WARM-UP:

- FIND $\int \sqrt{2x+1}\, dx$
- FIND $\int_0^6 e^{|x-3|} - 1\, dx$

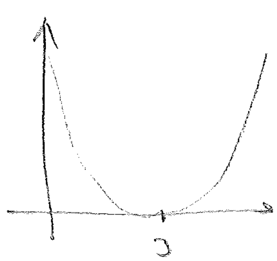
SOL:

- BY LINEAR SUBSTITUTION, $f(x) = \sqrt{x}$, $F(x) = \frac{2}{3}\sqrt{x^3}$

$$\int f(2x+1)\, dx = \frac{F(2x+1)}{2} + c = \frac{1}{3}\sqrt{(2x+1)^3} + c$$

VERIFY: $\frac{d}{dx}\frac{1}{3}\sqrt{(2x+1)^3} = 6(2x+1)^2 \cdot \frac{1}{3} \cdot \frac{1}{2\sqrt{(2x+1)^3}}$

$$= \frac{6}{6} \cdot (2x+1)^{2-\frac{3}{2}} = \sqrt{2x+1} \quad \checkmark$$

- 

$e^{|x-3|} - 1$ IS SYM W.R.T. $x = 3$

$$\int_0^6 e^{|x-3|} - 1\, dx \underset{c=-3}{=} \int_{-3}^{3} \underbrace{e^{|x|}}_{\text{EVEN!}} - 1\, dx = 2\int_0^3 e^x - 1\, dx$$

$$= 2\left[e^x - x\right]_0^3 = 2e^3 - 8$$

SUBSTITUTION

UP TO NOW WE'VE ONLY BEEN ABLE TO INTEGRATE VERY FEW FUNCTIONS; THOSE THAT WE KNOW ARE DERIVATIVES, CUT-AND-PASTE OF THOSE, AND THESE FUNCTIONS COMPOSED WITH A LINEAR FUNCTION. WE NEED A NEW TOOL.

NOTE: NO NUMBER ALL TOOLS WILL ALLOW US TO INTEGRATE EVERY FUNCTION. THE FUNCTION

$\int e^{-x^2} dx$, WHICH APPEARS EVERYWHERE IN PROBABILITY AND PHYSICS, CANNOT BE WRITTEN AS A FINITE COMBINATION OF OUR BASIC "BUILDING BLOCK" FUNCTIONS. IT CAN BE SEEN AS AN INFINITE SUM, THOUGH, AS WE'LL SEE LATER.

HOW DO WE FIND NEW TOOLS? LET'S LOOK AT THE FIRST RULE OF DIFFERENTIATION WE LEARNED IN MATH 100:

CHAIN RULE $F(x)$, $g(x)$ DIFFERENTIABLE, $F'(x) = f(x)$. THEN:

$$(F(g(x)))' = g'(x) f(g(x)).$$

LET'S TRY INTEGRATING BOTH SIDES:

$$\int (F(g(x)))' \, dx = \int g'(x) f(g(x)) \, dx$$

$$F(g(x)) + C = \int g'(x) f(g(x)) \, dx$$

<u>THM:</u> (SUBSTITUTION RULE)

$f(x)$ INTEGRABLE, $u(x)$ DIFFERENTIABLE.
THEN

$$\int f(u(x)) \cdot \frac{d}{dx} u(x) \, dx = \int f(u) \, du \Big|_{u=u(x)}$$

FOR THE MOMENT WE'LL BE USING THIS MOSTLY <u>LEFT TO RIGHT</u>, WHICH SEEMS REASONABLE AS IT <u>SIMPLIFIES THE EXPRESSION</u>.

<u>EX:</u> $\int 2x e^{x^2} dx \qquad \underbrace{2x}\,\underbrace{e^{x^2}}$

① LOOK FOR WAYS TO FACTOR EXPRESSION

② DOES A TERM LOOK LIKE THE $\frac{d}{dx}$ OF THE OTHER'S ARGUMENT?

$$2x = \frac{d}{dx} x^2 \quad \text{SO} \quad u = x^2; \; f = e^x$$

$$\int 2x e^{x^2} dx = \int e^u \, du \Big|_{u=x^2} = e^{x^2} + C$$

VERIFY: $(e^{x^2})' = 2x e^{x^2}$ ✓

EX: WE CAN MULTIPLY/DIVIDE BY CONSTANTS IF NEEDED

$$\int x e^{x^2} dx = \frac{1}{2}\int 2x e^{x^2} dx = \frac{e^{x^2}}{2} + C$$

EX: $\int 9\sin^8 x \cos x \, dx$ $\qquad \underbrace{9\sin^8 x}\ \underbrace{\cos x}_{=\frac{d}{dx}\sin x}$

DON'T FORGET THAT $f(x) = x^n$ IS A VALID CHOICE!

$$9\sin^8 x = f(\sin x), \quad f(x) = 9x^8$$

SO $u(x) = \sin x$

$$\int 9\sin^8 x \cos x \, dx = \int 9u^8(x)\, u'(x) \, dx$$

$$= \int 9u^8 \, du \Big|_{u=\sin x} = \frac{9u^9}{9}\Big|_{u=\sin x} + C = \sin^9 x + C$$

VERIFY (ALWAYS!): $\sin^9(x)' = 9\sin^8 x \cos x$ ✓

EX: SOMETIMES WE HAVE TO TRY A FEW OPTIONS

$$\int x^2(x^3+4)\, dx$$

$u'(x) = x^3+4$

$u(x) = \frac{x^4}{4} + 4x$?

$u'(x) = x^2$

$u(x) = \frac{x^3}{3}$ ✓

$$\frac{1}{3}\int 3x^2(x^3+4)\, dx \underset{u=x^3}{=} \frac{1}{3}\int u'(u+4)\, dx = \frac{1}{3}\int u+4 \, du \Big|_{u=x^3}$$

$$= \frac{1}{3}\left(\frac{u^2}{2} + 4u\right)\Big|_{u=x^3} + C = \frac{x^6}{6} + \frac{4}{3}x^3 + C$$

VERIFY $\frac{d}{dx}\left(\frac{x^6}{6} + \frac{4}{3}x^3\right) = \frac{6x^5}{6} + 4x^2 = x^2(x^3+4)$ ✓

Ex: $\int \cot x \, dx$ $\quad \cot x = \frac{\cos x}{\sin x} = \underbrace{\cos x}_{=(\sin x)'} \cdot \frac{1}{\sin x}$

$u(x) = \sin x$ $\quad \int \cot x \, dx = \int \frac{u'}{u} \, dx =$

$\int \frac{1}{u} \, du \Big|_{u=\sin x} = \log|u| \Big|_{u=\sin x} + C =$

$\log|\sin x| + C.$

VERIFY: $(\log|\sin x|)' = (\sin x)' \frac{1}{\sin x} = \frac{\cos x}{\sin x}$

NOTE: $\cot x$ IS DEF WHERE $\sin x > 0$

WHAT IF WE TOOK $u = \cos x$?

$\frac{\cos x}{\sin x} = \frac{u}{u'}$, NOT THE RIGHT FORM!

Ex: $\int \frac{x}{1+x^2} \, dx$ $\quad (1+x^2)' = 2x$ SO $u' = 2x$, $u = 1+x^2$

$\int \frac{x}{1+x^2} \, dx = \frac{1}{2} \int \frac{u'}{u} \, dx \underset{\text{AS ABOVE}}{=} \dots = \frac{\log|1+x^2|}{2} + C$

VERIFY: $(\log|1+x^2|)' = \frac{1}{\cdot} \cdot 2x \cdot \frac{1}{\cdot} = \frac{x}{\cdot}$

NOTE: ONE WAY TO REMEMBER THE SUBSTITUTION RULE IS THIS

$$dx = \frac{1}{u'(x)} du$$

SO $\int u'(x) f(u)\, dx = \int f(u) \frac{u'(x)}{u'(x)} du =$

$$\int f(u)\, du$$

A <u>MNEMONIC DEVICE</u> TO REMEMBER THIS IS

$$\frac{du}{dx} = u' \sim \frac{du}{u'} = dx$$

WHILE THE FIRST FORMULA DOES HAVE MATHEMATICAL MEANING, THE MNEMONIC DEVICE IS JUST THAT, A WAY TO REMBER THINGS WITH NO MATHEMATICAL MEANING. <u>NEVER USE IT AS IF IT WAS ACTUAL CORRECT MATH!</u>

$\frac{du}{dx}$ <u>IS NOT A FRACTION</u>.

HOW ABOUT DEFINITE INTEGRALS? WE NEED TO BE SLIGTHLY MORE CAREFUL:

THM: (SUBSTITUTION, DEFINITE VERSION)

$f(x)$ INTEGRABLE, $u(x)$ DIFFERENTIABLE, THEN

$$\int_a^b f(u(x))\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du .$$

Ex: $\displaystyle\int_0^1 \frac{3x}{(x^2+1)^2}\, dx$

$$\underbrace{3x}_{=\frac{3}{2}\frac{d}{dx}(x^2+1)} \cdot \frac{1}{(x^2+1)^2} \quad \text{SO } u = x^2+1$$

$$= \int_0^1 \frac{3}{2}\frac{2x}{(x^2+1)^2}\, dx = \frac{3}{2}\int_0^1 \frac{u'}{u^2}\, dx =$$

$$= \frac{3}{2}\int_{u(0)}^{u(1)} \frac{1}{u^2}\, du = \frac{3}{2}\int_1^2 \frac{1}{u^2}\, du = \frac{3}{2}\left[-\frac{1}{u}\right]_1^2 = \frac{3}{2}\left[-\frac{1}{2}+1\right]$$

$$= \frac{3}{4}.$$

Ex: $\displaystyle\int_0^{\pi} \sin(x)\, e^{\cos x}\, dx$ $\qquad \sin(x) = -\frac{d}{dx}\cos x$

SO $u = \cos x$

$$= \int_{\cos 0}^{\cos\pi} -u' e^{u}\, dx = -\int_1^{-1} e^{u}\, du = \int_{-1}^{1} e^{u}\, du = e - \frac{1}{e}.$$

NOTE: VERIFY THAT THE SIGN IS CORRECT.