DEF:

THE N^{TH} PARTIAL SUM OF THE SERIES

$\sum_{i=1}^{\infty} a_m$ IS $S_N = \sum_{n=1}^{N} a_m = a_1 + \dots + a_N$.

THE PARTIAL SUMS FORM A SEQUENCE

$\{S_N\}_{N=1}^{\infty}$. IF S_N CONVERGES TO A

NUMBER S WE SAY $\underline{\sum_{m=1}^{\infty} a_m = S}$.

IF S_N DIVERGES WE SAY $\sum_{m=1}^{\infty} a_m$ DIVERGES.

EXAMPLE: (GEOMETRIC SERIES)

CONSIDER THE SUM

$$a + ar + ar^2 + ar^3 + ar^4 + \dots + ar^m + \dots = \sum_{m=0}^{\infty} ar^m$$

$\left(\text{OR } \sum_{m=1}^{\infty} ar^{m-1}\right)$ $\left(\text{EXAMPLE IN THE EXAMPLE: } \sum_{m=0}^{\infty} \frac{3}{10^m}\right)$

WE CAN WRITE DOWN THE PARTIAL SUM S_N

NICELY IF $r \neq 1$

$$S_N = a + ar + \dots + ar^N = \frac{a(r^N - 1)}{r - 1} = \frac{a(1 - r^N)}{1 - r^N}$$

NOW NOTE: IF $|r| \geq 1$ THE SERIES

CLEARLY DIVERGES

SO WE ARE LEFT WITH $|r|<1$. WE HAVE

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} a \cdot \frac{1-r^{\overset{\to 0}{N}}}{1-r} = \frac{a}{1-r}$$

EXAMPLE: (RECURRING DECIMAL)

CONSIDER A RECURRING DECIMAL, SUCH AS

$0.\overline{3542} = 0.354235423542\ldots$

WE CAN WRITE IT AS

$$0.\overline{3542} = \sum_{m=1}^{\infty} \frac{3542}{10{,}000^m} = \frac{3542}{10{,}000} + \frac{3542}{10{,}000^2} + \frac{3542}{10{,}000^3} + \ldots$$

NOW WE KNOW THAT

$$\sum_{m=0}^{\infty} 3542 \cdot \left(\frac{1}{10^4}\right)^m = 3542 \cdot \frac{1}{1-\frac{1}{10^4}} = 3542 \cdot \frac{10^4}{10^4-1}$$

SO $0.\overline{3542} = \displaystyle\sum_{m=1}^{\infty} \frac{3542}{10^{4m}} = 3542 \cdot \frac{10^4}{10^4-1} \underset{\text{FIRST TERM } m=0}{-3542}$

$= \dfrac{3542}{10^4-1} = \dfrac{3542}{9999}$ A RATIONAL NUMBER!

NOTE: $\displaystyle\sum_{m=1}^{\infty} \frac{a}{r^m} = \frac{a}{1-r} - a = a\left(-1+\frac{1}{1-r}\right) = a\left(\frac{r}{1-r}\right)$

IT'S OFTEN IMPOSSIBLE TO FIND S_N EXACTLY.

BEFORE GOING IN TO THE GENERAL CASE, LET'S LOOK AT ONE MORE

(RATHER ARTIFICIAL) "EASY" CASE

EXAMPLE (TELESCOPING SERIES)

$\sum_{m=1}^{\infty} \frac{1}{m(m+1)}$. WE CAN REWRITE

$\frac{1}{m(m+1)}$ AS $\frac{1}{m} - \frac{1}{m+1}$. THIS SHOWS

THAT

$$S_N = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{N(N+1)} =$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{N-1} - \frac{1}{N}\right) + \left(\frac{1}{N} - \frac{1}{N+1}\right)$$

ALL TERMS CANCEL OUT EXCEPT

1 AND $-\frac{1}{N+1}$ SO

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} 1 - \frac{1}{N+1} = 1 \quad \text{SO} \quad \sum_{m=1}^{\infty} \frac{1}{m(m+1)} = 1$$

EXAMPLE: (DIVERGENT TELESCOPING)

$$\sum_{m=1}^{\infty} \log\left(1+\frac{1}{m}\right) = \log(2) + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \dots$$

$$S_N = \log(2) + \log\left(\frac{3}{2}\right) + \dots + \log\left(\frac{N+1}{N}\right)$$

NOW $\log(n) + \log\left(\frac{n+1}{n}\right) =$

$\log\left(\frac{n+1}{n} \cdot n\right) = \log(n+1)$, SO APPLYING

TO $S_N = \sum_{n=1}^{N} \log\left(\frac{n+1}{n}\right)$ WE GET

$S_N = \log(N+1)$ WHICH DIVERGES!

SOME (OBVIOUS) ARITHMETIC RULES

<u>THM</u>: SAY $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B$, THEN

$$\sum_{n=1}^{\infty} a_n + b_n = A + B \qquad \sum_{n=1}^{\infty} a_n - b_n = A - B$$

$$\sum_{n=1}^{\infty} C a_n = CA$$

<u>EXAMPLE</u> $\sum_{n=1}^{\infty} \frac{1}{3^n} - \frac{5}{n(n+1)}$

WE KOW THAT $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{\frac{2}{3}} - 1 = \frac{1}{2}$

AND THAT $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ SO $\sum_{n=1}^{\infty} \frac{5}{n(n+1)} = 5 \cdot 1$

THEN $\sum_{n=1}^{\infty} \frac{1}{3^n} - \frac{5}{n(n+1)} = \frac{1}{2} - 5 = -\frac{9}{2}$

EXAMPLE: $\sum_{m=1}^{\infty} \frac{4^m+5^m}{9^m}$

$\sum_{m=1}^{\infty} \frac{4^m}{9^m}$ AND $\sum_{m=1}^{\infty} \frac{5^m}{9^m}$ BOTH CONVERGE, SO

$$\sum_{m=1}^{\infty} \frac{4^m+5^m}{9^m} = \sum_{m=1}^{\infty} \frac{4^m}{9^m} + \sum_{m=1}^{\infty} \frac{5^m}{9^m} = \frac{4}{9} \cdot \frac{1}{1-\frac{4}{9}} + \frac{5}{9} \frac{1}{1-\frac{5}{9}}$$

$$= \frac{4}{9} \cdot \frac{9}{5} + \frac{5}{9} \cdot \frac{9}{4} = \frac{4}{5} + \frac{5}{4} = \frac{41}{20}$$

EXAMPLE: $\sum_{m=1}^{\infty} \frac{(-1)^m}{2^{2m}}$

$\frac{(-1)^m}{2^{2m}} = \frac{(-1)^m}{4^m}$ GEOM SERIES WITH $a=1$, $r=-\frac{1}{4}$

SO $\sum_{m=1}^{\infty} \frac{(-1)^m}{2^{2m}} = -\frac{1}{4} \cdot \frac{1}{1+\frac{1}{4}} = -\frac{1}{4} \cdot \frac{4}{5} = -\frac{1}{5}$

EXAMPLE $\sum_{m=3}^{\infty} \left(\cos\frac{\pi}{m} - \cos\frac{\pi}{m+1} \right)$

TELESCOPING: $S_N = \left(\cos\frac{\pi}{3} - \cos\frac{\pi}{4}\right) + \left(\cos\frac{\pi}{4} - \cos\frac{\pi}{5}\right) +$

$\ldots + \left(\cos\frac{\pi}{N} - \cos\frac{\pi}{N+1}\right) = \cos\frac{\pi}{3} - \cos\frac{\pi}{N+1}$

SO $\lim_{N\to\infty} S_N = \cos\frac{\pi}{3} - \lim_{N\to\infty} \cos\frac{\pi}{N+1} =$

$\cos\frac{\pi}{3} - \cos 0 = -\frac{1}{2}$

THE DIVERGENCE TEST

SUPPOSE OUR SEQUENCE a_m DOES NOT CONVERGE TO 0. THIS IS LIKE SAYING THAT GIVEN SOME SMALL NUMBER d WE WILL ALWAYS FIND SOME a_m WITH $|a_m| > d$, NO MATTER HOW FAR WE GO.

NOW CONSIDER THE SERIES $\sum_{m=1}^{\infty} a_m$.

TO SAY IT CONVERGES TO A IS EQUIVALENT TO SAYING THAT FOR ANY SMALL NUMBER c EVENTUALLY WE'LL HAVE $|S_N - A| < c$ PERMANENTLY.

BUT! TAKE $c = \frac{d}{2}$, AND A N SUCH THAT $|a_{N+1}| > d$. THEN $|\underbrace{S_N - S_{N+1}}_{= a_{N+1}}| > d$ AND IT CANNOT HAPPEN THAT BOTH $|S_N - A| < c$ AND $|S_{N+1} - A| < c$! WE MOVED TOO FAR! THIS SHOWS:

THM: FOR $\sum_{m=1}^{\infty} a_m$ TO CONVERGE IT IS NECESSARY THAT $a_m \to 0$.

CONSEQUENTLY, IF $\lim_{n\to\infty} a_n \neq 0$ THEN $\sum_{n=1}^{\infty} a_n$ DIVERGES. (DIVERGENCE TEST)

<u>EXAMPLE</u>: $a_n = \frac{n}{n+1}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} 1 - \frac{1}{n} = 1$ SO

$\sum_{n=1}^{\infty} a_n$ DIVERGES BY THE DIVERGENCE TEST.

<u>WARNING</u>: THE DIVERGENCE TEST ONLY TELLS US SOMETHING IF $a_n \not\to 0$. IT IS FULLY POSSIBLE THAT $a_n \to 0$ AND $\sum_{n=1}^{\infty} a_n$ STILL DIVERGES, AS IN

$\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right)$. $\log\left(1+\frac{1}{n}\right) \to 0$ BUT AS WE SAW THE SERIES DIVERGES.

EXAMPLE: $\sum_{n=1}^{\infty} \sin(n)$

THIS SERIES DOES NOT CONVERGE AS $\{a_n = \sin(n)\}$ DOES NOT CONVERGE TO 0, AN EASY WAY TO SEE IT IS THAT AS n WILL FALL ON $\left[\frac{1}{4}\pi + 2k\pi, \frac{3}{4}\pi + 2k\pi\right]$ INFINITELY MANY TIMES,

SIN(N) WILL BE $\geq \frac{\sqrt{2}}{2}$ INFINITELY MANY TIMES.

SO, THE DIVERGENCE TEST WILL SHOW US THAT

$\sum_{n=1}^{\infty} a_n$ DIVERGES SOMETIMES, BUT IT'S A VERY ROUGH TEST... THERE ARE PLENTY OF SERIES THAT PASS IT BUT DIVERGE!

<u>EXAMPLE:</u> $\sum_{n=1}^{\infty} \frac{1}{n}$

LET'S REGROUP THE TERMS LIKE THIS

$$\underbrace{1}+\underbrace{\frac{1}{2}}+\underbrace{\frac{1}{3}+\frac{1}{4}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}+\underbrace{\frac{1}{9}+\cdots+\frac{1}{16}}$$
$$\underbrace{+\cdots+\frac{1}{32}}+\underbrace{\cdots+\frac{1}{64}}$$

THE TERMS FROM $n=2^i+1$ TO $n=2^{i+1}$ ARE ALL GREATER THAN THE LAST, $\frac{1}{2^{i+1}}$...

HOW MANY ARE THERE? 2^i! SO THEIR SUM IS $\geq \frac{1}{2}$! THEN

$\sum_{n=1}^{\infty} \frac{1}{n} \geq \sum_{i=1}^{\infty} \frac{1}{2} = \infty$! MORE PRECISELY:

$$S_{2^N} \geq \frac{N+1}{2} \quad \text{SO} \quad \lim_{N\to\infty} S_N \quad \text{DIVERGES.}$$