WARM-UP:

EVALUATE $\displaystyle\int_{\pi/4}^{\frac{3}{4}\pi} \cos(x)^4\,dx$

SOL:

$$\int_{\pi/4}^{\frac{3}{4}\pi} \cos(x)^4\,dx = \int_{\pi/4}^{\frac{3}{4}\pi} \frac{\left(1+\cos(2x)\right)^2}{4}\,dx =$$

$$\frac{1}{4}\int_{\pi/4}^{\frac{3}{4}\pi} \cos(2x)^2 + 2\cos(2x) + 1\,dx =$$

$$\frac{1}{4}\int_{\pi/4}^{\frac{3}{4}\pi} \frac{1+\cos(4x)}{2}\,dx + \frac{1}{2}\int_{\pi/4}^{\frac{3}{4}\pi} \cos(2x) + \frac{\pi}{8} =$$

$$\frac{x}{8} + \frac{\sin(4x)}{32}\Bigg|_{\pi/4}^{\frac{3}{4}\pi} + \frac{\sin(2x)}{4}\Bigg|_{\pi/4}^{\frac{3}{4}\pi} + \frac{\pi}{8} =$$

$$\frac{\pi}{16} - \frac{\sin(\frac{\pi}{2})}{4} + \frac{\sin(\frac{3}{2}\pi)}{4} + \frac{\pi}{8} =$$

$$\frac{3}{16}\pi - \frac{1}{2}$$

WARM-UP

$$\int_0^{\pi/4} \tan^5 x \sec^{\frac{3}{2}} x \, dx = \int_0^{\pi/4} \frac{\sin^5 x}{\cos^{13/2} x} \, dx =$$

$$\int_0^{\pi/4} \sin x \frac{\sin^4 x}{\cos^{\frac{13}{2}} x} \, dx = \int_0^{\pi/4} \sin x \frac{(1-\cos^2 x)^2}{\cos^{\frac{13}{2}} x} \, dx$$

$$= -\int_0^{\pi/4} u' \left(u^{-\frac{13}{2}} - 2u^{-\frac{9}{2}} + u^{-\frac{5}{2}} \right) dx = -\int_{\cos(0)}^{\cos\frac{\pi}{4}} u^{-\frac{13}{2}} - 2u^{-\frac{9}{2}} + u^{-\frac{5}{2}} \, du$$

$$= -\left[u^{-\frac{11}{2}} \cdot \left(-\frac{2}{11}\right) - 2u^{-\frac{7}{2}}\left(-\frac{2}{7}\right) + u^{-\frac{3}{2}}\left(-\frac{2}{3}\right) \right]_1^{\frac{1}{\sqrt{2}}} =$$

$$= \frac{2}{11} \sqrt{2}^{\frac{11}{2}} - \frac{4}{7} \sqrt{2}^{\frac{7}{2}} + \frac{2}{3} \sqrt{2}^{\frac{3}{2}} - \left(\frac{2}{11} - \frac{4}{7} + \frac{2}{3} \right)$$

$$= \sqrt{2}^{\frac{3}{2}} \left(\frac{2}{11} \cdot 4 - \frac{4}{7} \cdot 2 + \frac{2}{3} \right) - \left(\frac{2}{11} - \frac{4}{7} + \frac{2}{3} \right)$$

$$\approx 0.145$$

THE INTEGRAL OF $\tan(x)^m \sec(x)^n$

RECALL THAT

$$\sec(x) = \frac{1}{\cos(x)}, \quad (\tan(x))' = \sec(x)^2,$$

$$\sec(x)' = \sec(x)\tan(x) \qquad 1 + \tan(x)^2 = \sec(x)^2$$

THERE ARE FIVE BASIC CASES:

1. m ODD, n ANY (EVEN FRACTIONAL!)

$$(\tan(x))^m (\sec(x))^n = \frac{\sin(x)^m}{\cos(x)^{m+n}} = \frac{\sin(x)^{m-1}}{\cos(x)^{m+n}} \sin(x)$$

$u = \cos(x)$

2. ALTERNATIVELY m ODD, $n \geq 1$

$$\tan(x)^m \sec(x)^n = \tan(x)^{m-1} \sec(x)^{n-1} \cdot \sec(x)\tan(x)$$

$u = \sec(x) \quad u' = \tan(x)\sec(x), \quad \tan(x)^{m-1} =$
$(\sec(x)^2 - 1)^{\frac{m-1}{2}}$

3. n EVEN ≥ 2 $\quad \tan(x)^m \sec(x)^n =$

$\tan(x)^m \sec(x)^{n-2} \sec(x)^2 \qquad u = \tan(x)$

$\qquad u' = \sec(x)^2$

THEN USE $\quad \sec(x)^2 = \tan(x)^2 + 1$

4. m EVEN, $n = 0$ $\qquad \tan(x)^m = \tan(x)^{m-2}(\sec(x)^2 - 1)$

REPEATEDLY AND THEN $u = \tan(x)$

(5) M ODD , m EVEN .

TOO COMPLICATED FOR NOW. WE'LL GET BACK TO IT LATER IN THE COURSE.

AN EXAMPLE FOR EACH CASE:

EXAMPLE:

$$\int \operatorname{Tan}(x)^3 \sec(x)^{\frac{1}{2}}\, dx = \int \frac{\sin(x)^3}{\cos(x)^{\frac{7}{2}}}\, dx$$

$u = \cos(x)$

$$\int \frac{\sin(x)^3}{\cos(x)^{\frac{7}{2}}}\, dx = \int \frac{-(1-u^2)}{u^{\frac{7}{2}}} \cdot u'\, dx \Big|_{u=\cos(x)}$$

$$= -\int u^{-\frac{7}{2}} - u^{-\frac{3}{2}}\, du \Big|_{u=\cos(x)} = -\frac{2}{5} u^{-\frac{5}{2}} + 2 u^{-\frac{1}{2}} + C \Big|_{u=\cos x}$$

$$= -\frac{2}{5}\cos(x)^{-\frac{5}{2}} + 2\cos(x)^{-\frac{1}{2}} + C \qquad \text{(VERIFY!)}$$

EXAMPLE:

$$\int \operatorname{Tan}(x)^3 \sec(x)^4\, dx = \int \operatorname{Tan}(x)^2 \sec(x)^2 \cdot \operatorname{Tan}(x) \sec(x)^2\, dx$$

$\boxed{u = \sec(x)}$

$$= \int (u^2-1)^2 u^3 \cdot u'\, dx = \int (u^2-1)^2 u^3\, du \Big|_{u=\sec x}$$

$$= \frac{u^6}{6} - \frac{u^4}{4} + C \Big|_{u=\sec(x)} = \frac{\sec(x)^6}{6} - \frac{\sec(x)^4}{4} + C \qquad \text{(VERIFY!)}$$

EXAMPLE:

$$\int \sec(x)^4\,dx = \int \sec(x)^2 \sec(x)^2\,dx =$$

$\boxed{u = \operatorname{Tan}(x)}$

$$= \int (u^2+1)\,u'\,dx =$$

$$\int (u^2+1)\,du \Big|_{u=\operatorname{Tan} x} = \frac{u^3}{3} + u \Big|_{u=\operatorname{Tan} x} + C$$

$$= \frac{\operatorname{Tan}(x)^3}{3} + \operatorname{Tan}(x) + C \qquad (\text{VERIFY!})$$

EXAMPLE:

$$\int \operatorname{Tan}(x)^4\,dx = \int \operatorname{Tan}(x)^2 \left(\sec(x)^2 - 1\right) dx$$

$$= \int \operatorname{Tan}(x)^2 \sec(x)^2 - \operatorname{Tan}(x)^2\,dx =$$

$$= \int \operatorname{Tan}(x)^2 \sec(x)^2 - \sec(x)^2 + 1\,dx =$$

$$\int \left(\operatorname{Tan}(x)^2 - 1\right) \sec(x)^2\,dx + \int 1\,dx = \boxed{u = \operatorname{Tan}(x)}$$

$$= \int (u^2-1)\,u'\,dx + x + C = \int u^2 - 1\,du \Big|_{u=\operatorname{Tan}(x)}$$

$$+ x + C = \frac{\operatorname{Tan}(x)^3}{3} - \operatorname{Tan}(x) + x + C \qquad (\text{VERIFY!})$$

X FUNCTION OF $(x = g(u))$

$$\frac{dx}{du} = g'(u) \rightarrow dx = g'(u)\,du \sim du = \frac{dx}{g'(u)}$$

* REALITY CHECK $\boxed{\begin{array}{l} x = \cos u \\ u = \arccos x \end{array}}$

BUT WHY WOULD WE WANT TO USE THE SECOND VERSION? IT LOOKS LIKE IT COMPLICATES THINGS!

EXAMPLE:

$$\int_0^1 \sqrt{1-x^2}\,dx \overset{x = \sin(u)}{\underset{\downarrow}{=}} \int_0^{\pi/2 = \arcsin(1)} \sqrt{1-\sin(u)^2}\cdot\cos(u)\,du =$$

$$= \int_0^{\pi/2} \cos(u)^2\,du = \int_0^{\pi/2} \frac{1+\cos(2u)}{2}\,du =$$

$$\frac{u}{2} + \frac{\sin(2u)}{4}\Bigg|_0^{\frac{\pi}{2}} = \frac{\pi}{4} - 0 + \frac{\sin(0)}{4} - \frac{\sin(\pi)}{4}$$

$$= \frac{\pi}{4} \checkmark$$

SO WE CAN USE IT TO GET RID OF SQUARE ROOTS!