WARM-UP

USE THE APPROPRIATE TEST TO DECIDE CONVERGENCE OF THE FOLLOWING SERIES:

i) $\displaystyle\sum_{n=1}^{\infty} \frac{4n-8}{n^2-4n+20}$

ii) $\displaystyle\sum_{n=1}^{\infty} \arctan\left(\sin\left(\frac{\pi n}{4}\right)\right)$

iii) $\displaystyle\sum_{n=1}^{\infty} \frac{3n-\sin(n^3)}{5n+1}$

iv) $\displaystyle\sum_{n=1}^{\infty} e^{-n}(n^2-5n+6)$

SOL: i) FOR $x > 6$ $f(x) = \dfrac{4x-8}{x^2-4x+20}$ IS DECREASING AND POSITIVE $\left(f(x)' = \dfrac{-4(x^2-4x-12)}{(x^2-4x+20)^2}\right)$ SO COMPARE

$$\sum_{n=6}^{\infty} \frac{4n-8}{n^2-4n+20} \quad \text{WITH} \quad \int_6^{\infty} \frac{4x-8}{x^2-4x+20}\,dx = \ldots =$$

$u = x^2-4x+20$

$$\lim_{R\to\infty} 2\log(x^2-4x+20)\Big|_6^R = \infty$$

SO $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2-3n+12}$ DIVERGES BY THE INTEGRAL TEST.

iii) $\sin\left(\frac{\pi n}{4}\right)$ CYCLES BETWEEN

$\frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0,$

THUS $\arcsin\left(\sin\left(\frac{\pi n}{4}\right)\right)$ CYCLES BETWEEN $\arctan(b_1), \dots, \arctan(b_8)$ WHERE $b_1, \dots, b_8$ ARE THE NUMBERS ABOVE. BUT $\arctan(1) = \frac{\pi}{4}$, $\arctan(-1) = \frac{-1}{4}\pi$ SO THE SEQUENCE $\arctan\left(\sin\left(\frac{\pi}{4} n\right)\right)$ CANNOT HAVE A LIMIT. THEN $\sum_{n=1}^{\infty} \arctan\left(\sin\left(\frac{\pi}{4} n\right)\right)$ DIVERGES BY THE <u>DIVERGENCE TEST</u>.

iii) $$\lim_{n\to\infty} \frac{3n - \sin(n^3)}{5n+1} = \lim_{n\to\infty} \underbrace{\frac{3n}{5n+1}}_{\to \frac{3}{5}} - \underbrace{\frac{\sin(n^3)}{5n+1}}_{\text{SQUEEZE!}}$$

$$\underbrace{\frac{-1}{5n+1}}_{\to 0} \leq \cdot \leq \underbrace{\frac{1}{5n+1}}_{\to 0}$$

$= \frac{3}{5} + 0$ SO

$$\sum_{n=1}^{\infty} \frac{3n - \sin(n^3)}{5n+1}$$ DIVERGES

BY THE <u>DIVERGENCE TEST</u>.

iv) $\left(e^{-x}(x^2-5x+6)\right)' = -e^{-x}(x^2-8x+11)$

WHICH IS <0 FOR $x > 4+\sqrt{5}$

SO $e^{-x}(x^2-5x+6)$ IS DECREASING

FOR $x > 4+\sqrt{5}$. $4+\sqrt{5} < 7$ SO

WE COMPARE

$$\sum_{n=7}^{\infty} e^{-n}(n^2-5n+6) \quad \text{WITH} \quad \int_7^{\infty} e^{-x}(x^2-5x+6)\,dx =$$

$$\ldots = -e^{-x}(x^2-3x+3)\Big|_7^{\infty} = e^{-7}(31) \quad \text{, SO}$$

BY PARTS

$\sum_{n=1}^{\infty} e^{-n}(n^2-5n+6)$ CONVERGES BY THE

INTEGRAL TEST

THE COMPARISON TEST

WE WANT TO BEABLE TO CHECK CONVERGENCE WITHOUT GOING THROUGH AN INTEGRAL (WHICH WE MIGHT NOT BE ABLE TO EVALUATE). THIS IS SIMILAR TO WHAT WE WANTED FOR IMPROPER INTEGRALS, AND WE'LL USE THE SAME IDEA:

<u>THM</u>: (COMPARISON TEST) LET $K>0$, $c_n \geq 0$.

a) IF $|a_n| \leq K c_n$ FOR ALL (SUFFICIENTLY LARGE) n AND $\sum_{n=1}^{\infty} c_n$ CONVERGES, THEN $\sum_{n=1}^{\infty} a_n$ CONVERGES.

b) IF $a_n \geq K d_n \geq 0$ FOR ALL (SUFFICIENTLY LARGE) n AND $\sum_{n=1}^{\infty} d_n$ DIVERGES THEN $\sum_{n=1}^{\infty} a_n$ DIVERGES

NOTE: STATEMENT (a) WORKS FOR A SEQUENCE $\{a_n\}$ WHOSE SIGN ALTERNATES, BUT STATEMENT (b) <u>DOES NOT</u>.

EXAMPLE $\displaystyle\sum_{n=1}^{\infty} \frac{n^{0.1}}{n^{0.99}+n^{1.1}+1}$

IDEA: $\displaystyle\frac{n^{0.1}}{n^{0.99}+n^{1.1}+1} \simeq \frac{1}{n}$ SO IT SHOULD DIVERGE

USING THE TEST: WE COMPARE WITH $\frac{1}{n}$.

WE HAVE $n^{0.99}+n^{1.1}+1 \leq 3n^{1.1}$ SO

$$\frac{n^{0.1}}{n^{0.99}+n^{1.1}+1} \geq \frac{n^{0.1}}{3n^{1.1}} = \frac{1}{3n^{1.1-0.1}} = \frac{1}{3n}.$$

THEN BY THE COMPARISON TEST $\displaystyle\sum_{n=1}^{\infty} \frac{n^{0.1}}{n^{0.99}+n^{1.1}+1}$ DIVERGES AS $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ DOES.

EXAMPLE: $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin(n)}{n^2}$

IDEA: $\displaystyle\frac{1+\sin(n)}{n^2} \simeq \frac{1}{n^2}$ SO IT SHOULD CONVERGE

USING THE TEST: $1+\sin(n) \leq 2$ SO

$\displaystyle\left|\frac{1+\sin(n)}{n^2}\right| \leq \frac{2}{n^2}$. THEN $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin(n)}{n^2}$ CONVERGES AS $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ DOES.

EXAMPLE: $\sum_{n=1}^{\infty} \frac{n-1}{n^3+3n+2}$

IDEA: $\frac{n-1}{n^3+3n+2} \approx \frac{1}{n^2}$ SO IT SHOULD CONVERGE

USING THE TEST: WE WANT TO COMPARE WITH $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

WE HAVE $\left|\frac{n-1}{n^3+3n+2}\right| \leq \frac{n}{n^3+3n+2} \leq \frac{n}{n^3} = \frac{1}{n^2}$

SO BY THE COMP. TEST $\sum_{n=1}^{\infty} \frac{n-1}{n^3+3n+2}$ CONVERGES

AS $\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGES

EXAMPLE: $\sum_{n=1}^{\infty} \frac{n+5\sin(n)}{n^2+1}$ (* HARDER. DO IT LAST)

IDEA: $\frac{n+5\sin(n)}{n^2+1} \approx \frac{1}{n}$ SO IT SHOULD DIVERGE

USING THE TEST: WE COMPARE WITH $\frac{1}{n}$.

WE HAVE $\frac{n+5\sin(n)}{n^2+1} \geq 0$ FOR $n \geq 5$

AND $n+5\sin(n) \geq n-5 \geq \frac{n}{2}$ FOR $n \geq 10$ SO

WE START AT $n=10$.

NOW $n^2+1 \leq 2n^2$ SO FOR $n \geq 10$ WE HAVE

$$\frac{n+5\sin(n)}{n^2+1} \geq \frac{n}{2}\cdot\frac{1}{2n^2} = \frac{n}{4n^2} = \frac{1}{4}\cdot\frac{1}{n}$$

SO BY THE COMPARISON TEST WE CAN CONCLUDE THAT $\frac{n+5\sin(n)}{n^2+1}$ DIVERGES

<u>EXAMPLE</u>: $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n}$

IDEA: $\frac{\sqrt{n}}{n^2+n} \approx \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}}$ SO IT SHOULD CONVERGE

<u>USING THE TEST</u>: WE COMPARE WITH $\frac{1}{n^{\frac{3}{2}}}$.

WE HAVE $n^2+n \leq n^2$ SO $\left|\frac{\sqrt{n}}{n^2+n}\right| \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}}$

SO $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n}$ CONVERGES AS $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ DOES.

<u>EXAMPLE</u>: $\sum_{n=1}^{\infty} \frac{e^{-n}}{2n-5}$

IDEA: $\frac{e^{-n}}{2n-5} \approx \frac{e^{-n}}{n} \leq e^{-n}$ SO IT SHOULD CONVERGE

<u>USING THE TEST</u>: WE START AT $n=5$ SO OUR FUNCTION IS POSITIVE, AND $2n-5 \geq n$, THUS

$\left|\frac{e^{-n}}{2n-5}\right| \leq \frac{e^{-n}}{n} \leq e^{-n}$; SO BY COMPARISON

$\sum_{n=1}^{\infty} \frac{e^{-n}}{2n-5}$ CONVERGES AS $\sum_{n=1}^{\infty} e^{-n}$ DOES.

BUT WAIT, WE LIKE FINDING HOW SOMETHING GOES TO 0, BUT HATE INEQUALITIES! ISN'T THERE A BETTER TOOL?

THM: (LIMIT COMPARISON TEST)

LET $\{a_n\}$, $\{b_n\}$ BE TWO SEQUENCES WITH $b_n > 0$ FOR ALL (SUFFICIENTLY LARGE) n.

ASSUME THAT $\lim_{n \to \infty} \frac{a_n}{b_n} = L$ $(\neq \pm\infty)$.

THEN:

(a) IF $\sum_{n=1}^{\infty} b_n$ CONVERGES SO DOES $\sum_{n=1}^{\infty} a_n$.

(b) IF $L \neq 0$ AND $\sum_{n=1}^{\infty} b_n$ DIVERGES, SO DOES $\sum_{n=1}^{\infty} a_n$

IN PARTICULAR, IF $L \neq 0$ THEN $\sum_{n=1}^{\infty} a_n$ CONVERGES IF AND ONLY IF $\sum_{n=1}^{\infty} b_n$.

~~THIS~~ GREATLY SIMPLIFIES OUR COMPARISONS.

EXAMPLE: $\sum_{n=1}^{\infty} \frac{n + 5\sin(n)}{n^2+1}$

IDEA: COMPARE WITH $\frac{1}{n}$.

USING THE TEST:

$$\lim_{n\to\infty} \frac{\frac{n+5\sin(n)}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+5n\sin(n)}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2+1} + \frac{5n\sin(n)}{n^2+1} = 1+0 = 1$$

SQUEEZE

$$\frac{-5n}{n^2+1} \leq \cdot \leq \frac{5n}{n^2+1}$$

$$\downarrow \qquad\qquad \downarrow$$

$$0 \qquad\qquad 0$$

SO IT DIVERGES BY LIM. COMP WITH $\sum_{n=1}^{\infty} \frac{1}{n}$.

EXAMPLE: $\sum_{n=1}^{\infty} \frac{\log(n)^{50}}{n^2}$

IDEA: $\frac{\log(n)^{50}}{n^2} \leq \frac{1}{n^{2-d}}$ FOR ANY d, EVENTUALLY

USING THE TEST: PICK $d=\frac{1}{2}$ SO WE COMPARE WITH $\frac{1}{n^{\frac{3}{2}}}$. WE HAVE

$$\lim_{n\to\infty} \frac{\frac{\log(n)^{50}}{n^2}}{\frac{1}{n^{\frac{3}{2}}}} = \lim_{n\to\infty} \frac{\log(n)^{50}}{n^{\frac{1}{2}}} = 0$$

ANY POWER OF $\log(x)$ LOSES TO ANY POWER OF x