THM: (SUBSTITUTION, DEFINITE VERSION)

$f(x)$ INTEGRABLE, $u(x)$ DIFFERENTIABLE, THEN

$$\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du .$$

Ex: $\displaystyle\int_0^1 \frac{3x}{(x^2+1)^2}\,dx$

$$\underbrace{3x}_{\frac{3}{2}\frac{d}{dx}(x^2+1)} \cdot \frac{1}{(x^2+1)^2} \qquad \text{so } u = x^2+1$$

$$= \int_0^1 \frac{3}{2}\,\frac{2x}{(x^2+1)^2}\,dx = \frac{3}{2}\int_0^1 \frac{u'}{u^2}\,dx =$$

$$= \frac{3}{2}\int_{u(0)}^{u(1)} \frac{1}{u^2}\,du = \frac{3}{2}\int_1^2 \frac{1}{u^2}\,du = \frac{3}{2}\left[-\frac{1}{u}\right]_1^2 = \frac{3}{2}\left[-\frac{1}{2}+1\right]$$

$$= \frac{3}{4}.$$

Ex: $\displaystyle\int_0^{\pi} \sin(x)\,e^{\cos x}\,dx$

$$\sin(x) = -\frac{d}{dx}\cos x$$

so $u = \cos x$

$$= \int_{\cos 0}^{\cos \pi} -u' e^{u}\,dx = -\int_1^{-1} e^{u}\,du = \int_{-1}^{1} e^{u}\,du = e - \frac{1}{e}.$$

NOTE: VERIFY THAT THE SIGN IS CORRECT.

A SLIGHTLY DIFFERENT SUBSTITUTION:

· FIND $\int \frac{1}{x^2+9}\,dx$

$$\underbrace{1}\cdot\frac{1}{x^2+9}$$

WE SHOULD LOOK FOR A LINEAR SUBSTITUTION

$\frac{1}{x^2+9}$ LOOK SIMILAR TO $\frac{1}{x^2+1}$, WHICH

WE KNOW HOW TO DO. IF WE HAD

$x = 3u$, THEN $\frac{1}{x^2+9} = \frac{1}{9u^2+9} = \frac{1}{9}\frac{1}{1+u^2}$

BY SUBSTITUTION $u = \frac{x}{3}$, $u' = \frac{1}{3}$

$$\int \frac{1}{x^2+9}\,dx = \int \frac{1}{9(u^2+1)}\,dx = 3\int \frac{u'}{9(u^2+1)}\,dx$$

$$= \frac{1}{3}\int \frac{1}{u^2+1}\,du\Big|_{u=\frac{x}{3}} = \frac{\arctan\left(\frac{x}{3}\right)}{3}$$

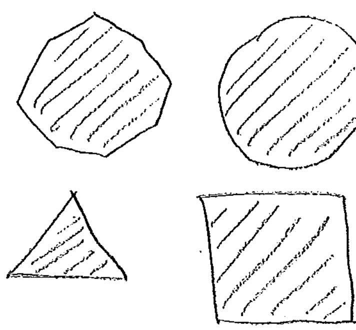
VERIFY: $\frac{d}{dx}\frac{\arctan\left(\frac{x}{3}\right)}{3} = \frac{1}{9}\cdot\frac{1}{\frac{x^2}{9}+1} =$

$$\frac{1}{9}\cdot\frac{1}{\frac{1}{9}(x^2+9)} = \frac{9}{9}\frac{1}{x^2+9} = \frac{1}{x^2+9}$$

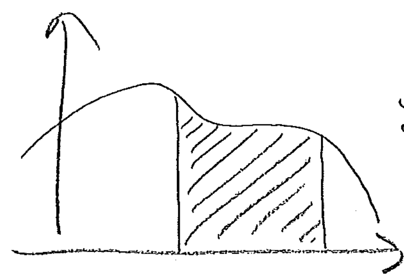
AREA BETWEEN CURVES

BEFORE WE DEVELOP SOME NEW INTEGRATION TOOL, WE ARE NOW ABLE TO EXPLORE SOME APPLICATIONS OF DEFINITE INTEGRALS.

ONE IMPORTANT QUESTION IN CALCULUS IS HOW DO WE COMPUTE AREAS? FOR NOW WE KNOW



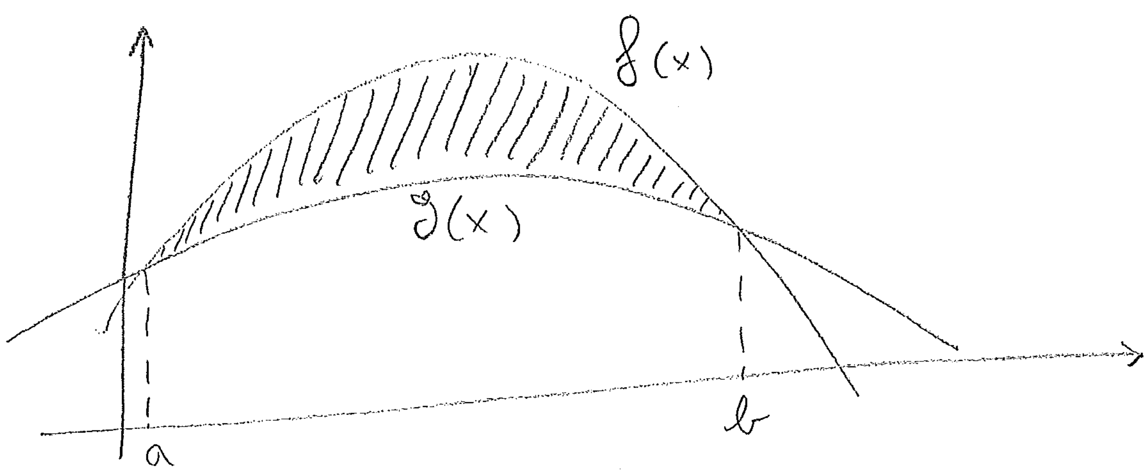
CLASSICAL PLANE AREAS



$\int_a^b f(x)\,dx$

AREA UNDER A CURVE

A SMALL, BUT POWERFUL GENERALIZATION IS:



THE AREA BETWEEN CURVES.

IN THE CASE ABOVE, AS WE KNOW THAT $f(x) \geq g(x)$ ON $[a, b]$, AND BOTH ARE POSITIVE, WE SEE THAT THE AREA MUST BE

$$\text{AREA BELOW } f(x) - \text{AREA BELOW } g(x) = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

EXAMPLE:

$$f(x) = 16 - 5(x-2)^2, \quad g(x) = 4 - 2(x-2)^2$$

$f(x)$ AND $g(x)$ INTERSECT AT THE SOLNS OF

$$f(x) - g(x) = 0 \sim 12 - 3(x-2)^2 = 0$$

$$\sim x = 0, 4$$

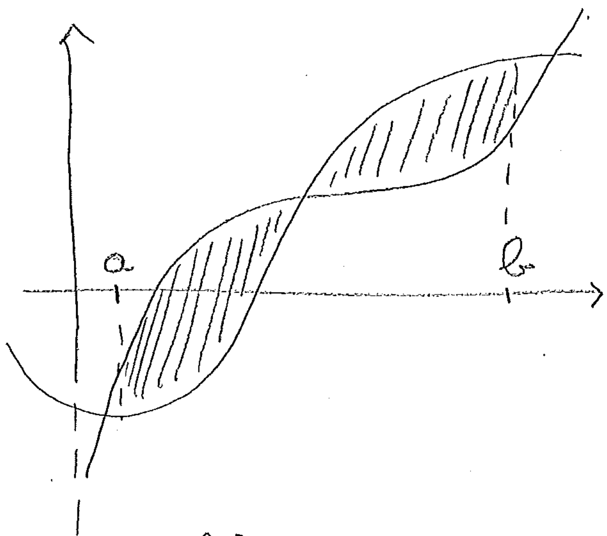
AT $x = 2$ $f(2) > g(2)$ SO THE AREA BETWEEN THEM (AS IN FIGURE) IS

$$\int_0^4 f(x) - g(x)\,dx = \int_0^4 12 - 3(x-2)^2\,dx \overset{u'=1}{=} \int_{-2}^{2} 12 - 3u^2\,du$$

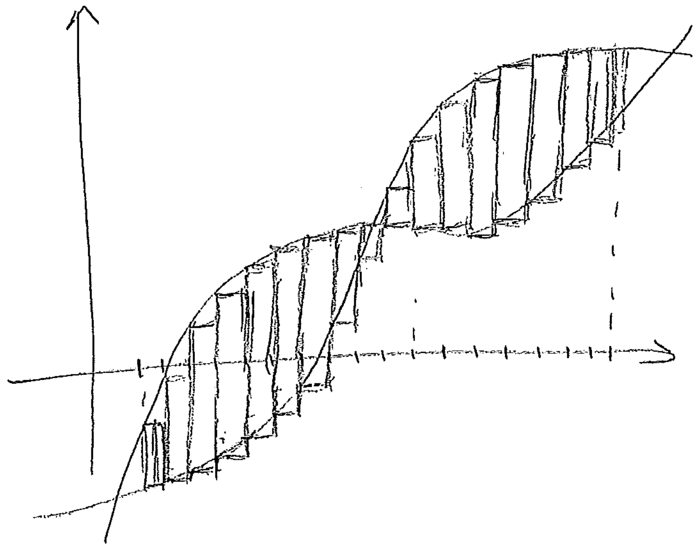
$$= 12u - u^3 \Big|_{-2}^{2} = 16 - (-16) = 32$$

BUT WHAT IF $f(x), g(x)$ INTERCEPT IN MULTIPLE POINTS AND/OR CHANGE SIGN?

GENERAL IDEA: DIVIDE AND CONQUER!

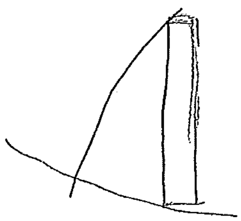


SUBDIVIDE $[a,b]$ IN n EQUAL PIECES



APPROXIMATE WITH RECTANGLES

WHAT'S THE HEIGHT OF A RECTANGLE?



$f(x_{i,n}^*) - g(x_{i,n}^*)$ WHICHEVER'S POSITIVE!

OR

$g(x_{i,n}^*) - f(x_{i,n}^*)$ SO IT'S $|f(x_{i,n}^*) - g(x_{i,n}^*)|$

WE GOT A RIEMANN SUM!

n-TH APPROX AREA $= \sum_{i=1}^{n} |f(x_{i,n}^*) - g(x_{i,n}^*)|$

TAKE THE LIMIT

$$\lim_{n \to \infty} \sum_{i=1}^{n} |f(x_{i,n}^* - g(x_{i,n}^*)| = \int_a^b |f(x) - g(x) dx \; !$$

DEF: AREA BETWEEN $f(x)$ AND $g(x)$ WITH x RUNNING FROM a TO b:

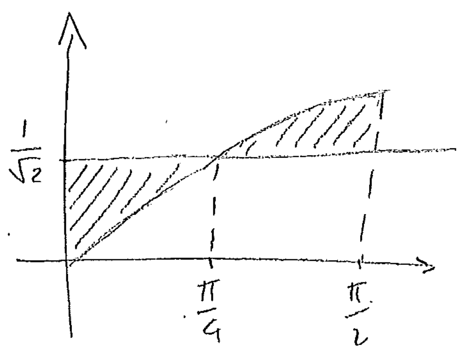
$$\int_a^b |f(x)-g(x)|\,dx$$

IN PRACTICE, TO COMPUTE IT WE'LL STILL HAVE TO DIVIDE AND CONQUER, I.E. FIND THE INTERSECTION POINTS.

EXAMPLE:
FIND THE AREA BETWEEN $y=\frac{1}{\sqrt{2}}$ AND $y=\sin(x)$ FOR x FROM 0 TO $\frac{\pi}{2}$.

SKETCH

$\sin(x)=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$ WHEN $x=\frac{\pi}{4}$



AREA A: x FROM 0 TO $\pi/4$

AREA B: x FROM $\frac{\pi}{4}$ TO $\frac{\pi}{2}$

TOTAL AREA = AREA A + AREA B =

$$\int_0^{\pi/4} \frac{1}{\sqrt{2}} - \sin x\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \frac{1}{\sqrt{2}}\,dx =$$

$$\frac{x}{\sqrt{2}} + \cos x\Big|_0^{\pi/4} + \left(-\cos x - \frac{x}{\sqrt{2}}\right)\Big|_{\pi/4}^{\frac{\pi}{2}} = \left(-1 + \frac{\pi}{4\sqrt{2}} + \frac{\sqrt{2}}{2}\right) +$$

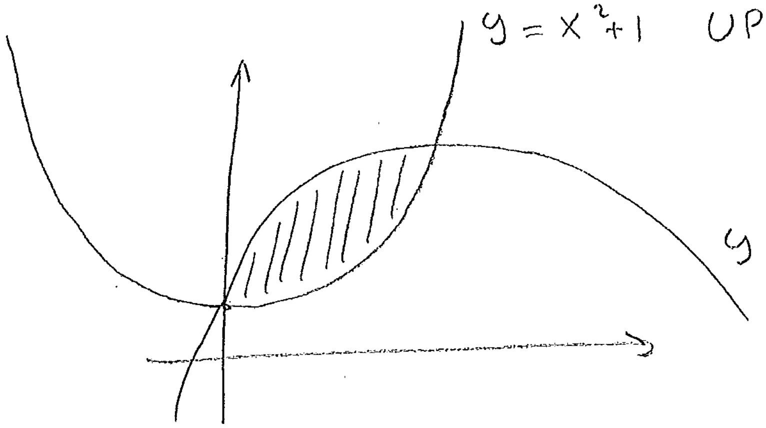
$$\left(\frac{\pi}{4\sqrt{2}} + \frac{\sqrt{2}}{2} - \frac{\pi}{2\sqrt{2}}\right) = \frac{\pi}{2\sqrt{2}} - \frac{\pi}{2\sqrt{2}} - 1 + \frac{2\sqrt{2}}{2} = \sqrt{2}-1$$

EXAMPLE:

FIND THE AREA OF THE REGION ENCLOSED BY

$$y = \underbrace{x^2+1}_{T(x)} \quad \text{AND} \quad y = \underbrace{6x - 3x^2 + 1}_{B(x)}$$

STEP 1: SKETCH



$y = x^2+1$ UPWARD FACING PARABOLA

$y = 6x - 3x^2 + 1$

DOWNWARD FACING PAR.

STEP 2: FIND INTERSECTIONS $T(x) = B(x) \sim$

$$x^2+1 = 6x - 3x^2 + 1 \sim 4x^2 - 6x = 0$$

$$\sim x(4x-6) = 0 \qquad x = 0, \tfrac{3}{2}$$

STEP 3: COMPUTE INTEGRAL (KNOWING THAT $B(x) \geq T(x)$ ON OUR INTERVAL)

$$\int_0^{\frac{3}{2}} B(x) - T(x)\, dx = \int_0^{\frac{3}{2}} 6x - 4x^2\, dx = 3x^2 - \frac{4}{3}x^3 \Big|_0^{\frac{3}{2}}$$

$$= 3 \cdot \frac{9}{4} - \frac{4}{3} \cdot \frac{27}{8} = \frac{27}{4} - \frac{18}{4} = \frac{9}{4}$$