WARM UP:

i) WE KNOW THAT $\sum_{n=2}^{\infty} \frac{(-1)^n}{n-\sqrt{n}}$ CONVERGES,

SAY TO $\sum_{n=2}^{\infty} \frac{(-1)^n}{n-\sqrt{n}} = S$.

FIND N SUCH THAT THE ERROR $|R_N| = |S - S_N|$ IS BELOW $\frac{1}{1000}$.

SOL: WE HAVE $|R_N| \leq a_{N+1} = \frac{1}{N+1-\sqrt{N+1}}$

OPTION 1): WE JUST BRUTALLY SAY $N+1-\sqrt{N+1} \geq \frac{N+1}{2}$ FOR $N \geq 3$ AND THUS

$$\frac{2}{N+1} \leq \frac{1}{1000} \sim N+1 \geq 2000 \sim N \geq 1999$$

WORKS

OPTION 2) $\frac{1}{N+1-\sqrt{N+1}} \leq \frac{1}{1000} \sim N+1-\sqrt{N+1}-1000 \geq 0$

SOLVE FOR $\sqrt{N+1}$: $\sqrt{N+1} \geq \frac{1+\sqrt{4001}}{2}$

$\frac{1+\sqrt{4001}}{2} \approx \frac{64.3}{2} = 32.15$ SO $N+1 \geq (32.15)^2$

$= 1033{,}6225$ SO $N+1 \geq 1034$ $N \geq 1033$

(ACTUALLY THE BEST SOLUTION IS 1032 AS WE ROUNDED UP)

ABSOLUTE AND CONDITIONAL CONVERGENCE

OUR INTUITIVE IDEA THAT

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots = S$$

WHEN THE SERIES CONVERGES SEEMS TO SUGGEST THAT CONVERGENT SERIES WOULD FOLLOW THE MAIN PROPERTIES OF USUAL ADDITION, IN PARTICULAR THAT THE ORDER OF SUMMANDS DOES NOT AFFECT THE END RESULT.

UNFORTUNATELY, THIS TURNS OUT TO BE <u>HORRIBLY WRONG</u> IN GENERAL.

<u>EXAMPLE</u>: WE KNOW THAT $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ CONVERGES $\left(\text{TO } \log(2)\right)$. WE CAN DIVIDE THE TERMS IN TWO TYPES:

- ODD TERMS $\frac{1}{n}$ WITH n ODD
- EVEN TERMS $\frac{-1}{n}$ WITH n EVEN

IN THE ORIGINAL SERIES THEY ARE ARRANGED AS

ODD + EVEN + ODD + EVEN + ODD + ...

LET'S ARRANGE THEM, KEEPING THE RELATIVE ORDER, AS

ODD + EVEN + EVEN + ODD + EVEN + EVEN + ... =

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots =$$

$$\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \dots =$$

$$\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{10} - \frac{1}{12}\right) + \dots =$$

$$\frac{1}{2}\left(1 - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{6}\right) + \dots =$$

$$\frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots\right) = \frac{\log(2)}{2} \;!!$$

WE JUST GOT <u>HALF</u> THE PREVIOUS VALUE!

IN FACT, WE CAN REARRANGE THE TERMS TO GET <u>ANY NUMBER</u> OR EVEN TO HAVE IT <u>DIVERGE</u> TO $\pm\infty$!!

SO WE CAN SEE THAT THIS KIND OF CONVERGENCE IS RATHER FLIMSY...

<u>BUT</u>! IF WE TRIED TO REARRANGE

FOR EXAMPLE $\sum_{n=1}^{\infty} \frac{1}{n^2}$ WE WOULD ALWAYS GET THE SAME NUMBER, AND THE SAME WOLD HAPPEN FOR $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$. SO WHICH ARE THE "BETTER" SERIES?

DEF: (ABSOLUTE/CONDITIONAL CONVERGENCE)

- A SERIES $\sum_{n=1}^{\infty} a_n$ IS SAID TO ABSOLUTELY CONVERGE IF $\sum_{n=1}^{\infty} |a_n|$ CONVERGES
- A SERIES $\sum_{n=1}^{\infty} a_n$ IS SAID TO CONDITIONALLY CONVERGE IF $\sum_{n=1}^{\infty} a_n$ CONVERGES BUT $\sum_{n=1}^{\infty} |a_n|$ DIVERGES.

THM: AN ABSOLUTELY CONVERGENT SERIES IS CONVERGENT, I.E. IF

$$\sum_{n=1}^{\infty} |a_n| \text{ CONVERGES THEN } \sum_{n=1}^{\infty} a_n \text{ CONVERGES}$$

FACT (HOPEFULLY MIND-BLOWING): REARRANGING AN ABSOLUTELY CONVERGENT SERIES WILL ALWAYS YIELD THE SAME VALUE. A COND. CONV. SERIES CAN BE REARRANGE TO YIELD ANY VALUE OR DIVERGE TO $+\infty$ OR $-\infty$

EXAMPLES

- $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ IS CONDITIONALLY CONVERGENT, AS $\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES.

- $\sum_{n=2}^{\infty} \frac{\sqrt{n}\cos(n)}{n^2-1}$ IS ABSOLUTELY CONVERGENT BY COMPARISON WITH $\sum_{n=2}^{\infty} n^{-\frac{3}{2}}$ AS

$$\left|\frac{\sqrt{n}\cos(n)}{n^2-1}\right| = \frac{\sqrt{n}}{n^2-1}|\cos(n)| \le \frac{\sqrt{n}}{n^2-1}\cdot 1 \le \frac{2\sqrt{n}}{n^2}$$

$$= 2n^{-\frac{3}{2}} \quad \left(n^2-1 \geq \frac{n^2}{2}\right)$$

- $\sum_{n=1}^{\infty} \frac{\sin(n)}{n}$ IS CONDITIONALLY CONVERGENT, BUT PROVING IT REQUIRES ADVANCED TOOLS

- $\sum_{n=1}^{\infty} (-1)^n \log\left(1+\frac{1}{n}\right)$ IS CONDITIONALLY CONVERGENT. IT CONV. BY ALT. TEST, BUT

$$\sum_{n=1}^{\infty} \left|(-1)^n \log\left(1+\frac{1}{n}\right)\right| = \sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right) =$$

$$\sum_{n=1}^{\infty} \left(\log(n+1)-\log(n)\right) \quad \text{DIVERGING TELESCOPIC.}$$

NOTE (IMPORTANT!):

OUR MAIN TOOLS, THAT IS THE INTEGRAL TEST, COMPARISON TEST AND LIMIT COMPARISON TEST ALL CHECK FOR <u>ABSOLUTE CONVERGENCE</u> AS THEY ALL REQUIRE POSITIVITY OF THIER COMPARISON TERMS.

THE RATIO TEST

THE FINAL TEST WE'LL USE IS A RATHER "ROUGH" TEST (IT ONLY SELECTS SERIES THAT HAVE A BEHAVOIR SIMLAR TO A GEOMETRIC SERIES) BUT IT HAS THE RELEVANT ADVANTAGE THAT WE WON'T HAVE TO CHOOSE A COMPARISON TERM. MOREOVER, IT WILL BE OUR PRIME TOOL WHEN DISCUSSING POWER SERIES.

<u>THM</u> (RATIO TEST):

LET a_n BE A SEQUENCE, AND SUPPOSE THAT $a_n \neq 0$ FOR ALL (LARGE ENOUGH) n. THEN:

- IF $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ THEN $\sum_{n=1}^{\infty} a_n$ <u>CONVERGES ABSOLUTELY</u>.
- IF $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ THEN $\sum_{n=1}^{\infty} a_n$ <u>DIVERGES</u>.

<u>WARNING</u>: THE RATIO TEST TELLS YOU <u>NOTHING</u> IF $L=1$.

<u>EXAMPLE</u>: $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{\sqrt{n}}$ (FUNCTION OF X!)

WE HAVE SEEN ALREADY THAT FOR $X=-1$ THE SERIES CONVERGES AND FOR $X=-3$ IT DIVERGES. HOW ABOUT ALL THE OTHER VALUES? LET'S USE THE RATIO TEST

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(x+2)^{n+1}}{\sqrt{n+1}}}{\frac{(-1)^n (x+2)^n}{\sqrt{n}}} \right| = \lim_{n\to\infty} \left| (x+2) \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \right|$$

$$= |x+2|$$

SO FOR $|x+2|<1$, THAT IS $-3<x<-1$ THE SERIES CONVERGES ABSOLUTELY, FOR $|x+2|>1$, THAT IS $x\leq -3$ OR $x>-1$ THE SERIES DIVERGES, AND FOR $x=-1$ IT CONV. CONDITIONALLY.

<u>EXAMPLE</u>: $\sum_{n=1}^{\infty} \frac{a^n}{n!}$

$$\lim_{n\to\infty} \left| \frac{\frac{a^{n+1}}{(n+1)!}}{\frac{a^n}{n!}} \right| = \lim_{n\to\infty} \frac{a}{n+1} = 0$$

SO THE SERIES CONVERGES ABSOLUTELY FOR <u>ANY</u> a.

EXAMPLE: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(n^2+1)(n!)^2}$

$$\lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(2n+2)!}{((n+1)^2+1)((n+1)!)^2}}{\dfrac{(-1)^n(2n)!}{(n^2+1)(n!)^2}} \right| = \lim_{n\to\infty} \overbrace{\frac{(2n+2)(2n+1)}{(n+1)^2}}^{\to 2} \cdot \overbrace{\frac{(n^2+1)}{((n+1)^2+1)}}^{\to 1}$$

$= 2$ SO THE SERIES DIVERGES

NOTE:

IDEA 1 - IF YOU SEE FACTORIALS, A RATIO TEST CAN PROBABLY HELP YOU

IDEA 2 - $\dfrac{(2n)!}{(n!)^2} = \dfrac{2n \cdot \ldots \cdot n+1 \cdot \cancel{n \cdot \ldots \cdot 1}}{n \cdot \ldots \cdot 1 \cdot \cancel{n \cdot \ldots \cdot 1}} =$

$$\frac{2n \cdot \ldots \cdot n+1}{n \cdot \ldots \cdot 1} = \underbrace{\frac{2n}{n} \cdot \frac{2n-1}{n-1} \cdot \ldots \cdot \frac{n+1}{1}}_{\text{ALL} \geq 2} \geq 2^n$$

EXAMPLE: $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n \cdot n^2}{5^n - n^3}$

$$\lim_{n\to\infty} \left| \frac{\dfrac{2^{n+1} + 3^{n+1} \cdot (n+1)^2}{5^{n+1} - (n+1)^3}}{\dfrac{2^n + 3^n \cdot n^2}{5^n - n^3}} \right| = \lim_{n\to\infty} \frac{2^{n+1} + 3^{n+1}(n+1)^2}{2^n + 3^n n^2} \cdot \frac{5^n - n^3}{5^{n+1} - (n+1)^3}$$

$$= \lim_{n\to\infty} 3 \underbrace{\left(\frac{\frac{2}{3}^{n+1} + (n+1)^2}{\frac{2}{3}^n + n^2} \right)}_{\to 1} \cdot \frac{1}{5} \underbrace{\left(\frac{1 - \frac{n^3}{5^n}}{1 - \frac{(n+1)^3}{5^{n+1}}} \right)}_{\to 1}$$

$= \dfrac{3}{5}$ SO THE SERIES CONVERGES.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\lim_{n\to\infty}\left|\frac{\frac{n+1!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\right| = \lim_{n\to\infty}\frac{(n+1)}{(n+1)^{n+1}}\cdot n^n = \lim_{n\to\infty}\frac{n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$

SO THE SERIES CONVERGES.

Ex: $\displaystyle\sum_{n=1}^{\infty} (-1)^n n\, e^{-n^2}$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)\,e^{-(n+1)^2}}{(-1)^n(n)\,e^{-n^2}}\right| = \lim_{n\to\infty}\frac{n+1}{n}\,e^{-(n+1)^2+n^2}$$

$$= \lim_{n\to\infty}\underbrace{\frac{n+1}{n}}_{\to 1}\,\underbrace{e^{-2n+1}}_{\to 0} = 0 \quad \text{SO THE } \Sigma \text{ CONVERGES.}$$