ONE SOLUTION. SAY WE KNOW $y(0)$. THEN

$$y(0) = ce^0 + b \sim c = b - y(0)$$

SO THE SOLUTION IS

$$\underline{y(x) = (b - y(0))e^{ax} + b}$$

<u>NOTE</u>: THIS IS A <u>UNIQUE</u> SOLUTION, I.E. ANY $y(x)$ SATISFYING THE EQUATION WITH A GIVEN $y(0)$ IS IN THIS FORM.

<u>EXAMPLE</u>: $\dfrac{dy}{dx} = (y^2+1)e^x \qquad y(0) = 0$

$$\sim \ldots \sim \int \frac{1}{y^2+1}\, dx = \int e^x\, dx \sim$$

$$\operatorname{arcTan}(y) = e^x + C \underset{\substack{\uparrow \\ \text{APPLY} \\ \text{Tan}(-)}}{\sim} y = \operatorname{Tan}(e^x + C)$$

NOW $\quad y(0) = 0 = \operatorname{Tan}(1 + C)$ SO $C = -1$

VERIFY: $\operatorname{Tan}(e^x - 1)' \underset{\substack{\uparrow \\ \text{CHAIN RULE}}}{=} e^x \cdot \sec^2(e^x - 1) = e^x(y^2+1)$ ✓

EXAMPLE:

FIND SOL FOR $\frac{dy}{dx} = \frac{y}{x^2+1}$, $y(1)=1$

$\frac{y'}{y} = \frac{1}{x^2+1}$ $\quad \int \frac{y'}{y}\,dx = \int \frac{1}{x^2+1}\,dx$

$\int \frac{1}{y}\,dy \Big|_{y=y(x)} = \arctan x + C$

$\log y = \arctan x + C$

$y = e^{\arctan x + C}$

NOW, $y(1) = e^{\arctan 1 + C} = 1$ $\quad \arctan 1 = \frac{\pi}{4}$

SO $C = -\frac{\pi}{4}$, $y = e^{-\frac{\pi}{4} + \arctan x}$ (VERIFY!)

EXAMPLE: $\frac{dy}{dx} = -x(y-1)^2$, $y(1) = 2$.

$\frac{y'}{(y-1)^2} \overset{*}{=} -x$ $\quad \int \frac{1}{(y-1)^2}\,dy = \int -x\,dx$

$*$ $y=1$ SOLVES GEN EQ. BUT NOT $y(1)=2$

$-\frac{1}{y-1} = -\frac{x^2}{2} + C$ $\quad$ FIND C: $-\frac{1}{y(1)-1} = -\frac{1}{2} + C$

$-\frac{1}{1} = -\frac{1}{2} + C$, $C = -\frac{1}{2}$. $\quad -\frac{1}{y-1} = -\frac{x^2}{2} - \frac{1}{2}$, $y = 1 + \frac{2}{x^2+1}$

EXAMPLE: $\dfrac{dy}{dx} = -x e^{y}, \quad y(0) = 0$

$$y' e^{-y} = -x \qquad \int e^{-y}\,dy \Big|_{y=y(x)} = \int -x\,dx$$

$$-e^{-y(x)} = -\frac{x^2}{2} + C \qquad \text{AT } x = 0$$

$$-e^{0} = C \qquad C = -1$$

$$-e^{-y} = -\frac{x^2}{2} - 1 \qquad e^{-y} = \frac{x^2}{2} + 1 \qquad \text{(VERIFY!)}$$

$$\log e^{-y} = \log\left(\frac{x^2}{2} + 1\right) \qquad y = -\log\left(\frac{x^2}{2} + 1\right)$$

EXAMPLE: $\dfrac{dy}{dx} = \dfrac{x e^{x^2}}{y} \qquad y(0) = -1$

$$y' \cdot y = x e^{x^2}$$

$$\frac{y^2}{2} = \frac{e^{x^2}}{2} + C \qquad y^2 = e^{x^2} + C \quad \Bigg| \begin{array}{l} \text{AT } 0 \\ +1 = 1 + C \\ C = 0 \end{array}$$

$$y = -\sqrt{e^{x^2}} = -e^{\frac{x^2}{2}}$$

BECAUSE $y(0) < 0$! (VERIFY!)

VARIANT:

$$\frac{dy}{dx} = \frac{xe^{x^2}}{y+1} \qquad y(0)=1$$

NOTE: $y(0)=-1$ IS INCOMPATIBLE WITH EQUATION!!

$$y' = \frac{xe^{x^2}}{y+1} \sim y'(y+1) = xe^{x^2}$$

$$\sim \frac{y^2}{2} + y = \frac{e^{x^2}}{2} + c \sim y^2 + 2y = e^{x^2} + c$$

AT $x=0$: $\quad 1 + 2 = 1 + c \qquad c = 2$

$$y^2 + 2y - e^{x^2} - 2 = 0$$

$$y = \frac{-2 \pm \sqrt{4 + 4e^{x^2} + 8}}{2} = -1 \pm \frac{\sqrt{12 + 4e^{x^2}}}{2}$$

PICK SOL THAT VERIFIES

$y(0)=1$

$$-1 + \frac{\sqrt{12+4\cdot 1}}{2} = -1 + \frac{\sqrt{16}}{2} = 1 \quad \checkmark$$

$$-1 - \frac{\sqrt{12+4\cdot 1}}{2} = -3 \quad \times$$

So $y = -1 + \dfrac{\sqrt{12+4e^{x^2}}}{2}$

EXAMPLE: $\dfrac{dy}{dx} - y\log(x) = \log\left(\dfrac{1}{x^2}\right)$

FIRST WE GET IT BACK TO STANDARD S.D.E. FORM

$$y' - y\log(x) = \log\left(\frac{1}{x^2}\right) \sim$$

$$y' = +y\log(x) + \log\left(\frac{1}{x^2}\right) \sim$$

$$y' = +y\log(x) - 2\log(x) \sim y' = \log(x)(y-2).$$

$$\sim \ldots \sim \quad \underset{*\, y \neq 2}{} \quad \int \frac{1}{y-2}\,dy = \int \log(x)\,dx$$

$$\log|y-2| = x\log x - x + C \sim$$

$$|y-2| = e^{x\log x - x + C} \sim y-2 = \overset{\pm e^{C}}{C}\, e^{x\log x - x}$$

LET'S FIND THE SOLUTION BASED ON $y(1)$ (IT HAS SOME PROBLEMS AT $x=0$)

$$y(1) = Ce^{-1} + 2 \sim C = e(y(1) - 2)$$

SEQUENCES

A SEQUENCE IS AN ORDERED INFINITE LIST OF NUMBERS. IT IS DENOTED

$$\{a_1, a_2, \dots, a_n, \dots\} \quad \text{OR} \quad \{a_n\} \quad \text{OR} \quad \{a_n\}_{n=1}^{\infty}$$

(NOTE: IT MAY ALSO START FROM A DIFFERENT NUMBER).

WHEN THERE IS AN EXPLICIT FUNCTION DESCRIBING THE SEQUENCE, WE'LL WRITE

$$\{a_n = f(n)\}_{n=1}^{\infty} \quad \text{OR SIMPLY} \quad \{f(n)\}_{n=1}^{\infty}$$

EXAMPLE:

- $\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$ OR $\{a_n = \frac{1}{n}\}_{n=1}^{\infty}$ OR $\{\frac{1}{n}\}_{n=1}^{\infty}$
- $\{1, -1, 1, \dots, (-1)^{n-1}, \dots\}$ OR $\{a_n = (-1)^{n-1}\}_{n=1}^{\infty}$ OR $\{(-1)^{n-1}\}_{n=1}^{\infty}$

NOTE THAT IN GENERAL THERE MAY BE NO REASONABLE FUNCTION TO DESCRIBE A SEQUENCE;

EXAMPLE:

$\{a_n\} = \{3, 1, 4, 1, 5, 9, 2, 6, 5, \dots\}$

$\{a_n = n\text{-th DECIMAL DIGIT OF } \pi\}$

A SEQUENCE $\{a_n\}$ <u>CONVERGES</u> TO A NUMBER A IF $\lim_{n\to\infty} a_n = A$

(FORMALLY: FOR ALL $d>0$ $|a_n - A| < d$ FOR ALL n SUFF. LARGE)

IT IS SAID TO <u>DIVERGE</u> IF $\lim_{n\to\infty} a_n$ DOES NOT EXIST OR IS $\pm\infty$.

EXAMPLE:

- $\{a_n = \frac{1}{n}\}$ CONVERGES TO $A=0$ AS

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

- $\{a_n = (-1)^{n-1}\}$ DIVERGES AS

$$\lim_{n\to\infty} (-1)^{n-1} \text{ DOES NOT EXIST}$$

- $\{a_n = n\}$ DIVERGES TO ∞ AS

$$\lim_{n\to\infty} n = \infty$$

- HOW ABOUT $\left\{\frac{n+3}{2n+5}\right\}_{n=1}^{\infty}$? WELL,

$$\frac{n+3}{2n+5} = \frac{n(1+\frac{3}{n})}{n\cdot(2+\frac{5}{n})} = \frac{1+\frac{3}{n}}{2+\frac{5}{n}} \quad \text{SO}$$

$$\lim_{n\to\infty} \frac{n+3}{2n+5} = \lim_{n\to\infty} \frac{1+\frac{3}{n}}{2+\frac{5}{n}} = \frac{1}{2}$$

THIS LOOKS LIKE WHAT WE WOULD DO FOR A REGULAR LIMIT... AND IN FACT:

THM: IF $\lim_{x\to\infty} f(x) = A$, THEN $\{a_n = f(n)\}$ CONVERGES TO A.

EXAMPLE: $\{a_n = e^{-n}\}$ THEN $\lim_{n\to\infty} a_n = \lim_{x\to\infty} e^{-x} = 0$

NOTE THAT THIS <u>DOES NOT</u> GO TWO WAYS!

EXAMPLE:

$f(x) = \sin(\pi x)$ $\quad \lim_{x\to\infty} \sin(\pi x)$ DNE AS $f(x)$ KEEPS OSCILLATING BETWEEN 1 AND -1.

BUT! $\{a_n = f(n)\}_{n=1}^{\infty} = \{a_n = \sin(\pi n)\}_{n=1}^{\infty}$ <u>DOES</u> CONVERGE AS $\sin(\pi n) = 0$ FOR ALL n!

LIMITS OF SEQUENCES BEHAVE AS WE GENERALLY EXPECT IN TERMS OF THE USUAL OPERATIONS: