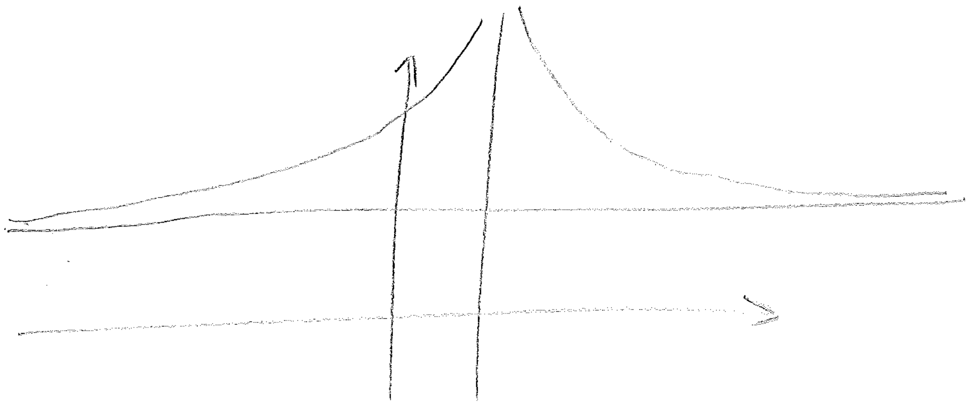
NOTE THAT THE DERIVATIVE <u>ONLY</u> CHANGES SIGN <u>AT LOCAL MIN/MAX</u>

<u>EXAMPLES</u>:

(1) CAN A FUNCTION HAVE POSITIVE DERIVATIVE <u>AND</u> HAVE $\lim_{x \to +\infty} f(x) = -\infty$

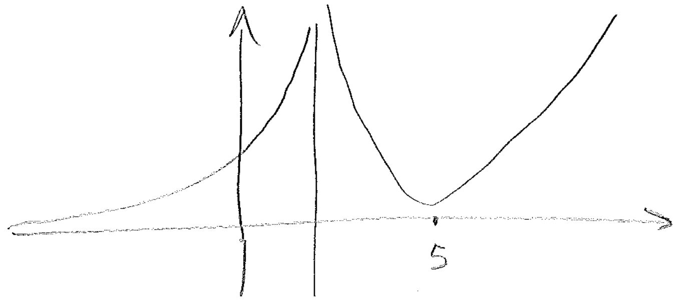
(2) SKETCH A FUNCTION $f(x)$ SUCH THAT

- $f(x)$ HAS A VERTICAL ASYMPTOTE AT $x = 1$
- $f(x)$ HAS A HORIZONTAL ASYMPTOTE $y = 1$ AT $\pm\infty$
- $f'(x) > 0$ FOR $-\infty < x < 0$, $f'(x) < 0$ FOR $0 < x < +\infty$



(3) SKETCH A FUNCTION $f(x)$ SUCH THAT

- $f(x)$ HAS A VERTICAL ASYMPTOTE AT $x = 1$
- $\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to +\infty} f(x) = +\infty$
- THE ONLY CRITICAL VALUE OF $f(x)$ IS AT $x = 5$, AND IT IS A LOCAL MINIMUM



(4) CAN THERE BE A FUNCTION $f(x)$ SUCH THAT

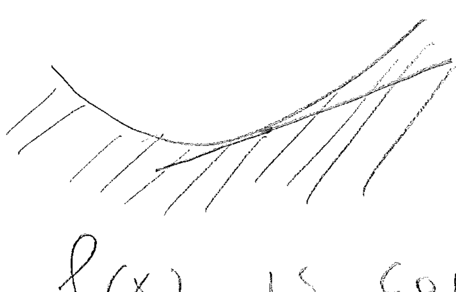
- $\lim_{x \to 0^-} f(x) = -\infty$
- $f(x)$ HAS A LOCAL MAXIMUM AT $x = -5$, AND $f(-5) = 3$. IT'S THE ONLY CRIT. VALUE.
- $\lim_{x \to -\infty} f(x) = 6$

DATA IN THE SECOND DERIVATIVE

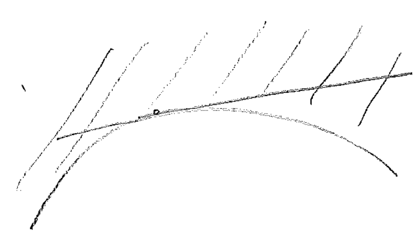
CAN WE TELL IF A CRIT POINT IS A MIN, MAX OR NEITHER AT A GLANCE?

DEF:

- $f(x)$ IS CONCAVE AT c IF THE TANGENT LINE TO $f(x)$ AT c STAYS UNDER THE GRAPH OF $f(x)$ CLOSE TO c. WE ALSO SAY $f(x)$ IS CONCAVE UP AT c.



- $f(x)$ IS CONVEX AT c IF THE TANGENT LINE TO $f(x)$ AT c STAYS OVER THE GRAPH OF $f(x)$ CLOSE TO c. WE ALSO SAY $f(x)$ IS CONCAVE DOWN AT c



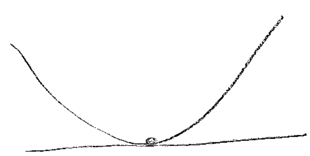
THM:

- IF $f''(c) > 0$ THEN $f(x)$ IS CONCAVE AT c.
- IF $f''(c) < 0$ THEN $f(x)$ IS CONVEX AT c.

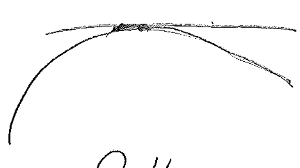
THM:

SUPPOSE $f'(c) = 0$. THEN

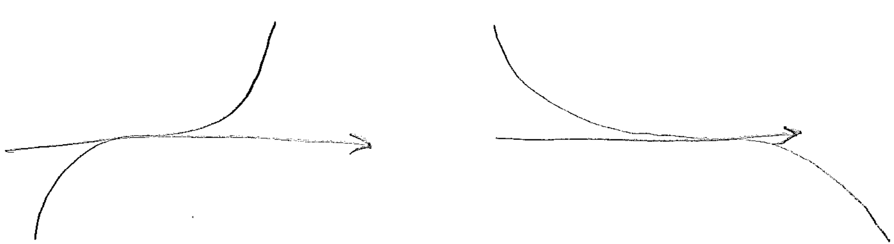
- IF $f''(c) > 0$ THEN c CORRESPONDS TO A LOCAL MINIMUM OF $f(x)$.



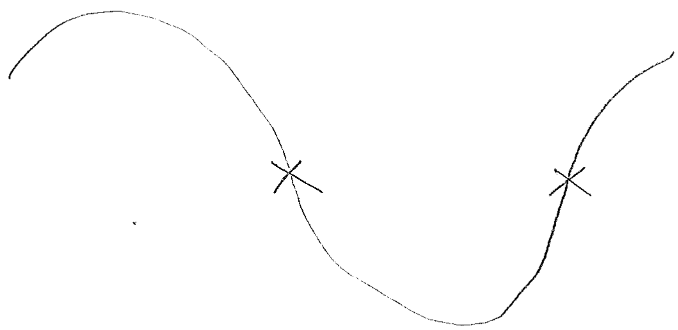
- IF $f''(c) < 0$ THEN c CORRESPONDS TO A LOCAL MAXIMUM OF $f(x)$



- IF $f''(c) = 0$ AND $f'''(c) \neq 0$ THEN c IS A SADDLE POINT OF $f(x)$



IN GENERAL, WHEN $f''(c)$ IS 0 AND $f'''(c) \neq 0$ (AND $f'(c)$ IS NOT NECESSARILY 0)
WE HAVE AN INFLEXION POINT, I.E. A POINT WHERE $f(x)$ CHANGES CONCAVITY



EXERCISES:

1. $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 6x + 1$. FIND ALL MIN/MAX, LIMITS FOR $\pm\infty$ AND INFLECTION PTS, THEN SKETCH $f(x)$.