WARM-UP

DOES $\int_1^{\infty} \frac{1}{x+2} - \frac{1}{x}\, dx$ CONVERGE?

IF SO, COMPUTE IT..

Sol:

TAKEN ONE BY ONE THE TWO TERMS DIVERGE, BUT TOGHETHER:

$$\lim_{R\to\infty} \int_1^R \frac{1}{x+2} - \frac{1}{x}\, dx = \lim_{R\to\infty} \log x+2 - \log x \Big|_1^R$$

$$= \lim_{R\to\infty} \log \frac{x+2}{x} \Big|_1^R = \lim_{R\to\infty} \log 1+\frac{2}{R} - \log 3$$

$$= \log 1 - \log 3 = -\log 3$$

IDEA / DEFINITION (UNBOUNDED INTEGRAND)

i) IF $\int_{T}^{b} f(x)\,dx$ EXISTS FOR ALL $c < T < b$

THEN $$\int_{c}^{b} f(x)\,dx = \lim_{T \to c^{+}} \int_{T}^{b} f(x)\,dx$$

WHEN LIMIT EXISTS AND IS FINITE.

(SO WE'RE THINKING OF A VERTICAL ASYMPTOTE AT c)

ii) IF $\int_{a}^{T} f(x)\,dx$ EXISTS FOR ALL $a < T < b$

THEN $$\int_{a}^{c} f(x)\,dx = \lim_{T \to c^{-}} \int_{a}^{T} f(x)\,dx$$

WHEN LIMIT EXISTS AND IS FINITE

iii) IF BOTH $\int_{a}^{c} f(x)\,dx$ AND $\int_{c}^{b} f(x)\,dx$

EXIST THEN $$\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$$

THE INTEGRAL IS CONVERGENT IF IT EXISTS, DIVERGENT OTHERWISE

WHAT **NOT** TO DO

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-1}^{1} = 2 \qquad \text{WRONG!}$$

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx \text{; NEITHER}$$

EXIST SO $\int_{-1}^{1} \frac{1}{x^2}\,dx$ IS DIVERGENT

EXAMPLE:

$$\int_0^1 \log(x)\,dx = \lim_{\tau \to 0^+} \int_\tau^1 \log(x) = \lim_{\tau \to 0^+} -\tau\log(\tau) + \tau$$

$$+ 1\log(1) - 1 = -1 + \lim_{\tau \to 0^+} -\tau\log(\tau) + \tau = -1$$

EXAMPLE:

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{\tau \to 0^+} \int_\tau^1 \frac{1}{x^p}\,dx = \begin{cases} p \neq 1 & \frac{x^{1-p}}{1-p} \\ p = 1 & \log|x|, \end{cases}$$

SO $p > 1$: $\lim_{\tau \to 0^+} \frac{x^{1-p}}{1-p}\Big|_\tau^1 = \frac{1}{1-p} - \lim_{\tau \to 0} \frac{\tau^{1-p}}{1-p} = \infty$ AS $1-p < 0$

$p < 1$: $\lim_{\tau \to 0^+} \frac{x^{1-p}}{1-p}\Big|_\tau^1 = \frac{1}{1-p} - \lim_{\tau \to 0} \frac{\tau^{1-p}}{1-p} = \frac{1}{1-p}$ AS $1-p > 0$

$p = 1$: $\lim_{\tau \to 0^+} \log|\tau|\Big|_\tau^1 = \lim_{\tau \to 0^+} -\log|\tau| = +\infty$

EXAMPLE: $\displaystyle\int_1^2 \frac{1}{x\log(x)^p}\,dx$, $p \leq 1$

$$\int_\tau^2 \frac{1}{x\log(x)^p}\,dx \underset{u=\log x}{=} \int_{\log\tau}^{\log 2} \frac{1}{u^p}\,du = \begin{cases} \left[\log u\right]_{\log\tau}^{\log 2} \\ \left[\dfrac{u^{-p+1}}{-p+1}\right]_{\log\tau}^{\log 2} \end{cases}$$

$$\lim_{\tau\to 1} \log(\log 2) - \log(\log\tau) = \lim_{s\to 0} \log(\log 2) - \log s$$

$$= -\infty \quad \text{DNE}$$

$$\lim_{\tau\to 1} \frac{\log 2^{-p+1}}{-p+1} - \frac{\log\tau^{-p+1}}{-p+1} = \frac{-\log 2^{1-p}}{p-1}$$

REMARK:

$\frac{1}{x^p}$, $0 < p$

	$p<1$	$p>1$	$p=1$
$\int_1^\infty$	DIV	CONV	DIV
$\int_0^1$	CONV	DIV	DIV

$\frac{1}{x\log(x)^p}$, $0 < p$

	$p<1$	$p>1$	$p=1$
$\int_2^\infty$	DIV	CONV	DIV
$\int_1^2$	CONV	DIV	DIV

EXAMPLE: $\displaystyle\int_2^6 \frac{1}{\sqrt{x-2}}\,dx$

UNBOUNDED AT $x=2$, SO

$$\int_2^6 \frac{1}{\sqrt{x-2}}\,dx = \lim_{\tau\to 2^+} \int_\tau^6 \frac{1}{\sqrt{x-2}}\,dx \underset{u=x-2}{=}$$

$$\lim_{\tau\to 0^+} \int_\tau^4 \frac{1}{\sqrt{u}}\,du = \lim_{\tau\to 0^+} 2\sqrt{u}\Big|_\tau^4 = 4$$

MIXING AN MATCHING:

WE WANT, SAY $\int_{-\infty}^{\infty} \frac{1}{(x+2)\,x^2}\,dx$. DOES IT MAKE SENSE?

IF AN INTEGRAL IS "IMPROPER" IN MORE THAN ONE WAY, WE JUST BREAK IT DOWN UNTIL IT IS A SUM OF THE BASIC TYPES OF IMPROPER INTEGRALS, SO:

- $f(x)$ IS UNBOUNDED AT $x=-2$, $x=0$
- WE HAVE $\pm\infty$

TO SOLVE THIS WE'LL PICK A $a<-2$, $-2<b<0$, $0<c$ AND WRITE IT AS

$$\int_{-\infty}^{a} \frac{1}{x^2(x+2)}\,dx + \int_{a}^{-2} \frac{1}{x^2(x+2)}\,dx + \int_{-2}^{b} \frac{1}{x^2(x+2)}\,dx +$$

$$\int_{b}^{0} \frac{1}{x^2(x+2)}\,dx + \int_{0}^{c} \frac{1}{x^2(x+2)}\,dx + \int_{c}^{\infty} \frac{1}{x^2(x+2)}\,dx$$

SEVERAL OF THESE ARE DIVERGENT, SO THE INTEGRAL IS DIVERGENT

EXAMPLE:

$$\int_0^\infty \frac{1}{x^p}\,dx \quad \text{DOES NOT EXIST FOR ANY } p$$

AS $\int_0^1 \frac{1}{x^p}\,dx$ CONV FOR $p<1$, $\int_1^\infty \frac{1}{x^p}\,dx$ CONV FOR $p>1$

EXAMPLE:

$$\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx = \int_0^1 \frac{e^{-x^2}}{\sqrt{x}}\,dx + \int_1^{+\infty} \frac{e^{-x^2}}{\sqrt{x}}\,dx$$

SHOULD EXIST AS FOR $|x| \geq 1$

$$\frac{e^{-x^2}}{\sqrt{x}} \leq e^{-x^2} \leq e^{-x}$$ AND FOR

$0<|x|<1$ $\quad \frac{e^{-x^2}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$, AND BOTH $\int_1^\infty e^{-x}\,dx$, $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ CONVERGE

BUT HOW DO WE SAY THIS PRECISELY?

WE NEED COMPARISONS

COMPARISONS

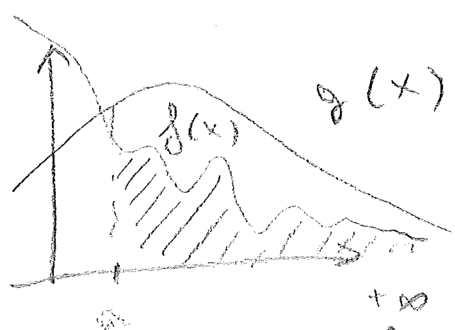
THM (COMPARISON TEST): a, b IN $\mathbb{R} \cup \pm\infty$.

i) SUPPOSE THAT

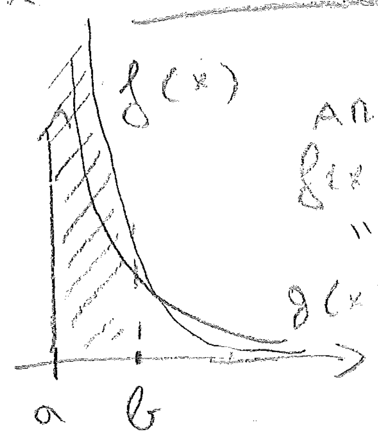
$0 \le f(x) \le g(x)$ OR $g(x) \le f(x) \le 0$

ON (a,b). THEN IF $\int_a^b g(x)\,dx$ CONVERGES $\int_a^b f(x)\,dx$ CONVERGES.

ii) SUPPOSE THAT $0 \le g(x) \le f(x)$ OR $f(x) \le g(x) \le 0$. THEN IF $\int_a^b g(x)\,dx$ DIVERGES $\int_a^b f(x)\,dx$ DIVERGES.

i) 

AREA UNDER $f(x) \le$ AREA UNDER $g(x)$

ii) 

AREA UNDER $f(x) \ge$ AREA UNDER $g(x)$.

EXAMPLE:

$\int_1^\infty \frac{1}{x^2+x}\,dx$ CONVERGES BY COMPARISON WITH $\int_1^\infty \frac{1}{x^2}\,dx$ AS $\frac{1}{x^2+x} < \frac{1}{x^2}$ FOR $x \ge 1$