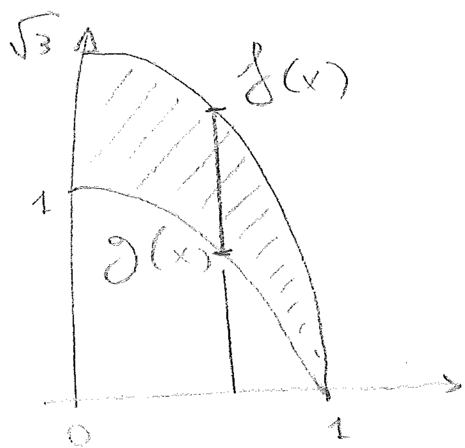
EXAMPLE: "BOWL"

SOLID OBTAINED BY ROTATING AROUND X-AXIS SHAPE BETWEEN

$$y = \sqrt{3-3x} \qquad y = \sqrt{1-x^2}$$
$$\quad \overset{\shortparallel}{f} \qquad\qquad\qquad \overset{\shortparallel}{g}$$



SLICE ALONG X-AXIS; SLICE IS A CIRCULAR CROWN OF AREA $\pi f(x)^2 - \pi g(x)^2$

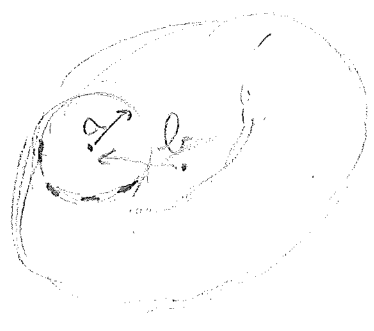
(NOTE: NOT $\pi(f(x)-g(x))^2$!)

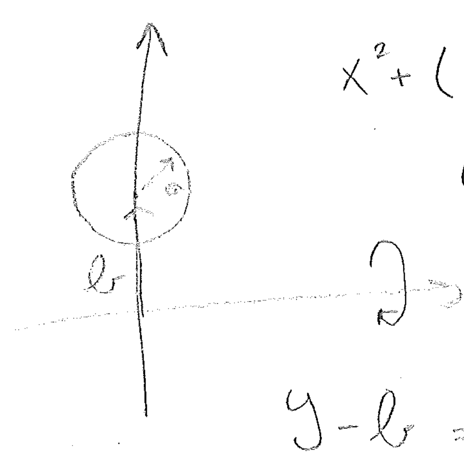
SO VOLUME $= \displaystyle\int_0^1 \pi\left(f(x)^2 - g(x)^2\right) dx =$

$$\pi \int_0^1 \left(3-3x-(1-x^2)\right) dx = \pi \int_0^1 x^2 - 3x + 2 \, dx =$$

$$\pi \left[\frac{x^3}{3} - \frac{3}{2}x^2 + 2x\right]_0^1 = \pi\left(\frac{1}{3} - \frac{3}{2} + 2\right) = \frac{5}{6}\pi$$

EXAMPLE: TORUS

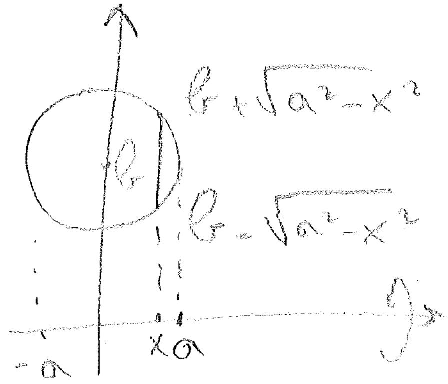




$$x^2 + (y-b)^2 = a^2$$
$$(b > a)$$

$$y - b = \pm\sqrt{a^2 - x^2}$$

$y = b \pm \sqrt{a^2 - x^2}$



SLICE: CIRC CROWN,

$R = b + \sqrt{a^2-x^2}$, $r = b - \sqrt{a^2-x^2}$

AREA $= \pi (R^2 - r^2) =$

$$\pi \left(b^2 - b^2 + 2b\sqrt{a^2-x^2} + 2b\sqrt{a^2-x^2} + a^2 - x^2 - (a^2 - x^2) \right)$$

$$= \pi \cdot 4b\sqrt{a^2-x^2}$$

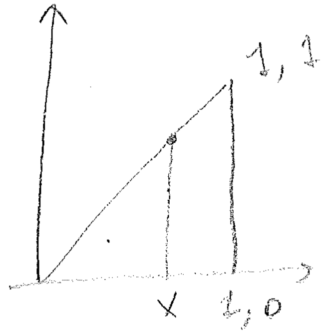
$$\text{VOL} = 4\pi b \int_{-a}^{a} \underbrace{\sqrt{a^2-x^2}}_{\frac{1}{2}\text{ OF CIRCLE OF RADIUS } a}\, dx = 4\pi b \cdot \frac{\pi a^2}{2} =$$

$2\pi^2 b a^2$.

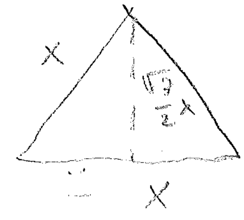
EXAMPLE: CONSIDER A SOLID S SUCH THAT

- THE BASE OF S IS THE LOCUS $y \leq x$, $0 \leq x \leq 1$
- EACH CROSS SECTION IS AN EQUILATERAL TRIANGLE

FIND THE VOLUME OF S



SLICE: EQ. TRIANG OF BASE x; AREA $= \frac{\sqrt{3}}{4} x^2$

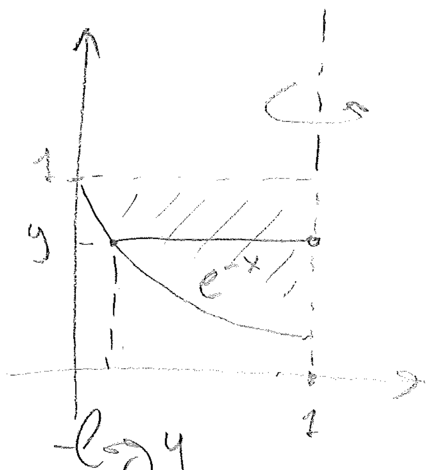


So $\text{VOL} = \int_0^1 \frac{\sqrt{3}}{4} x^2 \, dx = \frac{\sqrt{3}}{4} \left[\frac{x^3}{3} \right]_0^1 =$

$= \frac{\sqrt{3}}{12} = \frac{1}{4\sqrt{3}}$

EXAMPLE: DIFFERENT AXIS

CONSIDER $y = e^{-x}$. FIND VOL OF SHAPE BETWEEN $y = e^{-x}$, $y = 1$, $x = 1$ ROTATED ABOUT $x = 1$ (JUST WRITE FORMULA)



SLICE PERP TO $x = 1$ LINE

$y = e^{-x} \sim x = -\log y$

AREA OF SLICE

$\pi(1 + \log y)^2$ SO

$\text{VOL} = \pi \int_{\frac{1}{e}}^{1} 1 + 2\log y + (\log y)^2 \, dy$

BUT: WE CAN'T COMPUTE THIS. YET! WE NEED A NEW TOOL.

INTEGRATION BY PARTS

JUST AS WE REVERSE-ENGINEERED THE CHAIN RULE TO GET SUBSTITUTION, NOW WE'LL REVERSE-ENGINEER THE

PRODUCT RULE

$$\frac{d}{dx}\left(u(x)\,v(x)\right) = \left(\frac{d}{dx}u(x)\right)v(x) + u(x)\left(\frac{d}{dx}v(x)\right)$$

LET'S TRY INTEGRATING IT:

$$\int \frac{d}{dx}\left(u(x)\,v(x)\right)dx = \int u'v\,dx + \int uv'\,dx$$

$$u(x)\,v(x) + C = \int u'v\,dx + \int uv'\,dx$$

$$u(x)\,v(x) - \int u'v\,dx = \int uv'\,dx$$

INTEGRATION BY PARTS:

$F(x)$, $G(x)$ DIFFERENTIABLE, THEN

$$F(x)\,G(x) - \int f(x)\,G(x)\,dx = \int F(x)\,g(x)\,dx$$

DEFINITE VERS:

$$F(x)\,G(x)\Big|_a^b - \int_a^b f(x)\,G(x)\,dx = \int_a^b F(x)\,g(x)\,dx$$

HOW DO WE USE IT?

EXAMPLE: $\int x e^x \, dx$

WE WANT TO INTEGRATE ONE FACTOR AND DERIVE THE OTHER;

$$e^x \xrightarrow{\int} e^x \qquad e^x \xrightarrow{\frac{d}{dx}} e^x$$

NO DIFFERENCE

$$x \xrightarrow{\int} \frac{x^2}{2} \qquad x \xrightarrow{\frac{d}{dx}} 1$$

BETTER TO $\frac{d}{dx}$

SO $F(x) = x$, $g(x) = e^x$, THEN

$$\underbrace{x e^x}_{FG} - \underbrace{\int 1 \cdot e^x \, dx}_{\int f G} = \underbrace{\int x e^x \, dx}_{\int F g}$$

SO

$$\int x e^x \, dx = x e^x - e^x$$

VERIFY: $\frac{d}{dx}\left(x e^x - e^x\right) = (x)' e^x + x(e^x)' - (e^x)'$

$= e^x + x e^x - e^x = x e^x$ ✓

EX: $\int x^2 e^x \, dx$

SAME IDEA: $F(x) = x^2$, $g(x) = e^x$

$$\underbrace{x^2 e^x}_{FG} - 2\underbrace{\int x e^x dx}_{\int gG} = \int x^2 e^x \, dx$$

$$x^2 e^x - 2x e^x + 2e^x + C = \int x^2 e^x \, dx \quad \text{(VERIFY!)}$$

WE CAN INDUCTIVELY COMPUTE $\int x^m e^x \, dx$

FOR ALL m!

EX: $\int x \log x \, dx$

$$\begin{tikzcd}[row sep=small] & \frac{x^2}{2} \\ x \arrow[ur] \arrow[dr] & \\ & 1 \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & ? \\ \log x \arrow[ur] \arrow[dr] & \\ & \frac{1}{x} \end{tikzcd}$$

WE CAN ONLY $\frac{d}{dx}$

SO $F(x) = \log x$, $g(x) = x$

$$\frac{x^2 \log x}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx = \int x \log x \, dx$$

$$\frac{x^2 \log x}{2} - \frac{x^2}{4} + C = \int x \log x \, dx \quad \text{(VERIFY!)}$$