WARM UP

DISCUSS CONVERGENCE FOR THE FOLLOWING SERIES:

- $\sum_{n=1}^{\infty} \frac{5^n}{3^n+4^n}$ • $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{\sqrt{n^4+4}}$ • $\sum_{n=1}^{\infty} e^{-\left(\frac{n+1}{2n+1}\right)}$

- $a_n = \frac{5^n}{3^n+4^n} = \frac{1}{\underbrace{\frac{3^n}{5^n}+\frac{4^n}{5^n}}_{\to 0}}$ SO $a_n \to \infty$ AND

$\sum a_n$ DIVERGES.

- $a_n = \frac{(-1)^n n^2}{\sqrt{n^4+4}} = (-1)^n \frac{n^2}{\sqrt{n^4+4}} = (-1)^n \frac{1}{\underbrace{\sqrt{1+\frac{4}{n^4}}}_{\to 1}}$

$a_n \not\to 0$ SO $\sum a_n$ DIVERGES.

- $a_n = e^{-\left(\frac{n+1}{2n+1}\right)} = e^{-\frac{1}{2}\overbrace{\left(\frac{1+\frac{1}{n}}{1+\frac{1}{2n}}\right)}^{\to 1}}$ SO $a_n \to e^{-\frac{1}{2}}$

AND $\sum a_n$ DIVERGES.

~~$\sin(m)$ WILL BE $\geq \frac{\sqrt{2}}{2}$ INFINITELY MANY TIMES.~~

~~SO,~~ THE DIVERGENCE TEST WILL SHOW US THAT $\sum_{m=1}^{\infty} a_m$ DIVERGES SOMETIMES, BUT IT'S A VERY ROUGH TEST... THERE ARE PLENTY OF SERIES THAT PASS IT BUT DIVERGE!

EXAMPLE: $\sum_{m=1}^{\infty} \frac{1}{m}$

LET'S REGROUP THE TERMS LIKE THIS

$$\underbrace{1} + \underbrace{\frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}} + \underbrace{\cdots + \frac{1}{32}} + \underbrace{\cdots + \frac{1}{64}}$$

THE TERMS FROM $m = 2^i + 1$ TO $m = 2^{i+1}$ ARE ALL GREATER THAN THE LAST, $\frac{1}{2^{i+1}}$...

HOW MANY ARE THERE? 2^i! SO THEIR SUM IS $\geq \frac{1}{2}$! THEN

$$\sum_{m=1}^{\infty} \frac{1}{m} \geq \sum_{i=1}^{\infty} \frac{1}{2} = \infty!$$

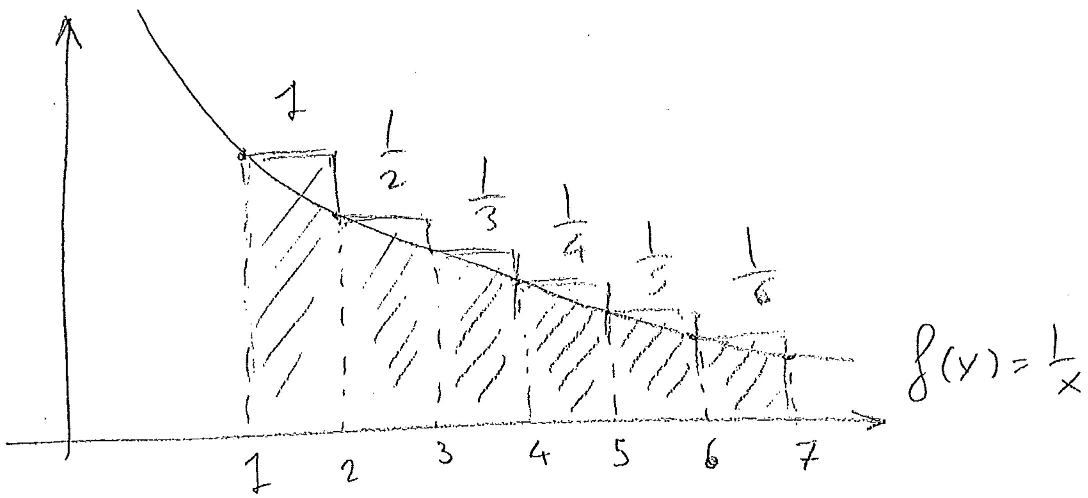
MORE PRECISELY:

$S_{2^N} \geq \frac{N+1}{2}$ SO $\lim_{N \to \infty} S_N$ DIVERGES.

THE INTEGRAL TEST

IS THERE A BETTER WAY TO SEE THAT $\sum_{m=1}^{\infty} \frac{1}{m}$ DIVERGES?

IDEA:

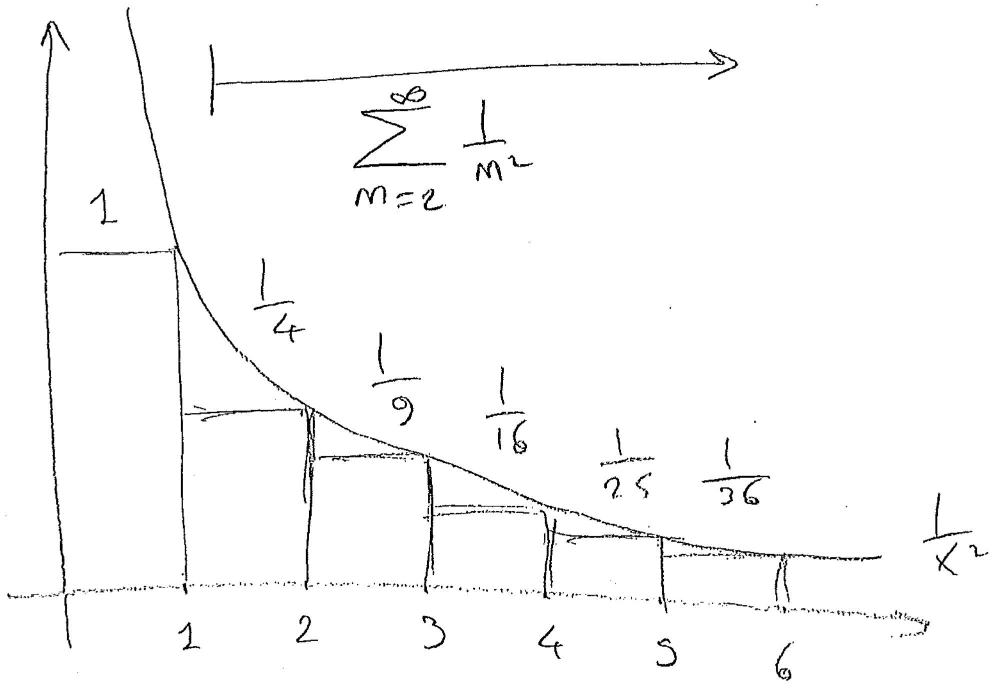


$$S_N \geq \int_1^N \frac{1}{x}\,dx = \log(N) \longrightarrow \infty \quad !!$$

WOW, THAT WAS EASY! BUT WHAT ABOUT SOMETHING THAT SHOULD CONVERGE, SUCH AS $\sum_{m=1}^{\infty} \frac{1}{m^2}$?

SMARTY-PANTS IDEA: COMPARE WITH TELESCOPING SERIES $\frac{1}{m(m+1)}$

MORE GENERAL IDEA: COMPARE WITH $\int_1^{\infty} \frac{1}{x^2}\,dx$



WE DON'T REALLY WANT TO LOOK AT

$\int_0^\infty \frac{1}{x^2}\,dx\ldots$ BUT WHAT IF WE REMOVE

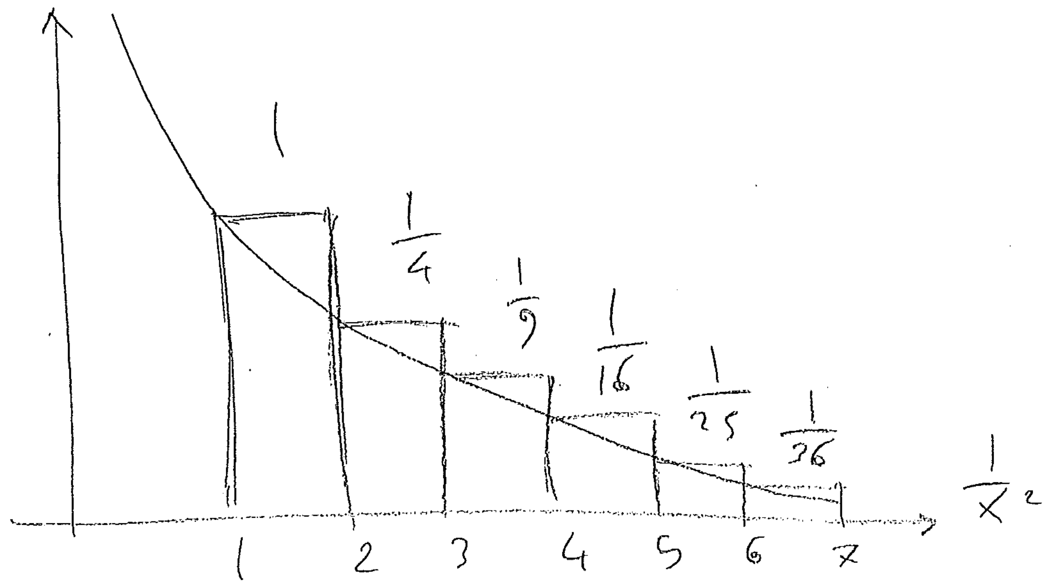
THE FIRST TERM? WE GET

$$S_N - 1 \le \int_1^N \frac{1}{x^2}\,dx \le \frac{1}{3} \quad !! \quad \text{SO } S_N \le \frac{3}{2}$$

AS $\frac{1}{m^2}$ IS ALWAYS POSITIVE WE CAN CONCLUDE THAT S_N MUST CONVERGE TO SOME $A \le \frac{3}{2}$

<u>NOTE</u>: AN <u>INCREASING</u> SEQUENCE CAN EITHER <u>CONVERGE</u> OR <u>DIVERGE AT ∞</u>

NOW NOTE ONE MORE THING:



IF WE PICK A LEFT RULE SUM WE GET THAT

$$S_N \geqslant \int_1^N \frac{1}{x^2}\, dx \;!$$

SO KNOWING THAT S_N CONVERGES TELLS US THAT $\int_1^\infty \frac{1}{x^2}\, dx$ CONVERGES! WE HAVE LAID DOWN ALL THE TOOLS FOR THE THEOREM:

THM: LET $f(x)$ A POSITIVE AND DECREASING FUNCTION, DEFINED ON $[N_0, \infty)$. THEN:

$$\int_{N_0}^\infty f(x)\, dx \text{ CONVERGES} \iff \sum_{n=N_0}^\infty f(n) \text{ CONVERGES}.$$

EXAMPLE $\sum_{n=1}^{\infty} \frac{1}{n^p}$

INTEGRAL TEST WITH $f(x) = \frac{1}{x^p}$

$$\int_1^{\infty} \frac{1}{x^p}\,dx \text{ CONVERGES } \iff p > 1$$

SO $\sum_{n=1}^{\infty} \frac{1}{n^p}$ CONVERGES $\iff p > 1$

EXAMPLE $\sum_{n=2}^{\infty} \frac{1}{n\log(n)^p}$

FOR $x \geq 2$ $f(x) = \frac{1}{x\log(x)^p}$ IS

POSITIVE AND DECREASING, SO

WE CAN COMPARE WITH

$$\int_2^{\infty} \frac{1}{x\log(x)^p}\,dx \underset{u=\log x}{=} \int_{\log 2}^{\infty} \frac{1}{u^p}\,du \text{ WHICH}$$

CONVERGES $\iff p > 1$. SO

$$\sum_{n=2}^{\infty} \frac{1}{n\log(n)^p} \text{ CONV } \iff p > 1.$$

EXAMPLE : $\sum_{m=1}^{\infty} \frac{\log(m)+10}{m^{\frac{3}{2}}}$

FIRST WE HAVE TO CHECK WETHER

$\left(\log(x)+10\right)x^{-\frac{3}{2}}$ IS DECREASING FOR $x > 1$

$$\left(\log(x)\cdot x^{-\frac{3}{2}}+10x^{-\frac{3}{2}}\right)' = x^{-\frac{5}{2}} - \frac{3}{2}x^{-\frac{5}{2}}\log(x) - 15x^{-\frac{5}{2}}$$

$$= x^{-\frac{5}{2}}\left(-\log(x)-14\right) < 0 \text{ FOR } x \geq 1$$

$\left(\log(x)+10\right)x^{-\frac{3}{2}}$ IS CLEARLY POSITIVE, SO WE CAN USE THE INTEGRAL TEST.

$$\int (\log(x)+10)x^{-\frac{3}{2}}dx = -20x^{-\frac{1}{2}} + \int \log(x)x^{-\frac{3}{2}}dx =$$

$$\underset{\substack{\uparrow\\ \text{BY PARTS}}}{=} -20x^{-\frac{1}{2}} - 2\log(x)x^{-\frac{1}{2}} + \int -\frac{2x^{-\frac{1}{2}}}{x}dx =$$

$$= -20x^{-\frac{1}{2}} - 2(\log(x)+2)x^{-\frac{1}{2}} = -2x^{-\frac{1}{2}}(12+2\log(x))$$

SO $$\int_1^{\infty} (\log x+10)x^{-\frac{3}{2}}dx = 24 + \lim_{R\to\infty} -2R^{-\frac{1}{2}}(12+2\log(R))$$

$= 24$ THEN

BY THE INTEGRAL TEST $\sum_{m=1}^{\infty} \frac{\log(m)+10}{m^{\frac{3}{2}}}$

CONVERGES.

EXAMPLE $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)\log(\log(n+1))}$

WE'D LIKE TO USE THE INTEGRAL TEST,

BUT $f(x) = \dfrac{1}{(x+1)\log(x+1)\log(\log(x+1))}$ IS NEGATIVE

AT $x=1$ ...

IDEA: THE INTEGRAL TEST WORKS FINE EVEN IF WE START FARTHER! WE CAN COMPARE

$$\sum_{n=2}^{\infty} f(n) \quad \text{WITH} \quad \int_2^{\infty} f(x)\,dx, \quad \text{AND}$$

FOR $x>2$ WE HAVE $x+1>3$ AND THE FUNCTIONS $(x+1)$, $\log(x+1)$, $\log(\log(x+1))$ ARE ALL POSITIVE ($\log(\log(3))>\log(1)=0$) AND INCREASING SO

$$\frac{1}{(x+1)\log(x+1)\log(\log(x+1))}$$

IS POSITIVE AND DECREASING.

NOW

$$\int_2^{\infty} \frac{1}{(x+1)\log(x+1)\log(\log(x+1))}\,dx = \int_3^{\infty} \frac{1}{x\log(x)\log(\log(x))}\,dx$$

$$\underset{u=\log(x)}{=} \int_{\log 3}^{\infty} \frac{1}{u\log u}\,du \underset{y=\log u}{=} \int_{\log(\log 3)}^{\infty} \frac{1}{y}\,dy \quad \text{WHICH}$$

DIVERGES. SO OUR SERIES DIVERGES.