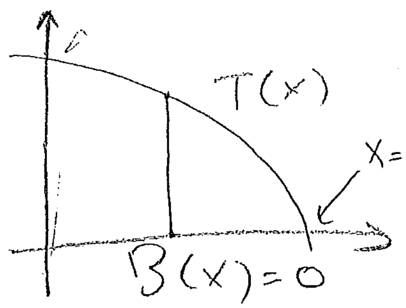
EXAMPLES:

(1) FIND THE C.O.M. OF THE REGION GIVEN BY THE INTERSECTION OF $2x^2+3y^2=12$ AND THE FIRST QUADRANT $\left(A=\frac{\pi\sqrt{6}}{2}\right)$



$$2x^2+3y^2=12$$

$$y^2=4-\frac{2}{3}x^2$$

$$y \underset{\substack{\uparrow \\ \text{FIRST Q.}}}{=} \sqrt{4-\frac{2}{3}x^2}=T(x)$$

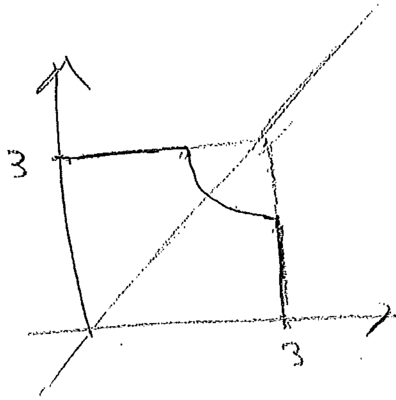
$$\hat{y}=\int_0^{\sqrt{6}}\frac{T(x)^2-B(x)^2}{2A}\,dx=\int_0^{\sqrt{6}}\frac{\left(4-\frac{2}{3}x^2\right)}{\pi\sqrt{6}}\,dx=$$

$$\frac{4}{\pi}-\left[-\frac{2}{9\sqrt{6}\pi}x^3\right]_0^{\sqrt{6}}=\frac{2}{\pi}\left(2-\left(\frac{2}{3}\right)\right)=\frac{8}{3\pi}$$

$$\hat{x}=\int_0^{\sqrt{6}}\frac{2x\sqrt{4-\frac{2}{3}x^2}}{\pi\sqrt{6}}\,dx\overset{u=x^2}{=}\int_0^{6}\frac{\sqrt{4-\frac{2}{3}u}}{\pi\sqrt{6}}\,du=\frac{4}{\pi}\cdot\sqrt{\frac{2}{3}}$$

(2) FIND THE C.O.M. OF THE REGION GIVEN BY A SQUARE OF SIDE 3 LEANING ON THE ORIGIN AT THE FIRST

QUADRANT TO WHICH WE REMOVE A QUARTER OF CIRCLE OF RADIUS 1 CENTERED ON THE RIGHT-UPPER CORNER

$R =$ 

BY SYMMETRY, $\hat{x} = \hat{y}$.

WE HAVE $A = 3^2 - \frac{\pi}{4}^{\,2}$.

$$T(x) = \begin{cases} 3 & 0 \leq x \leq 2 \\ 3 - \sqrt{1-(3-x)^2} & 2 \leq x \leq 3 \end{cases} \qquad B(x) = 0$$

$$\hat{x} = \int_2^3 \frac{x\left(3 - \sqrt{1-(3-x)^2}\right)}{A}\,dx + \int_0^2 \frac{3x}{A}\,dx =$$

$$\int_0^3 \frac{3x}{A}\,dx - \int_2^3 \frac{x\sqrt{1-(3-x)^2}}{A}\,dx \overset{u = x-3}{=} \frac{27}{2A} - \int_{-1}^{0} \frac{(u+3)\sqrt{1-u^2}}{A}\,du$$

$$= \dots = \frac{27}{2A} - \frac{3\pi}{4A} + \frac{1}{3A} = \hat{y}$$

SEPARABLE DIFFERENTIAL EQUATIONS

A (ORDINARY) DIFFERENTIAL EQUATION IS AN EQUATION IN THE FORM

$$\frac{d^n y}{dx} = f(x, y, y', \ldots, y^{(n-1)})$$

BASICALLY EVERY LAW OF NATURE CAN BE EXPRESSED IN TERMS OF D.E.s.
(OFTEN MORE GENERAL TYPES, WITH MULTIPLE VARIABLES, ETC...).
WE ARE GOING TO CONCENTRATE ON A SPECIAL TYPE OF D.E. CALLED SEPARABLE DIFFERENTIAL EQUATIONS (S.D.E.), WHICH CAN BE SOLVED WITH THE TOOLS AT OUR DISPOSAL.

DEF:

A S.D.E. IS A D.E. IN THE FORM

$$\frac{dy}{dx} = f(x)g(y(x))$$

HOW DO WE SOLVE FOR $y(x)$?

IDEA:

$$y' = f(x)\,g(y) \sim \frac{y'}{g(y)} = f(x)$$

$$\sim \int \frac{y'}{g(y)}\,dx = \int f(x)\,dx$$

SUBSTITUTION: $\displaystyle\int \frac{y'}{g(y)}\,dx = \int \frac{1}{g(y)}\,dy \Big|_{y=y(x)}$

SO WE GET THE NEW EQUATION

$$\int \frac{1}{g(y)}\,dy \Big|_{y=y(x)} = F(x) + C$$

THIS IS AN ORDINARY EQUATION!!!

(ASSUMING WE <u>CAN</u> INTEGRATE $\frac{1}{g(y)}$)

<u>MNEMONIC TRICK</u> (IT'S NOT FORMALLY CORRECT!)

$$\frac{dy}{dx} = g(y)\,f(x) \overset{\text{MULT BY } dx}{\sim} dy = g(y)\,f(x)\,dx \overset{\text{DIVIDE BY } g(y)}{\sim}$$

$$\frac{dy}{g(y)} = f(x)\,dx \;\sim\; \int \frac{dy}{g(y)} = \int f(x)\,dx$$

APPLY
INTEGRATION
SYMBOL

<u>EXAMPLE:</u> $\dfrac{dy}{dx} = a(y-b)$

$\underset{\sim}{*}$ $\dfrac{y'}{y-b} = a \sim \displaystyle\int \frac{1}{y-b}\,dy \Big|_{y=y(x)} = \int a\,dx$

$\sim \log|y-b| = ax$ NOW WE EXPONENTIATE

$|y-b| = e^{ax+c} \sim y-b = \overset{\pm e^c}{C}\, e^{ax}$

$*$ NOTE THAT WE JUST ASSUMED THAT $y(x) \neq b$ FOR ALL x. <u>BUT</u> $y(x)=b$ IS A PERFECTLY FINE SOLUTION, CORRESPONDING TO $C=0$

SO WE GOT <u>INFINITELY MANY</u> SOLNS, DEPENDING ON A NUMBER C. THIS IS NORMAL! WE ALSO HAVE TO SPECIFY THE <u>VALUE</u> OF y AT A POINT TO GET

ONE SOLUTION. SAY WE KNOW $y(0)$. THEN

$$y(0) = ce^{0} + b \sim c = b - y(0)$$

SO THE SOLUTION IS

$$\underline{y(x) = (b - y(0))e^{ax} + b}$$

<u>NOTE</u>: THIS IS A <u>UNIQUE</u> SOLUTION, I.E. ANY $y(x)$ SATISFYING THE EQUATION WITH A GIVEN $y(0)$ IS IN THIS FORM.

<u>EXAMPLE</u>: $\dfrac{dy}{dx} = (y^2+1)e^x \qquad y(0) = 0$

$$\sim \dots \sim \int \frac{1}{y^2+1}\,dx = \int e^x\,dx \sim$$

$$\arctan(y) = e^x + C \underset{\substack{\uparrow \\ \text{APPLY} \\ \text{Tan}(-)}}{\sim} y = \tan(e^x + C)$$

NOW $y(0) = 0 = \tan(1 + C)$ SO $C = -1$

VERIFY: $\tan(e^x - 1)' \underset{\substack{\uparrow \\ \text{CHAIN RULE}}}{=} e^x \cdot \sec^2(e^x - 1) = e^x(y^2 + 1)$ ✓

EXAMPLE:

FIND SOL FOR $\frac{dy}{dx} = \frac{y}{x^2+1}$, $y(1)=1$

$$\frac{y'}{y} = \frac{1}{x^2+1} \qquad \int \frac{y'}{y}\,dx = \int \frac{1}{x^2+1}\,dx$$

$$\int \frac{1}{y}\,dy \Big|_{y=y(x)} = \arctan x + C$$

$$\log y = \arctan x + C$$

$$y = e^{\arctan x + C}$$

NOW, $y(1) = e^{\arctan 1 + C} = 1$ $\qquad \arctan 1 = \frac{\pi}{4}$

SO $C = -\frac{\pi}{4}$, $y = e^{-\frac{\pi}{4} + \arctan x}$ (VERIFY!)

EXAMPLE: $\frac{dy}{dx} = -x(y-1)^2$, $y(1)=2$.

$$\frac{y'}{(y-1)^2} = -x \qquad \int \frac{1}{(y-1)^2}\,dy = \int -x\,dx$$

$$-\frac{1}{y-1} = -\frac{x^2}{2} + C \qquad \text{FIND } C: -\frac{1}{y(1)-1} = -\frac{1}{2} + C$$

$$-\frac{1}{1} = -\frac{1}{2} + C, \quad C = -\frac{1}{2}. \qquad -\frac{1}{y-1} = -\frac{x^2}{2} - \frac{1}{2}, \quad y = 1 + \frac{2}{x^2+1}$$