MIN-MAX REPRESENTATIONS OF VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS

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Abstract. In this paper a duality relation between the Mañé potential and the action functional is derived in the context of convex and state-dependent Hamiltonians. The duality relation is used to obtain min-max representations of viscosity solutions of first order evolutionary Hamilton-Jacobi equations. As a special case, for state-independent Hamiltonians the duality result provides a new way to derive the classical Hopf-Lax-Oleinik representation.

1. Introduction

In this paper we study viscosity solutions of Hamilton-Jacobi equations associated with Lagrangian dynamics arising in the theory of large deviations for stochastic processes. Suppose that the rate function associated with the large deviations of a sequence of stochastic processes $\{X^n(t); t \in [0,T]\}$ is of the form

$$\int_t^T \bar{L}(\psi(s), \dot{\psi}(s))ds,$$

where ψ is an absolutely continuous function and $\bar{L}$ is the local rate function, such that $v \mapsto \bar{L}(x,v)$ is convex for all $x \in \mathbb{R}^n$. This is typical, for example, for Markov processes [8, 17, 22]. To illustrate the connection between large deviations and Hamilton-Jacobi equations, consider the probability that the process X^n has exited an open set $\Omega \subset \mathbb{R}^d$ before time T, conditioned on $X^t = x$ for for $0 \leq t < T$, $x \in \Omega$: $P_{t,x}(X^n(T) \notin \Omega)$. If the rate function associated with X^n is of the aforementioned form, then the large deviations rate of this probability is given by

$$\bar{U}(t,x) = \inf_{\psi} \Big\{ \int_t^T \bar{L}(\psi(s), \dot{\psi}(s))ds, \psi(t) = x, \psi(T) \notin \Omega \Big\},$$

where the infimum is taken over all absolutely continuous functions. That is, for large n,

$$P_{t,x}(X^n(T) \notin \Omega) \approx e^{-n\bar{U}(t,x)}.$$

Since $\bar{U}$ is the value function of a variational problem, it satisfies, in the sense of a viscosity solution, a Hamilton-Jacobi terminal value problem of the form

$$\begin{cases} \bar{U}_t(t,x) - \bar{H}(x, -D\bar{U}(t,x)) = 0, & (t,x) \in [0,T) \times \Omega, \\ \bar{U}(T,x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

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where $\bar{H}$ is the Fenchel-Legendre transform of $\bar{L}$, see e.g. [17].

Furthermore, because of the connection between probabilities that become exponentially small asymptotically, in the above example as $n \to \infty$, and Hamilton-Jacobi equations, subsolutions of such equations are intrinsically linked to efficient Monte Carlo methods for so-called rare-event simulation; see e.g. [9, 10, 11]. Therefore, in addition to its relevance for Lagrangian dynamics in general and problems arising in, for example, partial differential equations, calculus of variations and control theory, the results of this paper lay the foundation for a systematic approach to solving the challenging task of finding subsolutions that define provably efficient rare-event methods, a topic pursued in forthcoming work.

In this paper we consider Lagrangians $(x, v) \mapsto L(x, v)$ that are convex in v. The main results, Proposition 3.2 and Theorem 4.1, prove a certain form of viscosity solutions to evolutionary Hamilton-Jacobi equations and a duality between Mañé's potential and the action functional, respectively; we now give a brief outline.

The Mañé potential at level c is given by the value of the variational problem

$$S^c(x, y) = \inf_{\psi,t} \left\{ \int_0^t c + L(\psi(s), \dot{\psi}(s)) ds, \psi(0) = x, \psi(t) = y \right\}, \quad x, y \in \mathbb{R}^d,$$

where the infimum is taken over all absolutely continuous functions $\psi : [0, \infty) \to \mathbb{R}^n$ and $t > 0$, see [20]. Whenever it is continuous, $y \mapsto S^c(x, y)$ is a viscosity subsolution of the stationary Hamilton-Jacobi equation

$$H(y, DS(y)) = c, \quad y \in \mathbb{R}^d,$$

where H denotes the Fenchel-Legendre transform of L and D denotes the gradient. An object similar to the Mañé potential is the action functional given by

$$M(t, y; x) = \inf_{\psi} \left\{ \int_0^t L(\psi(s), \dot{\psi}(s)) ds, \psi(0) = x, \psi(t) = y \right\}, \quad t > 0 \; x, y \in \mathbb{R}^d,$$

where the infimum is taken over all absolutely continuous functions $\psi : [0, t] \to \mathbb{R}^d$. This action functional is a well-studied object in large deviations theory and control-theory, see for example [19, 18] and the references therein. In the weak KAM and dynamical systems literature it is often referred to as Mather's action functional - see the overview paper [21] and references therein - even though the functional was known well before the papers by Mather. From the definition of the Mañé potential it is elementary to show that

$$S^c(x, y) = \inf_{t>0} \{ M(t, y; x) + ct \}.$$

The main result, Theorem 4.1, shows that, in the one-dimensional setting, $d = 1$, and for all $t < t_L$, the dual relation also holds:

$$M(t, y; x) = \sup_{c > c_L} \{ S^c(x, y) - ct \},$$

where t_L is a time that depends on the Lagrangian and (x, y) and c_L denotes the smallest c such that $S^c > -\infty$. An example is given that illustrates why the duality may fail for $t > t_L$. As a prelude to this duality result, in Proposition 3.2 it is shown that, for arbitrary dimension $d \geq 1$, the right-hand side of the last display is a viscosity solution whenever $y \neq x$.

The duality result is used to derive min-max representations of viscosity solutions of various time-dependent problems. For the initial value problem

$$\begin{cases} V_t(t,y) + H(y, DV(t,y)) = 0, & (t,y) \in (0,\infty) \times \mathbb{R}, \\ V(0,y) = g(y), & y \in \mathbb{R}, \end{cases}$$

the duality leads to a min-max representation of the form

$$V(t,y) = \inf_x \sup_{c > c_L} \{g(x) + S^c(x,y) - ct\}, \quad (t,y) \in [0, t_L) \times \mathbb{R}.$$

The min-max representation may be viewed as a generalization, to state-dependent Hamiltonians, of the classical Hopf-Lax-Oleinik formula, which states that if $H(x,p) = H(p)$, then the solution to the initial value problem is given by

$$V(t,y) = \inf_x \Big\{ g(x) + tL\Big(\frac{y-x}{t}\Big)\Big\}.$$

See [5, 2] for further details and generalizations of Hopf-Lax representation formulas to some state-dependent Hamiltonians. For state-independent Hamiltonians the duality result holds for arbitrary dimension d and we obtain the Hopf-Lax representation as a special case.

Similar min-max representations are stated for terminal value problems, problems on domains, and exit problems. For instance, the viscosity solution $\bar{U}$ to (1.1) can be represented as

$$\bar{U}(t,x) = \inf_{y \in \partial\Omega} \sup_{c > c_{\bar{L}}} \{\bar{S}^c(x,y) - c(T-t)\},$$

where $\bar{S}^c$ is the Mañé potential associated with the Lagrangian $\bar{L}$.

The paper is organized as follows. In Section 2 the Mañé potential and stationary Hamilton-Jacobi equations are introduced and we establish some relevant properties. In Section 3 a similar program is carried out for the action functional and evolutionary Hamilton-Jacobi equations. This is followed by a duality theorem involving the Mañé potential and the action functional in Section 4. In Section 5 it is shown how the duality leads to min-max representations for initial value problems, terminal value problems, problems on domains and exit problems. We conclude the paper by showing a direct relation between the min-max representation for the initial value problem and the Hopf-Lax-Oleinik formula for state-independent convex Hamiltonians (Section 6).

2. The Mañé potential and the stationary Hamilton-Jacobi equation

We begin by introducing the Mañé potential and establish some of its properties, as well as its relation to the stationary Hamilton-Jacobi equation. Throughout the paper the following assumption will be made. Let the *Langrangian* $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a locally bounded measurable function that is convex in the second coordinate and let the *Hamiltonian* H be the Fenchel-Legendre transform of L,

$$H(x,p) = \sup_v \{\langle p, v\rangle - L(x,v)\}. \tag{2.1}$$

By convex duality it follows that

$$L(x,v) = \sup_p \{\langle p, v\rangle - H(x,p)\}.$$

2.1. **The Mañé potential.** Originally introduced by Mañé in [20], the *Mañé potential at level* $c \in \mathbb{R}$, is the function $S^c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$S^c(x,y) = \inf_{\psi,t} \Big\{ \int_0^t c + L(\psi(s), \dot\psi(s))ds, \psi(0) = x, \psi(t) = y \Big\}, \quad x, y \in \mathbb{R}^d, \tag{2.2}$$

where the infimum is taken over all $t > 0$ and absolutely continuous paths $\psi : [0,\infty) \to \mathbb{R}^d$. Since L is locally bounded it follows that $S^c(x,y) < \infty$, for all $x, y \in \mathbb{R}^d$ and $c < \infty$. It is possible that S^c is identically $-\infty$ for small c. Indeed, if $L(x,v) = \frac{1}{2}|v|^2$ and $c < 0$, then it follows from the definition (2.2) that $S^c(x,y) = -\infty$ for all $x, y \in \mathbb{R}^d$, by taking $\psi(t) = 0$ for all $t > 0$. Next, for completeness, some elementary and well known properties of S^c are established.

Proposition 2.1. *The following properties hold.*

(i) For each $x, y \in \mathbb{R}^d$, the function $c \mapsto S^c(x,y)$ is nondecreasing.

(ii) For each $c \in \mathbb{R}$, the function $(x,y) \mapsto S^c(x,y)$ satisfies the triangle inequality:

$$S^c(x,z) \le S^c(x,y) + S^c(y,z), \quad x, y, z \in \mathbb{R}^d. \tag{2.3}$$

(iii) If $S^c(x_0, y_0) = -\infty$ for some $x_0, y_0 \in \mathbb{R}^d$ and $c \in \mathbb{R}$, then $S^c(x,y) = -\infty$ for all $x, y \in \mathbb{R}^d$.

(iv) If $S^c > -\infty$, then $S^c(x,x) = 0$, for each $x \in \mathbb{R}^d$.

Throughout the paper c_L denotes the infimum over all c such that $S^c > -\infty$.

Proof. (i) follows immediately from the definition of the Mañé potential. (ii) For the triangle inequality, if $S^c(x,z) = -\infty$ there is nothing to prove. Suppose $S^c(x,z) > -\infty$. Then $S^c(x,y) > -\infty$ and $S^c(y,z) > -\infty$ as well, for otherwise, if $S^c(x,y) = -\infty$, then there exists, for each $N > 0$, a $t_N > 0$ and an absolutely continuous path ψ_N with $\psi_N(0) = x$ and $\psi_N(t_N) = y$ such that

$$S^c(x,y) \le \int_0^{t_N} c + L(\psi_N(s), \dot\psi_N(s))ds \le -N.$$

Let $\tau > 0$ and φ be any absolutely continuous path with $\varphi(0) = y$ and $\varphi(\tau) = z$ and $\int_0^\tau c + L(\varphi(s), \dot\varphi(s))ds =: C < \infty$. Then, by concatenating ψ_N and φ as

$$\psi_N(s) I\{0 \le s \le t_N\} + \varphi(s - t_N) I\{t_N < s \le t_N + \tau\}$$

it follows that

$$S^c(x,z) \le \int_0^{t_N} c + L(\psi_N(s), \dot\psi_N(s))ds + \int_0^\tau c + L(\varphi(s), \dot\varphi(s))ds \le -N + C.$$

By sending $N \to \infty$ it follows that $S^c(x,z) = -\infty$, which is a contradiction. Consequently, $S^c(x,y) > -\infty$. A similar argument shows that $S^c(y,z) > -\infty$.

To proceed with the proof of the triangle inequality, take an arbitrary $\epsilon > 0$, and select $t_1, t_2 > 0$ and absolutely continuous paths ψ_1, ψ_2 with $\psi_1(0) = x$, $\psi_1(t_1) = y$, $\psi_2(0) = y$ and $\psi_2(t_2) = z$ such that

$$S^c(x,y) \geq \int_0^{t_1} c + L(\psi_1(s), \dot{\psi}_1(s))ds - \frac{\epsilon}{2},$$
$$S^c(y,z) \geq \int_0^{t_2} c + L(\psi_2(s), \dot{\psi}_2(s))ds - \frac{\epsilon}{2}.$$

Concatenate the two trajectories by

$$\psi(s) = \psi_1(s)I\{0 \leq s \leq t_1\} + \psi_2(s - t_1)I\{t_1 < s \leq t_1 + t_2\}.$$

It follows that

$$\begin{aligned} S^c(x,y) + S^c(y,z) &\geq \int_0^{t_1} c + L(\psi_1(s), \dot{\psi}_1(s))ds \\ &\quad + \int_0^{t_2} c + L(\psi_2(s), \dot{\psi}_2(s))ds - \epsilon \\ &= \int_0^{t_1+t_2} c + L(\psi(s), \dot{\psi}(s))ds - \epsilon \\ &\geq S^c(x,z) - \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary the triangle inequality follows.

(iii) follows from the triangle inequality.

To prove (iv), take $x \in \mathbb{R}^d$ and let $\epsilon > 0$, $h > 0$ be such that $h(c + L(x,0)) < \epsilon$ and $\psi(s) = x$ for each $0 \leq s \leq h$. By definition of the Mañé potential,

$$S^c(x,x) \leq h(c + L(x,0)) < \epsilon.$$

Since $\epsilon > 0$ is arbitrary it follows that $S^c(x,x) \leq 0$. The reverse inequality, $S^c(x,x) \geq 0$, follows from the triangle inequality. □

2.2. **The stationary Hamilton-Jacobi equation.** Given a Hamiltonian H and $c \in \mathbb{R}$, the stationary Hamilton-Jacobi equation is

$$H(y, DS(y)) = c, \quad y \in \mathbb{R}^d. \tag{2.4}$$

A continuous function $S : \mathbb{R}^d \to \mathbb{R}$ is a *viscosity subsolution (supersolution)* of the stationary Hamilton-Jacobi equation (2.4) if, for every function $v \in C^\infty(\mathbb{R}^d)$,

$$\left.\begin{array}{l} \text{if } S - v \text{ has a local maximum (minimum) at } y_0 \in \mathbb{R}^d, \\ \text{then } H(y_0, Dv(y_0)) \leq c \quad (\geq c). \end{array}\right\} \tag{2.5}$$

Such a function S is a *viscosity solution* if it is both a viscosity subsolution and a viscosity supersolution.

The *Mañé critical value* is the infimum over c for which (2.4) admits a viscosity subsolution. With some abuse of notation it will be denoted by c_H. The critical value admits the lower bound

$$c_H \geq \sup_y \inf_p H(y,p). \tag{2.6}$$

Indeed, if (2.4) admits a viscosity subsolution U^c at level c, then for every y there is a $v \in C^\infty(\mathbb{R}^d)$ such that $U^c - v$ has a local maximum at y and $\inf_p H(y,p) \le H(y, Dv(y)) \le c$. The claim follows by taking supremum over y. Examples where $c_H = \sup_y \inf_p H(y,p)$ are provided below.

The Mañé potential (2.2) is well studied within weak KAM theory, where it is commonly assumed that the Hamiltonian is uniformly superlinear: for each $K \ge 0$ there exists $C(K) \in \mathbb{R}$ such that $H(y,p) \ge K|p| - C(K)$ for each y, p. Under such an assumption there exist critical viscosity subsolutions, that is, there exists a global viscosity subsolution to (2.4) for $c = c_H$, see [15, 14]. In this paper it is assumed that the Hamiltonian is given by the Fenchel-Legendre transform of a Lagrangian L, as in (2.1), and consequently $p \mapsto H(y,p)$ is convex in p, for every $y \in \mathbb{R}^d$. For instance, the Hamiltonian associated with the unit rate Poisson process, which is of the form

$$H(p) = e^p - 1, \quad p \in \mathbb{R},$$

is covered by our assumptions. For this choice of H the Mañé critical value is $c_H = -1$, but there can be no critical subsolution S as it would have to satisfy $DS(y) = -\infty$ almost eveywhere, see Example 2.3 below.

The following properties of the Mañé potential are well known and similar statements appear in [14, 15, 16], see also the lecture notes [13, 4]. However, our assumptions on the Hamiltonian are different and thus a proof is included for completeness.

Proposition 2.2. *Assume* (2.1). *Take $c \in \mathbb{R}$, $x \in \mathbb{R}^d$ and suppose the function $y \mapsto S^c(x,y)$ is continuous.*

(i) Suppose that $S^c > -\infty$. Then $y \mapsto S^c(x,y)$ is a viscosity subsolution to $H(y, DS(y)) = c$ on $\mathbb{R}^d$ and a viscosity solution on $\mathbb{R}^d \setminus \{x\}$.

(ii) For each $y \in \mathbb{R}^d$, $S^c(x,y) = \sup_{S \in \mathcal{S}^c_x} S(y)$, where $\mathcal{S}^c_x$ is the collection of all continuous viscosity subsolutions to $H(y, DS(y)) = c$ that vanish at x.

Recall that c_L is the infimum over c such that $S^c > -\infty$. Take $x \in \mathbb{R}^d$ and suppose that, for each $c > c_L$, the function $y \mapsto S^c(x,y)$ is continuous. For $c > c_H$ there exist viscosity subsolutions to (2.4) and by Proposition 2.2(ii) it follows that $S^c > -\infty$. Consequently, $c_H \ge c_L$. Similarly, for $c < c_H$ there are no subsolutions and by Proposition 2.2(i) $S^c = -\infty$, which implies $c_H \le c_L$. This proves the following.

Corollary 2.1. *Take $x \in \mathbb{R}^d$ and suppose that, for each $c > c_L$, the function $y \mapsto S^c(x,y)$ is continuous. Then $c_H = c_L$.*

Before proceeding to the proof of Proposition 2.2 we state an important lemma that can be interpreted as a dynamic programming property of the Mañé potential.

Lemma 2.1. *Suppose that $S^c > -\infty$. For any $x, y_0 \in \mathbb{R}^d$ with $y_0 \neq x$ and $\epsilon > 0$ there exist $0 < \delta < |x - y_0|$, y with $|y - y_0| < \delta$, $h > 0$ and an absolutely continuous path ψ with $\psi(0) = y$, $\psi(h) = y_0$, and $|\psi(s) - y_0| < \delta$ for all $s \in [0,h]$, such that*

$$S^c(x, y_0) \ge S^c(x,y) + \int_0^h \Big(c + L(\psi(s), \dot\psi(s))\Big)\, ds - \epsilon.$$

Proof. Given $x, y_0 \in \mathbb{R}^d$ with $x \neq y_0$ and $\epsilon > 0$, take $t > 0$ and an absolutely continuous function φ with $\varphi(0) = x$, $\varphi(t) = y_0$ such that

$$S^c(x, y_0) \geq \int_0^t (c + L(\varphi(s), \dot{\varphi}(s)))\, ds - \epsilon.$$

Let $0 < \delta < |x - y_0|$ and take $h > 0$ such that $|\varphi(s) - y_0| < \delta$ for each $s \in [t-h, t]$. With $y = \varphi(t-h)$ and $\psi(s) = \varphi(s + t - h)$, $s \in [0, h]$, it follows that

$$\begin{aligned} S^c(x, y_0) &\geq \int_0^t (c + L(\varphi(s), \dot{\varphi}(s)))\, ds - \epsilon \\ &= \int_0^{t-h} (c + L(\varphi(s), \dot{\varphi}(s)))\, ds + \int_{t-h}^t (c + L(\varphi(s), \dot{\varphi}(s)))\, ds - \epsilon \\ &\geq S^c(x, y) + \int_0^h \Big(c + L(\psi(s), \dot{\psi}(s))\Big)\, ds - \epsilon. \end{aligned}$$

This completes the proof. $\square$

Proof of Proposition 2.2. Proof of (i). Suppose that $S^c > -\infty$, take $x \in \mathbb{R}^d$ and suppose that $y \mapsto S^c(x, y)$ is continuous. First we prove the viscosity subsolution property. For $v \in C^\infty(\mathbb{R}^d)$, suppose that $S^c(x, \cdot) - v$ has a local maximum at y_0 and, contrary to what we want to show, that $H(y, Dv(y)) - c \geq \theta > 0$ for $|y - y_0| \leq \delta$, for some $\delta > 0$. We may assume that δ is sufficiently small that

$$S^c(x, y) - v(y) \leq S^c(x, y_0) - v(y_0), \quad \text{for } |y - y_0| \leq \delta.$$

Take any y with $|y - y_0| \leq \delta$ and consider any absolutely continuous path ψ such that $\psi(0) = y$, $\psi(h) = y_0$ and $|\psi(s) - y_0| \leq \delta$ for all $s \in [0, h]$. By the triangle inequality (2.3) and the last inequality

$$\begin{aligned} 0 &\geq S^c(x, y_0) - S^c(x, y) - \int_0^h \Big(c + L(\psi(s), \dot{\psi}(s))\Big)\, ds \\ &\geq v(y_0) - v(y) - \int_0^h \Big(c + L(\psi(s), \dot{\psi}(s))\Big)\, ds \\ &= \int_0^h \left(\frac{d}{ds} v(\psi(s)) - L(\psi(s), \dot{\psi}(s)) - c\right) ds \\ &= \int_0^h \Big(\langle Dv(\psi(s)), \dot{\psi}(s)\rangle - L(\psi(s), \dot{\psi}(s)) - c\Big)\, ds. \end{aligned}$$

We may assume that $\dot{\psi}$ is chosen such that, using the conjugacy between H and L,

$$H(\psi(s), Dv(\psi(s))) \leq \langle Dv(\psi(s)), \dot{\psi}(s)\rangle - L(\psi(s), \dot{\psi}(s)) + \frac{\theta}{2},$$

for all $s \in [0, h]$. Then

$$\frac{\theta h}{2} \geq \int_0^h (H(\psi(s), Dv(\psi(s))) - c)\, ds \geq \theta h,$$

which is a contradiction. Thus, it must hold that $H(y_0, Dv(y_0)) \leq c$.

Next, we prove the supersolution property on $\mathbb{R}^d \setminus \{x\}$. Take $v \in C^\infty(\mathbb{R}^d)$ and suppose $S^c(x,\cdot) - v$ has a local minimum at $y_0 \neq x$ and, contrary to what we want to show, that $H(y, Dv(y)) - c \leq -\theta < 0$ for $|y - y_0| \leq \delta$, for some $\delta > 0$. We may assume that δ is sufficiently small that $|x - y_0| > \delta$ and

$$S^c(x,y) - v(y) \geq S^c(x,y_0) - v(y_0), \quad \text{for } |y - y_0| \leq \delta.$$

By Lemma 2.1 we may select y with $|y - y_0| \leq \delta$ and an absolutely continuous path ψ such that $\psi(0) = y$, $\psi(h) = y_0$ and $|\psi(s) - y_0| \leq \delta$ for all $s \in [0,h]$, with the property that

$$S^c(x,y_0) \geq S^c(x,y) + \int_0^h c + L(\psi(s), \dot{\psi}(s)) ds - \frac{\theta h}{2}.$$

The last inequality implies that

$$\begin{aligned}
\frac{\theta h}{2} &\geq S^c(x,y) - S^c(x,y_0) + \int_0^h \Big(c + L(\psi(s), \dot{\psi}(s))\Big) ds \\
&\geq v(y) - v(y_0) + \int_0^h \Big(c + L(\psi(s), \dot{\psi}(s))\Big) ds \\
&= \int_0^h \left(-\frac{d}{ds} v(\psi(s)) + L(\psi(s), \dot{\psi}(s)) + c\right) ds \\
&= \int_0^h \Big(-\langle Dv(\psi(s)), \dot{\psi}(s)\rangle + L(\psi(s), \dot{\psi}(s)) + c\Big) ds \\
&\geq \int_0^h -\Big(H(\psi(s), Dv(\psi(s))) - c\Big) ds.
\end{aligned}$$

We conclude that

$$-\frac{\theta h}{2} \leq \int_0^h \left(H(\psi(s), Dv(\psi(s))) - c\right) ds \leq -\theta h.$$

This is a contradiction and thus it must indeed hold that $H(y_0, Dv(y_0)) \geq c$, which completes the proof of (i).

Proof of (ii). Let $c \in \mathbb{R}$. If there are no viscosity subsolutions at level c, then by (i) $S^c = -\infty$ and $\mathcal{S}_x^c = \emptyset$, which implies that $\sup_{S \in \mathcal{S}_x^c} S(y) = -\infty$ as well. If there exist continuous viscosity subsolutions at level c, take $x \in \mathbb{R}^d$ and let S be a continuous viscosity subsolution of $H(y, DS(y)) = c$ on $\mathbb{R}^d$. It is sufficient to show that for any $y \in \mathbb{R}^d$, $t > 0$ and absolutely continuous function ψ with $\psi(0) = x$ and $\psi(t) = y$,

$$S(y) - S(x) \leq \int_0^t \Big(c + L(\psi(s), \dot{\psi}(s))\Big) ds. \tag{2.7}$$

To show (2.7), fix $t > 0$, $y \in \mathbb{R}^d$, an absolutely continuous path ψ with $\psi(0) = x$ and $\psi(t) = y$ and take an arbitrary $\epsilon > 0$. For every $s \in [0,t]$, let $v_s \in C^\infty(\mathbb{R}^d)$ be such that $S - v_s$ has a local maximum at $\psi(s)$. Then, there exists $\delta_s > 0$ such that

$$S(z) - v_s(z) \leq S(\psi(s)) - v_s(\psi(s)), \quad \text{for } |z - \psi(s)| < \delta_s,$$

and consequently that

$$S(z) - S(\psi(s)) \leq v_s(z) - v_s(\psi(s)), \qquad \text{for } |z - \psi(s)| < \delta_s. \tag{2.8}$$

By continuity of H and Dv_s we may, in addition, assume that δ_s is sufficiently small that

$$H(z, Dv_s(z)) \leq c + \frac{\epsilon}{t}, \qquad \text{for } |z - \psi(s)| < \delta_s.$$

For every $s \in [0,t]$, let $h_s > 0$ be such that $|\psi(u) - \psi(s)| < \delta_s$ for every u with $|u - s| < h_s$. This is possible due to the continuity of ψ. The union

$$[0, h_0) \cup \bigcup_{s \in (0,t]} (s, s + h_s),$$

is an open cover of $[0,t]$. Since $[0,t]$ is compact there is a finite subcover, which we can assume is of the form

$$[0, h_0) \cup \bigcup_{k=1}^{n-1} (s_k, s_k + h_{s_k}),$$

where $0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = t$. Since the finite union is a subcover, it must hold that $s_{k-1} < s_k < s_{k-1} + h_{s_{k-1}}$ for each $k = 1, \ldots, n$. It follows that, using (2.8) and the conjugacy between H and L,

$$\begin{aligned}
S(y) - S(x) &= \sum_{k=1}^{n} S(\psi(s_k)) - S(\psi(s_{k-1})) \\
&\leq \sum_{k=1}^{n} v_{s_{k-1}}(\psi(s_k)) - v_{s_{k-1}}(\psi(s_{k-1})) \\
&= \sum_{k=1}^{n} \int_{s_{k-1}}^{s_k} \langle Dv_{s_{k-1}}(\psi(s)), \dot{\psi}(s) \rangle ds \\
&\leq \sum_{k=1}^{n} \int_{s_{k-1}}^{s_k} \Big(H(\psi(s), Dv_{s_{k-1}}(\psi(s))) + L(\psi(s), \dot{\psi}(s)) \Big) ds \\
&\leq \sum_{k=1}^{n} \int_{s_{k-1}}^{s_k} \Big(c + \frac{\epsilon}{t} + L(\psi(s), \dot{\psi}(s)) \Big) ds \\
&= \epsilon + \int_0^t \Big(c + L(\psi(s), \dot{\psi}(s)) \Big) ds.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary the claim follows. □

We proceed by computing Mañé's critical value, c_H for some Hamiltonians arising in the theory of large deviations of stochastic processes; in all three examples there is equality in the lower bound for c_H.

Example 2.1 (Critical diffusion process)**.** Let $U : \mathbb{R}^d \to \mathbb{R}$ be a potential function and $b(y) = -DU(y)$. Consider the Hamiltonian $H(y,p) = \langle b(y), p \rangle + \frac{1}{2}|p|^2$. Then $c_H = \sup_y \inf_p H(y,p) = -\frac{1}{2} \inf_y |b(y)|^2$. Indeed, from (2.6), $c_H \geq -\frac{1}{2} \inf_y |b(y)|^2$ and U is a subsolution to $H(y, DS(y)) = -\frac{1}{2} \inf_y |b(y)|^2$, which implies $c_H \leq -\frac{1}{2} \inf_y |b(y)|^2$. In particular, if $DU(y) = 0$ for some y, then $c_H = 0$. In this setting the Mañé potential can be viewed as a generalization of Freidlin and Wentzell's quasi-potential, described in [19, Ch. 4].

Example 2.2 (Birth-and-death process)**.** Consider an interval $(a, b) \subset \mathbb{R}$ and functions $\mu : (a, b) \to [0, \infty)$, $\lambda : (a, b) \to [0, \infty)$ satisfying $\int_a^b \log(\sqrt{\mu(y)/\lambda(y)})dy < \infty$. Consider the Hamiltonian

$$H(y, p) = \lambda(y)(e^p - 1) + \mu(y)(e^{-p} - 1).$$

In this case $c_H = \sup_y \inf_p H(y, p) = -\inf_y(\sqrt{\mu(y)} - \sqrt{\lambda(y)})^2$. To see this, recall from (2.6) that $c_H \geq -\inf_y(\sqrt{\mu(y)} - \sqrt{\lambda(y)})^2$. A subsolution of

$$H(y, DS(y)) = -\inf_y(\sqrt{\mu(y)} - \sqrt{\lambda(y)})^2,$$

is given by

$$U(y) = \int_a^y \log(\sqrt{\mu(z)/\lambda(z)})dz.$$

Indeed,

$$H(y, DU(y)) = -(\sqrt{\mu(y)} - \sqrt{\lambda(y)})^2 \leq -\inf_y(\sqrt{\mu(y)} - \sqrt{\lambda(y)})^2.$$

Example 2.3 (Pure birth process)**.** Let $\lambda : [0, \infty)^d \to [0, \infty)^d$ and put

$$H(y, p) = \sum_{j=1}^d \lambda_j(y)(e^{p_j} - 1).$$

In this case $c_H = \sup_y \inf_p H(y, p) = -\inf_y \sum_{j=1}^d \lambda_j(y) =: -\lambda_*$. Indeed, from (2.6) it follows that $c_H \geq -\lambda_*$ and for any $c \in (-\lambda_*, 0)$ and $\alpha \leq \log(1 + c/\lambda_*)$, the function $\alpha\langle 1, y\rangle$ is a subsolution to $H(y, DS(y)) = c$, which implies $c_H \leq -\lambda_*$.

We end this subsection by proving a sufficient condition for the continuity of $y \mapsto S^c(x, y)$.

Proposition 2.3. *Suppose that the Lagrangian L is continuous at $(y, 0)$ for each $y \in \mathbb{R}^d$. Then, for each $x \in \mathbb{R}^d$ and $c > c_L$ the function $y \mapsto S^c(x, y)$ is continuous.*

Proof. Take $y_0 \in \mathbb{R}^d$ and $\epsilon > 0$. To prove continuity at y_0 we show that there exists a $\delta > 0$ such that $|y - y_0| < \delta$ implies

$$S^c(x, y_0) \leq S^c(x, y) + \epsilon, \tag{2.9}$$

$$S^c(x, y) \leq S^c(x, y_0) + \epsilon. \tag{2.10}$$

We begin to prove (2.9). By assumption L is continuous at $(y_0, 0)$ and we may select δ' such that $L(y_0 + z, v) \leq L(y_0, 0) + 1$ for all $|z| < \delta'$ and $|v| < \delta'$. Pick $h > 0$ such that $h(c + L(y_0, 0) + 1) < \epsilon/2$ and let $\delta = h\delta'$. For $|y - y_0| < \delta$, take $t > h$ and an absolutely continuous path ψ with $\psi(0) = x$, $\psi(t - h) = y$ such that

$$S^c(x, y) \geq \int_0^{t-h} \Big(c + L(\psi(s), \dot{\psi}(s))\Big)\, ds - \frac{\epsilon}{2},$$

and $\dot\psi(s) = h^{-1}(y_0 - y)$ for $t - h \le s \le t$. Then,

$$\begin{aligned} S^c(x, y_0) &\le \int_0^{t-h} \Big(c + L(\psi(s), \dot\psi(s))\Big)\, ds + \int_{t-h}^{t} \Big(c + L(\psi(s), \dot\psi(s))\Big)\, ds \\ &\le S^c(x, y) + \frac{\epsilon}{2} + h(c + L(y_0, 0) + 1) \\ &\le S^c(x, y) + \epsilon, \end{aligned}$$

by the choice of h. The proof of (2.10) is similar.

□

3. The action functional and the evolutionary Hamilton-Jacobi equation

In this section we establish the connection between the action functional, the Mañé potential and viscosity solutions to the evolutionary Hamilton-Jacobi equation. The key result is Proposition 3.2, which shows how to construct certain solutions via the Mañé potential. The results in this section are derived in $\mathbb{R}^d$ for arbitrary $d \ge 1$, whereas for the duality theorem in Section 4 we move to the one-dimensional setting ($d = 1$). For a more thorough introduction to Hamilton-Jacobi equations we refer to [1, 3, 12, 14, 7, 6].

3.1. **The action functional.** For any $x \in \mathbb{R}^d$ and $(t, y) \in (0, \infty) \times \mathbb{R}^d$, let M be the action functional

$$M(t, y; x) = \inf_{\psi} \Big\{ \int_0^t L(\psi(s), \dot\psi(s)) ds, \psi(0) = x, \psi(t) = y \Big\}, \tag{3.1}$$

where the infimum is taken over all absolutely continuous $\psi : [0, \infty) \to \mathbb{R}^d$. M is the action functional of Mather, see [21], viewed as a function of (t, y).

3.2. **The evolutionary Hamilton-Jacobi equation.** Consider a Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ as in (2.1). The evolutionary Hamilton-Jacobi equation is

$$V_t(t, y) + H(y, DV(t, y)) = 0, \quad (t, y) \in (0, \infty) \times \mathbb{R}^d, \tag{3.2}$$

where $V_t = \partial V/\partial t$ and $DV = (\partial V/\partial y_1, \dots, \partial V/\partial y_n)$. A continuous function $V : (0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ is a *viscosity subsolution (supersolution)* of (3.2) if, for every $v \in C^\infty((0, \infty) \times \mathbb{R}^d)$,

$$\left.\begin{array}{l} \text{if } V - v \text{ has a local maximum (minimum) at } (t_0, y_0) \in (0, \infty) \times \mathbb{R}^d, \\ \text{then } v_t(t_0, y_0) + H(x_0, Dv(t_0, y_0)) \le 0 \quad (\ge 0). \end{array}\right\}$$

V is a *viscosity solution* if it is both a subsolution and a supersolution of (3.2).

Proposition 3.1 shows how the action functional M in (3.1) plays a similar role for the evolutionary Hamilton-Jacobi equation as the Mañé potential does for the stationary Hamilton-Jacobi equation.

Proposition 3.1. *Take $x \in \mathbb{R}^d$ and assume that $(t, y) \mapsto M(t, y; x)$ is continuous.*

(i) $M(\cdot\,; x)$ is a viscosity solution to (3.2) on $(0, \infty) \times \mathbb{R}^d$.

(ii) $M(t, y; x) = \sup_{V \in \mathcal{S}_{0,x}} V(t, y)$, where $\mathcal{S}_{0,x}$ is the collection of all continuous viscosity subsolutions to (3.2) vanishing at $(0, x)$.

The proof of Proposition 3.1 is almost identical to that of Proposition 2.2 and is therefore omitted.

We now show how viscosity solutions to (3.2) can be constructed from the Mañé potential. Suppose that L and H satisfy (2.1) and $y \mapsto S^c(x,y)$ is continuous for $x \in \mathbb{R}^d$ and $c > c_L$. By Proposition 2.2(i), $y \mapsto S^c(x,y)$ is a viscosity subsolution to $H(y, DS(y)) = c$ for each $x \in \mathbb{R}^d$ and $c > c_L$. It follows that the function $(t,y) \mapsto S^c(x,y) - ct$ is a viscosity subsolution of the evolutionary Hamilton-Jacobi equation (3.2). Perron's method, see [1, Theorem V.2.14], implies that the function $U(\cdot\,; x)$ given by

$$U(t,y;x) = \sup_{c > c_L}\{S^c(x,y) - ct\}, \quad (t,y) \in [0,\infty) \times \mathbb{R}^d, \tag{3.3}$$

is also a viscosity subsolution to (3.2). Moreover, $(t,y) \mapsto S^c(x,y) - ct$ is a viscosity solution to (3.2) on $\mathbb{R}^d \setminus \{x\}$. This property also transfers to $U(t,y;x)$ as the following proposition shows.

Proposition 3.2. *For $x \in \mathbb{R}^d$, the function U in* (3.3) *is a viscosity solution to* (3.2) *on $(0,\infty) \times \mathbb{R}^d \setminus \{x\}$.*

Proof. Since $y \mapsto S^c(x,y)$ is a viscosity subsolution to $H(y, DS(y)) = c$, for any $c > c_L$, it follows by Perron's method that $U(t,y;x)$ is a viscosity subsolution to (3.2). It remains to show the supersolution property.

Fix $x \in \mathbb{R}^d$ and take $v \in C^\infty((0,\infty) \times \mathbb{R}^d \setminus \{x\})$. Suppose that $U(\cdot\,;x) - v$ has a local minimum at (t_0, y_0) where $t_0 > 0$ and $y_0 \neq x$. We must show that $v_t(t_0,y_0) + H(y_0, Dv(t_0,y_0)) \geq 0$.

Suppose, on the contrary, that there exist $\theta > 0$ and $\delta > 0$ such that

$$v_t(t,y) + H(y, Dv(t,y)) \leq -\theta,$$

for all (t,y) with $|t - t_0| + |y - y_0| < \delta$. We will arrive at a contradiction by showing that there is a $c > c_L$ for which the viscosity supersolution property is violated for the function $(t,y) \mapsto S^c(x,y) - ct$ at some point (t,y), $t > 0$, $y \neq x$.

We may assume that the δ above is sufficiently small that $|x - y_0| > \delta$ and

$$U(t,y;x) - v(t,y) \geq U(t_0,y_0;x) - v(t_0,y_0),$$

for all (t,y) with $|t-t_0| + |y-y_0| < \delta$. Then, the subsolution property is strict at (t_0,y_0) in the sense that there is a $\theta_1 > 0$ such that, for all $w \in C^\infty((0,\infty) \times \mathbb{R}^d \setminus \{x\})$ such that $U(\cdot\,;x) - w$ has a local maximum at (t_0,y_0)

$$w_t(t,y) + H(y, Dw(t,y)) \leq -\theta_1, \tag{3.4}$$

for all $|t - t_0| + |y - y_0| < \delta_1$, some $\delta_1 > 0$. To prove (3.4), observe first that since $U(\cdot\,;x) - w$ has a local maximum at (t_0,y_0) and $U(\cdot\,;x) - v$ has a local minimum at (t_0,y_0) we may select δ_1 such that

$$\begin{aligned} U(t,y;x) - w(t,y) &\leq U(t_0,y_0;x) - w(t_0,y_0), \\ U(t,y;x) - v(t,y) &\geq U(t_0,y_0;x) - v(t_0,y_0), \end{aligned}$$

for all (t,y) with $|t - t_0| + |y - y_0| < \delta_1$. Consequently, $w - v$ satisfies

$$w(t,y) - v(t,y) \geq w(t_0,y_0) - v(t_0,y_0),$$

for all (t, y) with $|t - t_0| + |y - y_0| < \delta_1$ so $w - v$ has a local minimum at (t_0, y_0). Since both v and w are in $C^\infty((0, \infty) \times \mathbb{R}^d \setminus \{x\})$ it follows that $w_t(t_0, y_0) = v_t(t_0, y_0)$ and $Dw(t_0, y_0) = Dv(t_0, y_0)$. We conclude that

$$w_t(t_0, y_0) + H(y_0, Dw(t_0, y_0)) = v_t(t_0, y_0) + H(y_0, Dv(t_0, y_0)) \leq -\theta.$$

Now (3.4) follows by taking $\theta_1 \in (0, \theta)$ and using continuity of H, w_t and Dw.

Without loss of generality we assume that w in (3.4) is such that $w(t_0, y_0) = U(t_0, y_0; x)$ and $w(t, y) > U(t, y; x)$, $|t - t_0| + |y - y_0| < \delta_1$. Take $0 < \epsilon < \delta_1$ so that (3.4) holds on $\overline{N}_\epsilon = \{(t, y) : |t - t_0| + |y - y_0| \leq \epsilon\}$. Since $w(t, y) > U(t, y; x)$ on ∂N_ϵ there is an $\eta > 0$ such that

$$w(t, y) - \eta \geq U(t, y; x), \quad (t, y) \in \partial N_\epsilon.$$

Moreover, we may select $c > c_L$ such that

$$S^c(x, y_0) - ct_0 > U(t_0, y_0; x) - \eta = w(t_0, y_0) - \eta.$$

Rewriting the last two displays we find that

$$\begin{aligned} S^c(x, y_0) - ct_0 - w(t_0, y_0) &> -\eta, \\ S^c(x, y) - ct - w(t, y) &\leq -\eta, \quad (t, y) \in \partial N_\epsilon. \end{aligned}$$

It follows that the maximum of the continuous function $(t, y) \mapsto S^c(x, y) - ct - w(t, y)$ over the compact set $\overline{N}_\epsilon$ is attained at some (t_ϵ, y_ϵ) in the open neighborhood N_ϵ and by (3.4)

$$w_t(t_\epsilon, y_\epsilon) + H(y_\epsilon, Dw(t_\epsilon, y_\epsilon)) \leq -\theta_1.$$

Let $v^\epsilon \in C^\infty((0, \infty) \times \mathbb{R}^d \setminus \{x\})$ be such that the function $(t, y) \mapsto S^c(x, y) - ct - v^\epsilon(t, y)$ has a local minimum at (t_ϵ, y_ϵ). Then, there is a $\delta_2 > 0$ such that

$$\begin{aligned} S^c(x, y) - ct - w(t, y) &\leq S^c(x, y_\epsilon) - ct_\epsilon - w(t_\epsilon, y_\epsilon), \\ S^c(x, y) - ct - v^\epsilon(t, y) &\leq S^c(x, y_\epsilon) - ct_\epsilon - v^\epsilon(t_\epsilon, y_\epsilon), \end{aligned}$$

for all (t, y) with $|t - t_\epsilon| + |y - y_\epsilon| < \delta_2$. Consequently, $w - v^\epsilon$ has a local minimum at (t_ϵ, y_ϵ) and $w_t(t_\epsilon, y_\epsilon) = v^\epsilon_t(t_\epsilon, y_\epsilon)$ and $Dw(t_\epsilon, y_\epsilon) = Dv^\epsilon(t_\epsilon, y_\epsilon)$. We conclude that

$$v^\epsilon_t(t_\epsilon, y_\epsilon) + H(y_\epsilon, Dv^\epsilon(t_\epsilon, y_\epsilon)) = w_t(t_\epsilon, y_\epsilon) + H(y_\epsilon, Dw(t_\epsilon, y_\epsilon)) \leq -\theta_1 < 0.$$

The last display contradicts the viscosity supersolution property of $(t, y) \mapsto S^c(x, y) - ct$ at (t_ϵ, y_ϵ). We conclude that the supersolution property holds for $U(\cdot\,; x)$ on $(0, \infty) \times \mathbb{R}^d \setminus \{x\}$. □

Take $x \in \mathbb{R}^d$ and assume the required continuity. By Proposition 3.1 and Proposition 3.2 both $(t, y) \mapsto M(t, y; x)$ and $(t, y) \mapsto U(t, y; x)$ are viscosity solutions to (3.2) on $(0, \infty) \times \mathbb{R}^d \setminus \{x\}$. At $t = 0$, $M(0, y; x) = U(0, y; x) = \infty$ if $y \neq x$ and $= 0$ if $y = x$. However, in the present setting there is no valid comparison principle so equality between M and U need not hold for $t > 0$. Indeed, by Theorem 4.1, when $d = 1$ $M(t, y; x) = U(t, y; x)$ for $t \leq t_L$, but it may happen that $M(t, y; x) > U(t, y; x)$ for $t > t_L$ as illustrated in Example 4.3.

4. The duality theorem

We now move to state and prove a duality result between between the Mañé potential (2.2) and the action functional (3.1) in $\mathbb{R}$. The duality may fail when constraints on the time t in the action functional is not satisfied and we include an example that illustrates this; we also compute the Mañé potential for the Hamiltonians used in Examples 2.1 and 2.2.

From the definition (2.2) of the Mañé potential it follows immediately that, for $x, y \in \mathbb{R}^d$,

$$S^c(x,y) = \inf_{t>0}\{M(t,y;x) + ct\}.$$

The dual relationship holds in $\mathbb{R}$ for all t not too large. Let c_L denote the infimum over all c such that $S^c > -\infty$.

Theorem 4.1. *Assume* (2.1). *For each $x, y \in \mathbb{R}^d$ and $c \in \mathbb{R}$,*

$$S^c(x,y) = \inf_{t>0}\{M(t,y;x) + ct\}, \tag{4.1}$$

For $x, y \in \mathbb{R}$ and $t < t_L = \lim_{c \downarrow c_L} \partial_{c+} S^c(x,y)$,

$$M(t,y;x) = \sup_{c>c_L}\{S^c(x,y) - ct\}. \tag{4.2}$$

Moreover, if $S^{c_L} > -\infty$, then (4.2) *holds for $t = t_L$ and, in addition, if either $x \in \mathcal{A} := \{x : L(x,0) = -c_L\}$ or $y \in \mathcal{A}$, then* (4.2) *holds for all $t > 0$ and*

$$M(t,y;x) = S^{c_L}(x,y) - c_L t,$$

for $t \geq t_L$.

Before proceeding with the proof ot Theorem 4.1, we give an intuitive physical interpretation of the duality between $S^c(x,y)$ and $M(t,y;x)$. The optimal t in the representation (4.1) is the optimal time it takes to move from x to y in a system with energy level c. Similarly, the optimal c in the representation (4.2) is the energy level at which it takes precisely time t to move from x to y along the most-cost efficient path. The duality may fail if t is sufficiently large that it exceeds the optimal time in the definition of $S^{c_L}(x,y)$. In convex analysis terms: $c \mapsto S^c(x,y)$ is always concave but $t \mapsto M(t,y;x)$ is convex only for $t < t_L$. Example 4.3 serves as an illustration.

Proof of Theorem 4.1. As mentioned above (4.1) follows from the definition (2.2) of the Mañé potential. Let us prove (4.2). By (4.1) it follows that

$$\sup_{c>c_L}\{S^c(x,y) - ct\} = \sup_{c>c_L}\inf_{s>0}\{M(s,y;x) - c(t-s)\} \leq M(t,y;x),$$

by taking $s = t$, so it is sufficient to prove $M(t,y;x) \leq \sup_{c>c_L}\{S^c(x,y) - ct\}$. Take $x, y \in \mathbb{R}$ and suppose that $y > x$ and $t_L > 0$. The argument for $y < x$ is similar. The proof relies on the construction of a convex upper bound $\tilde{M}(\cdot, y; x)$ of $M(\cdot, y; x)$. Let

$$\begin{aligned}\mathcal{M}(t, y; x) &= \{\psi : [0, t] \to \mathbb{R}, \text{ abs. cont.}, \psi(0) = x, \psi(t) = y, \psi \text{ strictly increasing}\}, t > 0,\\ \mathcal{M} &= \cup_{t>0}\mathcal{M}(t, y; x),\\ \mathcal{M}^{-1} &= \{\psi^{-1} : \psi \in \mathcal{M}\},\\ \mathcal{M}_1^{-1} &= \{\xi : \xi(z) = \frac{d}{dz}\psi^{-1}(z), \psi^{-1} \in \mathcal{M}^{-1}\}.\end{aligned}$$

Since each $\psi \in \mathcal{M}$ is absolutely continuous and strictly increasing, so is ψ^{-1}. Moreover, $\psi^{-1}(x) = 0$ and $\psi^{-1}(t) = y$ for some $t > 0$. Consequently, each $\xi \in \mathcal{M}_1^{-1}$ is strictly positive a.e. and $\int_x^y \xi(z)dz = t$ for some $t > 0$. By a change of variables it follows that, for all $t > 0$,

$$\begin{aligned}M(t, y; x) &\le \inf_{\psi\in\mathcal{M}} \Big\{ \int_0^t L(\psi(s), \dot\psi(s))\, ds, \psi(t) = y\Big\}\\ &= \inf_{\xi\in\mathcal{M}_1^{-1}} \Big\{ \int_x^y L\Big(z, \frac{1}{\xi(z)}\Big)\xi(z)dz, \int_x^y \xi(z)dz = t\Big\} =: \tilde{M}(t, y; x).\end{aligned}$$

By the convexity of $v \mapsto L(x, v)$ it follows that $F : \xi \mapsto \int_x^y L(z, \frac{1}{\xi(z)})\xi(z)dz$ is convex and, consequently, $\tilde{M}(t, y; x)$ is the value of the following convex optimization problem: minimize the convex functional F over the convex set $\mathcal{M}_1^{-1}$, subject to the linear constraint $G(\xi) := \int_x^y \xi(z)dz = t$.

For $c \in \mathbb{R}$, let

$$\tilde{S}^c(x, y) = \inf_{\psi\in\mathcal{M}} \Big\{ \int_0^t c + L(\psi(s), \dot\psi(s))ds, t > 0\Big\} = \inf_{\xi\in\mathcal{M}_1^{-1}} \{F(\xi) + cG(\xi)\}.$$

The proof proceeds by showing the relation

$$\tilde{M}(t, y; x) = \sup_{c\in\mathbb{R}}\{\tilde{S}^c(x, y) - ct\}, \quad \text{for all } t > 0. \tag{4.3}$$

To prove (4.3), let A be the convex set

$$A = \{(r, s) \in (-\infty, \infty) \times (0, \infty) : r \ge F(\xi), s = G(\xi), \text{ some } \xi \in \mathcal{M}_1^{-1}\}.$$

The following representation then holds for any $c \in \mathbb{R}$:

$$\tilde{S}^c(x, y) = \inf_{\xi\in\mathcal{M}_1^{-1}} \{F(\xi) + cG(\xi)\} = \inf_{(r,s)\in A} \{\langle(1, c), (r, s)\rangle\}.$$

Take $t > 0$, let $\mu_t = \tilde{M}(t, y; x) = \inf_{\xi\in\mathcal{M}_1^{-1}}\{F(\xi), G(\xi) = t\}$ and $(1, c_t)$ be the normal vector to the tangent plane of A at (μ_t, t). If A has a corner at (μ_t, t) so that c_t is not unique, take the largest c_t. By the choice of c_t

$$0 \le \langle(1, c_t), (r, s) - (\mu_t, t)\rangle, \quad (r, s) \in A.$$

The inequality in the last display can be rewritten as $\mu_t \le r + c_t(s - t)$ and consequently,

$$\mu_t \le \inf_{(r,s)\in A}\{r + c_t(s - t)\} \le \inf_{\xi\in\mathcal{M}_1^{-1}}\{F(\xi) + c_t(G(\xi) - t)\} \le \inf_{\xi\in\mathcal{M}_1^{-1}}\{F(\xi), G(\xi) = t\} = \mu_t.$$

It follows that all inequalities in the last display are in fact equalities and, in particular,

$$\tilde{S}^{c_t}(x,y) - c_t t = \inf_{(r,s)\in A}\{\langle (1,c_t),(r,s)\rangle\} - c_t t = \inf_{(r,s)\in A}\{\langle (1,c_t),(r,s-t)\rangle\} = \tilde{M}(t,y;x).$$

This completes the proof of (4.3) and we conclude that

$$M(t,y;x) \le \tilde{M}(t,y;x) = \tilde{S}^{c_t}(x,y) - c_t t.$$

We proceed by showing that

$$S^c(x,y) = \tilde{S}^c(x,y), \quad c > c_L. \tag{4.4}$$

To prove (4.4) take $c > c_L$. The inequality $\tilde{S}^c(x,y) \ge S^c(x,y)$ is trivial, so it is sufficient to show $\tilde{S}^c(x,y) \le S^c(x,y)$. Suppose, on the contrary, that there exist $\epsilon > 0$, $T > 0$ and an absolutely continuous path ψ with $\psi(0) = x$ and $\psi(T) = y$ such that

$$\int_0^T \Big(c + L(\psi(s),\dot{\psi}(s))\Big)\, ds \le \tilde{S}^c(x,y) - \epsilon.$$

Let

$$\psi^*(s) = \sup_{0\le u\le s}\psi(u) = \int_0^s \dot{\psi}(u)\vee 0\, ds,$$

and let $B_\psi = \{s > 0 : \psi^*(s) = \psi(s) \text{ and } \dot{\psi}(s) > 0\}$ be the points of increase of ψ. Then

$$\begin{aligned}
\int_0^T \Big(c + L(\psi(s),\dot{\psi}(s))\Big)\, ds &= \int_{B_\psi} \Big(c + L(\psi(s),\dot{\psi}(s))\Big)\, ds + \int_{B_\psi^c} \Big(c + L(\psi(s),\dot{\psi}(s))\Big)\, ds \\
&= \int_{B_\psi} \Big(c + L(\psi^*(s),\dot{\psi}^*(s))\Big)\, ds + \int_{B_\psi^c} \Big(c + L(\psi(s),\dot{\psi}(s))\Big)\, ds \\
&\ge \tilde{S}^c(x,y) + \int_{B_\psi^c} \Big(c + L(\psi(s),\dot{\psi}(s))\Big)\, ds.
\end{aligned}$$

Consequently,

$$\int_{B_\psi^c} \Big(c + L(\psi(s),\dot{\psi}(s))\Big)\, ds \le -\epsilon.$$

It follows that ψ has some excursion with negative cost. Repeating this excursion N times implies that

$$S^c(x,y) \le \tilde{S}^c(x,y) + N\int_{S_\psi^c} \Big(c + L(\psi(s),\dot{\psi}(s))\Big)\, ds \le \tilde{S}^c(x,y) - N\epsilon.$$

Letting $N \to \infty$ implies $S^c(x,y) = -\infty$, which contradicts $c > c_L$. We conclude that $S^c(x,y) = \tilde{S}^c(x,y)$ for $c > c_L$. This proves (4.4).

Next we show that

$$t < t_L \text{ implies } c_t > c_L. \tag{4.5}$$

To see this, suppose $t < t_L$. By (4.4) it follows that

$$\lim_{c\downarrow c_L} \partial_{c+}\tilde{S}^c(x,y) = \lim_{c\downarrow c_L} \partial_{c+} S^c(x,y) = t_L > t.$$

Since $c \mapsto \tilde{S}^c(x,y) - ct$ is concave with with supremum at c_t, it follows that

$$\partial_{c-}\tilde{S}^{c_t}(x,y) \geq t \geq \partial_{c+}\tilde{S}^{c_t}(x,y),$$

and, furthermore, that

$$t_L > t \geq \partial_{c+}\tilde{S}^{c_t}(x,y).$$

Concavity of $c \mapsto \tilde{S}^c(x,y)$ implies that $c \mapsto \partial_{c+}\tilde{S}^{c_t}(x,y)$ is non-increasing and we conclude that $c_t > c_L$. This proves (4.5).

The proof of (4.2) is completed by combining (4.3), (4.4) and (4.5). Indeed, with $x < y$ and $t < t_L$, by (4.3)

$$M(t,y;x) \leq \tilde{S}^{c_t}(x,y) - c_t t.$$

By (4.5) it follows that $c_t > c_L$ and finally (4.4) shows that

$$M(t,y;x) \leq \tilde{S}^{c_t}(x,y) - c_t t \leq S^{c_t}(x,y) - c_t t.$$

This completes the proof of (4.2).

Suppose $S^{c_L} > -\infty$. Then, by similar arguments, (4.4) holds for $c \geq c_L$ and (4.5) can be restated as

$$t \leq t_L \text{ implies } c_t \geq c_L,$$

which implies that (4.2) holds for all $t \leq t_L$. To prove the final statement, take $x \in \mathcal{A}$, $y \in \mathbb{R}$ and $t \geq t_L$. Since $x \in \mathcal{A}$ it follows that $L(x,0) = -c_L$ and therefore,

$$\begin{aligned} M(t,y;x) &\leq M(t-t_L,x;x) + M(t_L,y;x) \\ &\leq \int_0^{t-t_L} L(x,0)ds + M(t_L,y;x) \\ &= -c_L(t-t_L) + M(t_L,y;x). \end{aligned}$$

The proof is completed by showing

$$M(t_L,y;x) = S^{c_L}(x,y) - c_L t_L.$$

Since $S^{c_L} > -\infty$ it follows that

$$M(t_L,y;x) = \sup_{c \geq c_L}\{S^c(x,y) - ct_L\}.$$

As $c \mapsto \partial_{c+}S^c(x,y)$ is non-decreasing and, by definition of t_L,

$$\partial_{c+}S^c(x,y) - t_L \leq 0, \quad c > c_L$$

it follows that the concave function $c \mapsto S^c(x,y) - ct_L$ achieves its maximum over $[c_L,\infty)$ at c_L. Consequently

$$M(t_L,y;x) = S^{c_L}(x,y) - c_L t_L.$$

If instead $y \in \mathcal{A}$ and $x \in \mathbb{R}$, then, similarly

$$\begin{aligned} M(t,y;x) &\le M(t_L,y;x) + M(t-t_L,y;y) \\ &\le M(t_L,y;x) + \int_0^{t-t_L} L(x,0)ds \\ &= M(t_L,y;x) - c_L(t-t_L) \\ &= S^{c_L}(x,y) - c_L t. \end{aligned}$$

This completes the proof. □

Remark 4.2. For the set $\mathcal{A}$ we have the representation $\mathcal{A} = \{x : L(x,0) = -c_L\} = \{x : \inf_p H(x,p) = c_L\}$. In many of the examples considered in this paper this set is identical to the projected Aubry set, see for example [14, 13].

Example 4.3. Consider the Lagrangian

$$L(x,v) = \frac{v^2}{2} + \frac{x^2}{2},$$

with convex conjugate

$$H(x,p) = \frac{p^2}{2} - \frac{x^2}{2}. \tag{4.6}$$

Suppose that $x > 0$ and $y > x$. The claim is that for t sufficiently large, $M(t,y;x) > U(t,y;x)$.

The Euler-Lagrange equation associated with M is

$$\psi(s) - \frac{d}{ds}\dot{\psi}(s) = 0,$$

with boundary conditions $\psi(0) = x$, $\psi(t) = y$. The solution to this ODE is given by

$$\psi(s) = \frac{y - xe^{-t}}{2\sinh(t)}e^s + \frac{xe^t - y}{2\sinh(t)}e^{-s},$$

and the associated time derivative is

$$\dot{\psi}(s) = \frac{y - xe^{-t}}{2\sinh(t)}e^s - \frac{xe^t - y}{2\sinh(t)}e^{-s}.$$

It follows that the cost associated with this (optimal) trajectory is

$$M(t,y;x) = \frac{1}{2}\int_0^t \left(\psi(s)^2 + \dot{\psi}(s)^2\right) ds = \frac{x^2+y^2}{2}\frac{\cosh(t)}{\sinh(t)} - \frac{xy}{\sinh(t)}.$$

The Mañé potential is given by

$$\begin{aligned} S^c(x,y) &= \int_x^y \operatorname{sign}(z-x)\sqrt{z^2+2c}dz \\ &= \frac{1}{2}\left(y\sqrt{2c+y^2} - x\sqrt{2c+x^2} + 2c\log\left(\frac{\sqrt{2c+y^2}+y}{\sqrt{2c+x^2}+x}\right)\right). \end{aligned}$$

One way to see this is via Proposition 2.2. The stationary Hamilton-Jacobi equation takes the form

$$\frac{DS^c(x,y)^2}{2} - \frac{y^2}{2} = c,$$

where D denotes gradient with respect to y. The expression for $S^c(x,y)$ is obtained by solving for $DS^c(x,y)$, integrating from x to y, and using that S^c is the maximal subsolution, see Proposition 2.2(ii). From (2.6) it follows that $c_L \geq 0$ and it is easy to check that equality holds in this case, that is, $c_L = 0$.

To illustrate that the duality of Theorem 4.1 does not necessarily hold for $t > t_L(x,y)$, we compare the derived expressions for $M(t,y;x)$ and $U(t,y;x) = \sup_{c>c_L}\{S^c(x,y) - ct\}$ for specific choices of x, y, and t; the optimization over c in $U(t,y;x)$ is solved numerically. Figure 1 shows U and M as functions of t for $x = 0.5, y = 1$. Note that this is an arbitrary choice of x and y and it is easily checked that similar characteristics appear for other choices. A closer look at U and M for this particular choice of x and y reveals that the

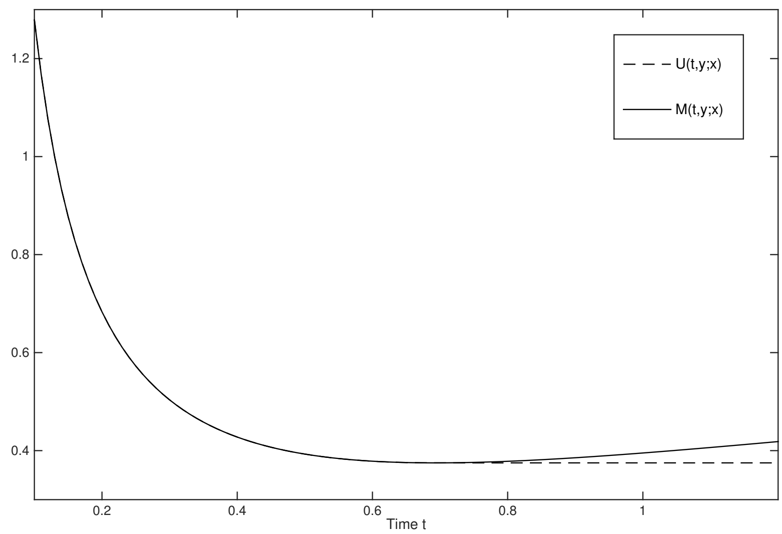

FIGURE 1. $U(t,y;x)$ and $M(t,y;x)$ for $x = 0.5$, $y = 1$.

duality ceases to hold at (roughly) $t \approx 0.6931$.

For the specific choice (4.6) of Hamiltonian H, $S^c(x,y)$ is clearly continuous in c and it is elementary to compute $\partial_{c+}S^c(x,y) = (\partial/\partial c)S^c(x,y)$ as well as the (right-hand) limit as $c \to c_L$:

$$\begin{aligned} t_L(x,y) &= \lim_{c\to c_L} \partial_{c+} S^c(x,y) \\ &= \frac{y}{2\sqrt{2c_L+y^2}} - \frac{x}{2\sqrt{2c_L+x^2}} + \log\left(\frac{y+\sqrt{2c_L+y^2}}{x+\sqrt{2c_L+x^2}}\right) \\ &\quad + c_L\left(\frac{1}{2c_L+y^2+y\sqrt{2c_L+y^2}} - \frac{1}{2c_L+x^2+x\sqrt{2c_L+x^2}}\right). \end{aligned}$$

For the choice $x = 0.5, y = 1$ the limit is $t_L \approx 0.6931$, and we have already seen that $M(t,y;x) > U(t,y;x)$ for $t > 0.6931$. This illustrates that the duality can cease to hold for $t > t_L(x,y)$.

Next, consider the choice $x = 0 \in \mathcal{A}$. Then

$$M(t,y;0) = \frac{y^2}{2}\frac{\cosh(t)}{\sinh(t)},$$

and

$$S^c(0,y) = \frac{y}{2}\sqrt{2c+y^2} + c\log\left(\frac{y}{\sqrt{2c}} + \sqrt{\frac{y^2}{2c}+1}\right).$$

It is straightforward to check that in this case $M(t,y;0)$ and $U(t,y;0)$ agree for all choices of y and $t > 0$. In fact, differentiating $S^c(0,y)$ with respect to c reveals that $t_L = \infty$ in this case.

We end this section by computing the Mañé potential for two examples of the Hamiltonian H connected to common stochastic processes: critical diffusion and birth-and-death process (see Examples 2.1-2.2).

Example 4.1 (Critical diffusion process)**.** Consider the Hamiltonian $H(y,p) = b(y)p + \frac{1}{2}|p|^2$, where $b(y) = -DU(y)$ for some potential function $U : \mathbb{R} \to \mathbb{R}$ such that $c_H = 0$. The function $y \mapsto S^c(x,y)$ is a viscosity solution to $H(y, DS(y)) = c$ for all $y \neq x$ and all solutions p to this equation are of the form

$$p(y) = \Big(DU(y) \pm \sqrt{DU(y)^2 + 2c}\,\Big).$$

$\bar{S}^c(x,\cdot)$ is a primitive function of p, and the maximal of all subsolutions vanishing at x, see Proposition 2.2(ii). Therefore the $\pm$ sign must be selected as $\operatorname{sign}(z-x)$. Consequently, the Mañé potential is given by

$$\bar{S}^c(x,y) = \int_x^y \Big(DU(z) + \operatorname{sign}(z-x)\sqrt{DU(z)^2+2c}\,\Big)dz.$$

Example 4.2 (Birth-and-death process)**.** We consider the setting of Example 2.2: $(a,b) \subset \mathbb{R}$ and $\mu : (a,b) \to [0,\infty)$ and $\lambda : (a,b) \to [0,\infty)$ satisfy $\int_a^b \log(\sqrt{\mu(y)/\lambda(y)})dy < \infty$. The Hamiltonian of a birth-and-death process with intensity functions λ and μ is given by

$$H(y,p) = \lambda(y)(e^p - 1) + \mu(y)(e^{-p} - 1).$$

To compute the Mañé potential, observe that in this example the function

$$p^c(y) = \log\left[\frac{c+\lambda(y)+\mu(y)}{2\lambda(y)} \pm \sqrt{\Big(\frac{c+\lambda(y)+\mu(y)}{2\lambda(y)}\Big)^2 - \frac{\mu(y)}{\lambda(y)}}\,\right],$$

is the solution to $H(y, p^c(y)) = c$. The Mañé potential $y \mapsto \bar{S}^c(x,y)$ is a primitive function of p^c, and the maximal of all viscosity subsolutions vanishing at x, see Proposition 2.2. Therefore the $\pm$ sign must be taken as positive for trajectories to the right, $y > x$, and negative for trajectories to the left, $y < x$. Consequently, the Mañé potential is given by

$$\bar{S}^c(x,y) = \int_x^y \log\left[\frac{c+\lambda(z)+\mu(z)}{2\lambda(z)} + \operatorname{sign}(z-x)\sqrt{\Big(\frac{c+\lambda(z)+\mu(z)}{2\lambda(z)}\Big)^2 - \frac{\mu(z)}{\lambda(z)}}\,\right] dz.$$

5. Min-max representation of viscosity solutions

In this section we demonstrate how the duality in Theorem 4.1 leads to min-max representations of viscosity solutions of initial value problems, terminal value problems and problems on domains. Since Theorem 4.1 is established in the one-dimensional setting, all the results of this section are also be restricted to the one-dimensional setting. The starting point is the evolutionary Hamilton-Jacobi equation

$$V_t(t,y) + H(y, DV(t,y)) = 0, \quad (t,y) \in (0,\infty)\times\mathbb{R}, \tag{5.1}$$

which is (3.2) with $n = 1$.

5.1. **Min-max representation for initial value problems.** Given an initial function $g : \mathbb{R} \to \mathbb{R}$, the initial value problem for the Hamilton-Jacobi equation is to find $V : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ such that

$$V \text{ satisfies (5.1) and } V(0,y) = g(y),\ y \in \mathbb{R}. \tag{5.2}$$

A continuous function $V : (0,\infty)\times\mathbb{R} \to \mathbb{R}$ is a viscosity subsolution (supersolution) if it is a viscosity subsolution (supersolution) to (3.2) and $V(0,y) \le g(y)$ ($\ge g(y)$).

If V is uniformly continuous and H satisfies Condition 5.1 below, then the comparison principle holds and the solution of the initial value problem is unique, see e.g. Theorem 3.7 and Remark 3.8 in Chapter II of [1].

Condition 5.1. *H is uniformly continuous on $\mathbb{R}^d \times B_0(R)$ for each $R > 0$ and*

$$|H(x,p) - H(y,p)| \le \omega(|x-y|(1+|p|)), \text{ for } x,y,p \in \mathbb{R}^d,$$

where $B_0(R) = \{p \in \mathbb{R}^n : |p| < R\}$ and $\omega : [0,\infty) \to [0,\infty)$ is a continuous nondecreasing function with $\omega(0) = 0$.

Moreover, viscosity solutions can be given a variational representation; we state this representation on $\mathbb{R}$ but it holds for arbitrary dimension d. Given an initial function $g : \mathbb{R} \to \mathbb{R}$, let V be the value function of the variational problem

$$V(t,y) = \inf_\psi \Big\{ g(\psi(0)) + \int_0^t L(\psi(s), \dot{\psi}(s))ds, \psi(t) = y \Big\}, \tag{5.3}$$

where $(t,y)\in[0,\infty)\times\mathbb{R}$ and the infimum is taken over all absolutely continuous functions $\psi:[0,\infty)\to\mathbb{R}$. It is well known that if V is continuous, then it is a continuous viscosity solution to (5.2), see e.g. [1, Ch. III, Sec. 3].

From Theorem 4.1 the following min-max representation of V is obtained.

Proposition 5.1. *Suppose that L and H are as in* (2.1)*, with $d=1$, and V is given by* (5.3)*. If either $y\in\mathcal{A}$ and $t>0$ or $y\in\mathbb{R}$ and $t<t_L$, then*

$$V(t,y)=\inf_x\sup_{c>c_L}\{g(x)+S^c(x,y)-ct\}. \tag{5.4}$$

Moreover, if V is continuous, then it is a viscosity solution to (5.2) *on* $[0,t_L)\times\mathbb{R}$.

Proof. It follows from (4.2) that

$$V(t,y)=\inf_x\{g(x)+M(t,y;x)\}=\inf_x\sup_{c>c_L}\{g(x)+S^c(x,y)-ct\}.$$

□

5.2. **Min-max representation for terminal value problems.** Let the following be given: A time $T>0$, a Lagrangian $\bar{L}$ and Hamiltonian $\bar{H}$ as in Section 4, and a terminal cost function g. Consider a terminal value problem with value function

$$\bar{V}(t,x)=\inf_\psi\left\{\int_t^T\bar{L}(\psi(s),\dot{\psi}(s))ds+g(\psi(T)),\psi(t)=x\right\},$$

where the infimum is taken over all absolutely continuous functions ψ on $[0,T]$, with $\psi(t)=x$. By changing the direction of the paths it follows that $\bar{V}(t,x)$ is equal to

$$\inf\left\{g(\psi(0))+\int_0^{T-t}\bar{L}(\psi(s),-\dot{\psi}(s))ds,\psi(T-t)=x\right\}=V(T-t,x),$$

where V is the value function of the forward problem (5.3) with $L(x,v)=\bar{L}(x,-v)$.

For $c>c_{\bar{L}}$, let $\bar{S}^c(x,y)$ denote the Mañé potential associated with $\bar{L}$. Then, it holds that $\bar{S}^c(x,y)=S^c(y,x)$ and the min-max representation of Proposition 5.1 can be expressed as

$$\begin{aligned}\bar{V}(t,x)=V(T-t,x)&=\inf_y\sup_{c>c_{\bar{L}}}\{g(y)+S^c(y,x)-c(T-t)\}\\&=\inf_y\sup_{c>c_{\bar{L}}}\{g(y)+\bar{S}^c(x,y)-c(T-t)\},\end{aligned}$$

if either $x\in\mathcal{A}$ and $t>0$ or $x\in\mathbb{R}^n$ and $T-t<t_L$.

The Hamiltonian of the corresponding forward problem is

$$H(x,p)=\sup_v\{\langle p,v\rangle-L(x,v)\}=\sup_v\{\langle -p,-v\rangle-\bar{L}(x,-v)\}=\bar{H}(x,-p).$$

If $\bar{V}$ is continuous, then so is V and since V is a continuous viscosity solution to (5.2) on $[0,t_L)$ it follows that $\bar{V}$ is a continuous viscosity solution to

$$\begin{cases}\bar{V}_t(t,x)-\bar{H}(x,-D\bar{V}(t,x))=0, & (t,x)\in[0,T)\times\mathbb{R},\\ \bar{V}(T,x)=g(x), & x\in\mathbb{R},\end{cases}$$

if $T\le t_L$.

In general, it is not possible to interchange the inf and sup in the min-max representation as the following example shows. More generally, results such as Sion's general minimax theorem can be applied on a case-by-case basis to check whether or not interchanging inf and sup is allowed.

Example 5.1. Consider a one-dimensional terminal value problem, with Hamiltonian $\bar{H}(x,p) = \bar{H}(p) = p + \frac{1}{2}p^2$ and $g(x) = 0$ on $\partial(a,b)$ and $g(x) = \infty$ on (a,b), where $a < 1 < b$ and $b - 1 < 1 - a$. Since $\bar{H}$ does not depend on x we have $\mathcal{A} = \mathbb{R}$. The Mañé critical value is $c_{\bar{H}} = c_L = -1/2$ and the Mañé potential is given by

$$\bar{S}^c(x,y) = \begin{cases} (y-x)(-1+\sqrt{1+2c}), & y \geq x, \\ (x-y)(1+\sqrt{1+2c}), & y < x. \end{cases}$$

By performing the optimization it follows that

$$\sup_{c > c_{\bar{L}}} \{\bar{S}^c(x,y) - c(T-t)\} = \begin{cases} \frac{T-t}{2}(\frac{y-x}{T-t} - 1)^2, & y \geq x, \\ \frac{T-t}{2}(\frac{x-y}{T-t} - 1)^2, & y < x. \end{cases}$$

and, for $x < a$, we have

$$\bar{V}(t,x) = \inf_{y \in \{a,b\}} \sup_{c > c_{\bar{L}}} \{\bar{S}^c(x,y) - c(T-t)\} = \inf_{y \in \{a,b\}} \frac{T-t}{2}\Big(\frac{y-x}{T-t} - 1\Big)^2.$$

In particular, with $T = 1$, we have

$$\bar{V}(0,0) = \inf_{y \in \{a,b\}} \frac{1}{2}(y-1)^2 = \frac{1}{2}(b-1)^2.$$

Consider interchanging the order of the inf and sup. For any $c > c_{\bar{L}}$ the infimum over the boundary is

$$\inf_{y \in \{a,b\}} \{\bar{S}^c(0,y) - c\} = \begin{cases} a(-1+\sqrt{1+2c}) - c, & \text{for } c \geq 0, \\ b(-1+\sqrt{1+2c}) - c, & \text{for } c < 0. \end{cases}$$

An elementary calculation shows that $\sup_{c > c_L} \inf_{y \in \{a,b\}} \{\bar{S}^c(0,y) - c\}$ is equal to

$$\Big(\sup_{c \geq 0}\{a(-1+\sqrt{1+2c}) - c\}\Big) \vee \Big(\sup_{c<0}\{b(-1+\sqrt{1+2c}) - c\}\Big) = 0.$$

We conclude that

$$\bar{V}(0,0) = \inf_{y \in \{a,b\}} \sup_{c > c_{\bar{L}}} \{\bar{S}^c(0,y) - c\} > \sup_{c > c_{\bar{L}}} \inf_{y \in \{a,b\}} \{\bar{S}^c(0,y) - c\}.$$

5.3. **Min-max representation for problems on domains.** Let $\Omega := (a,b) \subset \mathbb{R}$ be an open interval, $\partial\Omega := \{a,b\}$, $g : \partial\Omega \to \mathbb{R}$ a function representing the boundary condition and, for $(t,y) \in (0,\infty) \times (a,b)$, let

$$V(t,y) = \inf_{\psi} \Big\{ g(\psi(0)) + \int_0^t L(\psi(s), \dot{\psi}(s))ds, \psi(0) \in \partial\Omega, \psi(t) = y \Big\},$$

where the infimum is taken over all absolutely continuous functions $\psi : [0,\infty) \to \overline{\Omega}$, with $\psi(0) \in \partial\Omega$ and $\psi(t) \in \Omega$, $t > 0$. If either $y \in \mathcal{A}$ and $t > 0$ or $y \in \mathbb{R}$ and $t < t_L$, then the min-max representation is given by

$$V(t,y) = \inf_{x \in \{a,b\}} \sup_{c > c_L} \{g(x) + S^c(x,y) - ct\}. \tag{5.5}$$

If V is continuous, then it is a continuous viscosity solution to

$$\begin{cases} V_t(t,y) + H(y, DV(t,y)) = 0, & (t,y) \in (0,\infty) \times \Omega, \\ V(0,y) = g(y), & y \in \partial\Omega. \end{cases}$$

Similarly, the terminal value problem on Ω is

$$\bar{V}(t,x) = \inf \left\{ \int_t^T \bar{L}(\psi(s), \dot{\psi}(s))ds + g(\psi(T)), \psi(t) = x, \psi(T) \in \partial\Omega \right\},$$

where $(t,x) \in [0,T) \times \Omega$. If either $x \in \mathcal{A}$ and $t > 0$ or $x \in \mathbb{R}$ and $T - t < t_L$, then the min-max representation is given by

$$\bar{V}(t,x) = \inf_{y \in \partial\Omega} \sup_{c > c_{\bar{L}}} \{g(y) + \bar{S}^c(x,y) - c(T-t)\} \tag{5.6}$$

If $\bar{V}$ is continuous, then it is a continuous viscosity solution to

$$\begin{cases} \bar{V}_t(t,x) - \bar{H}(x, -D\bar{V}(t,x)) = 0, & (t,x) \in [0,T) \times \Omega, \\ \bar{V}(T,x) = g(x), & x \in \partial\Omega. \end{cases} \tag{5.7}$$

5.4. **Min-max representation for exit problems.** Let $\Omega : (a,b) \subset \mathbb{R}$ be an open interval, $\partial\Omega := \{a,b\}$, $g : \partial\Omega \to \mathbb{R}$ be the boundary condition and take $T > 0$. Consider the minimal cost $\bar{W}$ of leaving the interval before time T, when starting from $(t,x) \in [0,T) \times \Omega$. The function $\bar{W}$ is given by

$$\bar{W}(t,x) = \inf_{\psi,\sigma} \left\{ \int_t^\sigma \bar{L}(\psi(s), \dot{\psi}(s))ds + g(\psi(\sigma)), \psi(t) = x, \psi(\sigma) \in \partial\Omega \right\},$$

where $t \le \sigma \le T$. By the change of variables, $\tau = T - \sigma + t$, and, for $t \le s \le T$, $\varphi(s) = \psi(t + s - \tau)$.

$$\begin{aligned} \bar{W}(t,x) &= \inf_{\psi, t \le \tau \le T} \left\{ \int_\tau^T \bar{L}(\varphi(s), \dot{\varphi}(s))ds + g(\varphi(T)), \varphi(\tau) = x, \varphi(T) \in \{a,b\} \right\} \\ &= \inf_{t \le \tau \le T} \bar{V}(\tau, x), \quad (t,x) \in [0,T) \times (a,b), \end{aligned}$$

with $\bar{V}$ as in (5.7).

If either $x \in \mathcal{A}$ or $T < t_{\bar{L}}$, then $\bar{W}$ can be represented as

$$\bar{W}(t,x) = \inf_{t \le \tau \le T} \inf_{y \in \partial\Omega} \sup_{c > c_{\bar{L}}} \{g(y) + \bar{S}^c(x,y) - c(T-\tau)\} \tag{5.8}$$

Obviously $\bar{W}(t,x) \le \bar{V}(t,x)$. If $c_{\bar{L}} \ge 0$, then it follows that

$$\begin{aligned} \bar{W}(t,x) &= \inf_{t \le \tau \le T} \inf_{y \in \partial\Omega} \sup_{c > c_{\bar{L}}} \{g(y) + \bar{S}^c(x,y) - c(T-\tau)\} \\ &\ge \inf_{y \in \partial\Omega} \sup_{c > c_{\bar{L}}} \inf_{t \le \tau \le T} \{g(y) + \bar{S}^c(x,y) - c(T-\tau)\} \\ &\ge \inf_{y \in \partial\Omega} \sup_{c > c_{\bar{L}}} \{g(y) + \bar{S}^c(x,y) - c(T-t)\} \\ &= \bar{V}(t,x). \end{aligned}$$

We have proved the following.

Proposition 5.2. *If $c_{\bar{L}} \geq 0$ and either $x \in \mathcal{A}$ or $T < t_{\bar{L}}$, then $\bar{W}(t,x) = \bar{V}(t,x)$, $0 \leq t \leq T$.*

Note also that if $\bar{W}$ is continuous, then it is a continuous viscosity solution to

$$\begin{cases} \bar{W}_t(t,x) - \bar{H}(x, -D\bar{W}(t,x)) = 0, & (t,x) \in [0,T) \times \Omega, \\ \bar{W}(t,x) = g(x), & (t,x) \in [0,T] \times \partial\Omega. \end{cases} \tag{5.9}$$

6. The Hopf-Lax-Oleinik representation

Suppose the Hamiltonian H is state-independent, $H(x,p) = H(p)$, and convex. Then $\mathcal{A} = \mathbb{R}^d$ ($d \geq 1$). If g is uniformly continuous, then the Hopf-Lax-Oleinik representation, see [12, Ch. X], states that the function

$$V(t,y) = \inf_x \Big\{ g(x) + tL\Big(\frac{y-x}{t}\Big)\Big\}, \tag{6.1}$$

is the unique continuous viscosity solution to

$$\begin{cases} V_t(t,y) + H(DV(t,y)) = 0, & (t,y) \in (0,\infty) \times \mathbb{R}^d, \\ V(0,y) = g(y), & y \in \mathbb{R}^d. \end{cases}$$

We will demonstrate a direct relation between the Hopf-Lax-Oleinik representation and n-dimensional versions of the duality theorem and min-max representation (5.4) (available due to the state-independence of the Hamiltonian).

Proposition 6.1. *If H is convex and state-independent, then, for all $y \in \mathbb{R}^d$,*

$$\sup_{c > c_L} \{S^c(x,y) - ct\} = tL\Big(\frac{y-x}{t}\Big).$$

Moreover, if the initial function g is uniformly continuous, then

$$V(t,y) = \inf_x \sup_{c > c_L} \{g(x) + S^c(x,y) - ct\} = \inf_x \Big\{ g(x) + tL\Big(\frac{y-x}{t}\Big)\Big\}.$$

is the unique continuous viscosity solution to (5.2).

Proof. We begin by proving the folowing inequality: for each x,

$$\sup_{c > c_L} \{S^c(x,y) - ct\} \geq tL\Big(\frac{y-x}{t}\Big).$$

Take $x \in \mathbb{R}^d$, $c > c_L$ and observe that for p such that $H(p) = c$

$$\begin{aligned} S^c(x,y) &= \inf_{\psi,t} \Big\{ \int_0^t H(p) + L(\dot{\psi}(s)) ds, \psi(0) = x, \psi(t) = y \Big\} \\ &\geq \inf_{\psi,t} \Big\{ \int_0^t \langle p, \dot{\psi}(s) \rangle ds, \psi(0) = x, \psi(t) = y \Big\} \\ &= \langle p, y - x \rangle, \end{aligned}$$

where the inequality holds due to the convex conjugacy between L and H. It follows that

$$\begin{aligned} S^c(x,y) - ct &\geq \sup_{p:H(p)=c} \{\langle p, y-x\rangle - tH(p)\} \\ &= t \sup_{p:H(p)=c} \Big\{\langle p, \frac{y-x}{t}\rangle - H(p)\Big\}. \end{aligned}$$

By Proposition 2.2, $S^c(x,y) = -\infty$ for $c < c_L$, which implies that the supremum over $c > c_L$ can be extended to the whole of $\mathbb{R}$. That is,

$$\begin{aligned} \sup_{c>c_L}\{S^c(x,y) - ct\} &= \sup_{c\in\mathbb{R}}\{S^c(x,y) - ct\} \\ &\geq t \sup_{c\in\mathbb{R}} \sup_{p:H(p)=c} \Big\{\langle p, \frac{y-x}{t}\rangle - H(p)\Big\} \\ &= tL\Big(\frac{y-x}{t}\Big). \end{aligned}$$

The reverse inequality

$$\sup_{c>c_L}\{S^c(x,y) - ct\} \leq tL\Big(\frac{y-x}{t}\Big),$$

follows immediately by taking $\dot{\psi}(s) = (y-x)/t$ and observing that

$$S^c(x,y) \leq \int_0^t \Big(c + L(\dot{\psi}(s))\Big)\, ds = \Big[c + L(\frac{y-x}{t})\Big]t.$$

□

Remark 6.1. When H is state-independent the action functional reduces to $tL((y-x)/t)$, and Proposition 6.1 is the d-dimensional version of Theorem 4.1 for this setting.

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