KEY MACLAURIN SERIES:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots \quad \text{FOR ALL } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{2n!} + \dots \quad \text{FOR ALL } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots \quad \text{FOR ALL } x$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + \dots \quad \text{FOR } |x| < 1$$

ADDITIONAL: (OBTAINED FROM $\frac{1}{1-x}$)

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{n-1} x^n}{n} + \dots \quad x \text{ IN } (-1, 1]$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + \frac{(-1)^{n-1} x^{2n+1}}{2n+1} + \dots \quad x \text{ IN } [-1, 1]$$

ALSO: $(1+x)^p = 1 + px + \dots$

EX: COMPUTE THE MACLAURIN SERIES OF $x^3 \sin(x^2)$, FIND I.O.C.

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{12}}{6!} + \dots + \frac{(-1)^n x^{4n+2}}{2n+1!} + \dots$$

$$x^3 \sin x^2 = x^5 - \frac{x^9}{3!} + \frac{x^{13}}{5!} - \frac{x^{15}}{6!} + \dots + \frac{(-1)^n x^{4n+5}}{2n+1!} + \dots$$

I.O.C.: $\sin x \to \sin x^2 \quad R \to \sqrt{R}$ SO STILL ∞

$\sin x^2 \to x^3 \sin x^2$ SAME R, STILL ∞

EX: FIND $\dfrac{d^4 e^{-x^2}}{dx^4}(0)$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}, \quad \text{SO} \quad e^{-x^2} \underset{\substack{\uparrow \\ \text{SUBSTITUTE}}}{=} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{m!}$$

WITH $R = \infty$ BY SUBST. THM.

$$e^{-x^2} = 1 + 0 - x^2 + 0 + \underset{\substack{\uparrow \\ a_4}}{\frac{x^4}{2}} + \dots$$

$$\frac{d^4 e^{-x^2}(0)}{dx^4} = a_4 \cdot 4! = \frac{24}{2} = 12$$

EX: FIND $L = \lim_{x \to 0} \dfrac{2x - \log(1+2x)}{x^2}$

$$\log(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + \dots$$

$$\lim_{x \to 0} \frac{2x - \log(1+2x)}{x^2} = \lim_{x \to 0} \frac{2x - 2x + 2x^2 + x^3(\dots)}{x^2}$$

$$= \lim_{x \to 0} 2 + \underset{\substack{\uparrow \\ \text{CONT. FUNCTION}}}{\overset{\nearrow 0}{x}(\dots)} = 2$$

EX: EXPAND $f(x) = \frac{1}{x}$ AROUND $c = 3$

$$\frac{d^m \frac{1}{x}}{dx^m} = \frac{(-1)^m m!}{x^{m+1}} \qquad \frac{d^m \frac{1}{x}}{dx^m}(3) = \frac{(-1)^m m!}{3^m}$$

$$\frac{1}{x} = \sum_{m=0}^{\infty} \frac{(-1)^m}{3^m}(x-3)^m \qquad R = 3 \quad (\text{CHECK!})$$

EX: FIND a SUCH THAT

$$\int_0^1 \frac{\sqrt{3+x^2} - a}{\sqrt{x^3}}\,dx \quad \text{CONVERGES}$$

$y = x^2$

$$\sqrt{3+y} \underset{\substack{\uparrow \\ \text{MACLAURIN}}}{=} f(0) + f'(0)\,y + \dots = \underset{\substack{\text{BOUNDED} \\ \downarrow}}{\sqrt{3} + \frac{1}{2\sqrt{3}}\,y + y^2(\dots)}$$

$$= \sqrt{3} + \frac{x^2}{2\sqrt{3}} + x^4(\dots) \qquad a = \sqrt{3}$$

$$\int_0^1 \frac{\sqrt{3+x^2} - \sqrt{3}}{\sqrt{x^3}}\,dx = \int_0^1 \frac{\sqrt{3} - \sqrt{3} + \frac{x^2}{2\sqrt{3}} + x^4(\dots)}{\sqrt{x^3}}\,dx$$

$$= \int_0^1 \frac{x^{\frac{1}{2}}}{2\sqrt{3}} + x^{\frac{5}{2}}(\dots)\,dx \quad \text{WHICH CONV.}$$

EX: FIND $\lim\limits_{x \to \infty} \sqrt{x}\left(\sqrt{x+4} - \sqrt{x+1}\right)$

$$\sqrt{x}\left(\sqrt{x+4} - \sqrt{x+1}\right) = x\left(\sqrt{1+\frac{4}{x}} - \sqrt{1+\frac{1}{x}}\right)$$

$$\sqrt{1+\frac{4}{x}} = 1 + \frac{2}{x} + \frac{(\dots)}{x^2} \qquad \sqrt{1+\frac{1}{x}} = 1 + \frac{1}{2x} + \frac{(\dots)}{x^2}$$

$$\lim_{x \to \infty} \sqrt{x}\left(\sqrt{x+4} - \sqrt{x+1}\right) = \lim_{x \to \infty} x\left(\frac{3}{2x} + \frac{(\dots)}{x^2}\right) =$$

$$= \frac{3}{2}$$

EX: FIND A MACLAURIN SERIES

FOR F DEFINED BY $F'(x) = \frac{\sin x}{x}$, $F(0) = 3$

$$\frac{\sin x}{x} = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2m+1!} \cdot \frac{1}{x} = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2m+1!}$$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots + \frac{(-1)^m x^{2m}}{2m+1!} + \dots$$

$$\int \frac{\sin x}{x}\,dx = C + x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} + \dots + \frac{(-1)^m x^{2m+1}}{(2m+1)(2m+1!)} + \dots$$

$$= C + \sum \frac{(-1)^m x^{2m+1}}{(2m+1)(2m+1!)}$$, WE PICK $C = 3$

EX: SHOW THAT $\int_0^{\infty} \left(1 - \frac{x}{\sqrt{x^2+1}}\right) dx$ CONVERGES

USING A MCLAURIN SERIES.

$$1 - \frac{x}{\sqrt{x^2+1}} \overset{x>0}{=} 1 - \frac{x}{x\sqrt{1+\frac{1}{x^2}}} = 1 - \left(1 + \frac{1}{x^2}\right)^{-\frac{1}{2}}$$

$y = \frac{1}{x^2}$ $\qquad (1+y)^{-\frac{1}{2}} \underset{\text{MCLAURIN}}{=} 1 - \frac{y}{2} + y^2(\dots)$

$$\left(1 + \frac{1}{x^2}\right) = 1 - \frac{1}{2x^2} + \frac{1}{x^4}(\dots) \quad \text{SO}$$

$$\int_1^\infty 1 - \frac{x}{\sqrt{x^2+1}}\,dx = \int_1^\infty 1 - 1 + \frac{1}{2x^2} + \frac{1}{x^4}\underbrace{(\dots)}_{\text{BOUNDED FUNCTION}}\,dx$$

$$= \int_1^\infty \frac{1}{2x^2} + \underbrace{(\dots)}_{\text{SMALLER STUFF}}\,dx$$

SO $\int_1^\infty 1 - \frac{x}{\sqrt{x^2+1}}\,dx$ CONV.

$\int_0^1 1 - \frac{x}{\sqrt{x^2+1}}\,dx$ HAS NO PROBLEMS.

EX: FIND a SUCH THAT

$$\int_1^\infty \left(\frac{ax}{\sqrt{4x^4+1}} - \frac{1}{x} \right) dx \quad \text{EXISTS}$$

$$\frac{ax}{\sqrt{4x^4+1}} = \frac{ax}{x^2\sqrt{4+\frac{1}{x^4}}} = \frac{a}{2x\sqrt{1+\frac{1}{4x^4}}} = \frac{a}{2x}\left(1+\frac{1}{4x^4}\right)^{-\frac{1}{2}}$$

$$y = \frac{1}{4x^4} \qquad (1+y)^{-\frac{1}{2}} = 1 - \frac{y}{2} + y^2(\dots) = 1 - \frac{1}{8x^4} + \dots$$

$$\frac{a}{2x}\left(1+\frac{1}{4x}\right)^{-\frac{1}{2}} = \frac{a}{2x} - \frac{a}{16x^5} + \dots \qquad a = 2$$

$$\int_1^\infty \frac{2x}{\sqrt{4x^4+1}} - \frac{1}{x}\,dx = \int_1^\infty -\frac{1}{8x^5} + \dots\,dx \quad \text{CONVERGES.}$$