Negative Dependence and
the Geometry of Polynomials

Petter Brändén (KTH)

joint with
Julius Borcea (SU) and
Thomas M. Liggett (UCLA)
Notation

- $P$ is a probability measure on $\mathbb{N}^E$ where $E$ is a finite set and $P$ has finite support.

- Identify $\{0, 1\}^E$ with $\{S : S \subseteq E\}$

- $X$ is the identity random variable:

$$P[X = \alpha] = P[\{\alpha\}]$$

- The generating polynomial of $P$ is the multivariate polynomial in the variables $z_e, e \in E$

$$g_P(z) = \mathbb{E}[z^\alpha] = \sum_{\alpha \in \mathbb{N}^E} P[X = \alpha]z^\alpha$$

where

$$z^\alpha := \prod_{s \in E} z_s^\alpha(s)$$
Spanning trees and Rayleigh monotonicity

Let $G = (V, E)$ be a connected graph

Spanning tree polynomial

$$T(G) = \sum_{T \subseteq E} z^T, \quad z^T := \prod_{e \in T} z_e$$

summed over all spanning trees

**Theorem** [Kirchhoff].

Effective conductance relative to $a, b \in V$, $i = ab \in E$:

$$C_{ab}(z) = \frac{T(G; z)}{\frac{\partial T(G; z)}{\partial z_i}}$$

$C_{ab}(z)$ is monotone increasing in $z_j$ when $z_j > 0$:

$$\frac{\partial T}{\partial z_i} \cdot \frac{\partial T}{\partial z_j} - \frac{\partial T}{\partial z_i \partial z_j} \cdot \frac{T}{(\frac{\partial T}{\partial z_i})^2} = \frac{\partial}{\partial z_j} \left( C_{ab}(z) \right) \geq 0$$
Rayleigh polynomials and measures

**Rayleigh polynomial**
Square free polynomial \( f(z_1, \ldots, z_n) \) with nonnegative coefficients satisfying

\[
\frac{\partial f}{\partial z_i} \cdot \frac{\partial f}{\partial z_j} - \frac{\partial f}{\partial z_i \partial z_j} \cdot f \geq 0,
\]
for all \( 1 \leq i, j \leq n \) and \( z \in \mathbb{R}^n_+ \)

**Rayleigh measure (h-NLC+)**
Probability measure on \( \{0, 1\}^E \) with Rayleigh generating polynomial

\[
M := \# \text{ spanning trees in } G
\]

\[
P[X = S] = \begin{cases} 
1/M & \text{if } S \text{ spanning tree} \\
0 & \text{otherwise}
\end{cases}
\]

\( P \) is the **uniform random spanning tree measure**

**Fact.**
The uniform spanning tree measure is Rayleigh
Pairwise negative correlation

\[ \frac{\partial g_P}{\partial z_e} = \sum_{S \ni e} P[X = S] z^{S \setminus \{e\}} \]

\[ \frac{\partial g_P}{\partial z_e}(1, \ldots, 1) = P[e \in X] \]

**Pairwise Negative Correlation**

Rayleigh Monotonicity \(\implies\)

\[ P[e, f \in X] \leq P[e \in X] \cdot P[f \in X], \quad \forall e \neq f \in E \]
Negative association

**Negative Association (NA)**

\[ \mathbb{E}[FG] \leq \mathbb{E}[F] \cdot \mathbb{E}[G] \]

for all increasing (w.r.t. product order on \( \mathbb{N}^E \)) functions \( F, G : \mathbb{N}^E \to \mathbb{R} \) depending on disjoint set of variables

Take \( \{0, 1\}^E \) and

\[ X_e(S) = \begin{cases} 1 & \text{if } e \in S, \\ 0 & \text{if } e \notin S \end{cases} \]

\[ \mathbb{E}[X_e] = \mathbb{P}[e \in X], \quad \mathbb{E}[X_eX_f] = \mathbb{P}[e, f \in X] \]

**Theorem**[Feder-Mihail].
The uniform spanning tree measure is negatively associated
Positive dependence

Positive Association
\[ \mathbb{E}[FG] \geq \mathbb{E}[F] \cdot \mathbb{E}[G] \]
for all increasing \( F, G : \mathbb{N}^E \rightarrow \mathbb{R} \)

Positive Lattice Condition
\[ P[X = \alpha \lor \beta] \cdot P[X = \alpha \land \beta] \geq P[X = \alpha] \cdot P[X = \beta] \]

FKG Theorem
Positive Lattice Condition \( \implies \) Positive Association

Pemantle (and others): Under what conditions do we have Negative Association? We need a theory of Negative Dependence?
Rota’s philosophy

G.-C. Rota: ”The one contribution of mine that I hope will be remembered has consisted in just pointing out that all sorts of problems of combinatorics can be viewed as problems of location of the zeros of certain polynomials and in giving these zeros a combinatorial interpretation.”

**Stable Polynomial**

\[ f \in \mathbb{C}[z_1, \ldots, z_n] \text{ is stable if} \]
\[ \forall j, \Im(z_j) > 0 \implies f(z_1, \ldots, z_n) \neq 0 \]

**Example.** If \( f(z) \in \mathbb{R}[z] \), then \( f \) is stable iff all zeros of \( f \) are real

**Example.** Spanning tree polynomials are stable
Strongly Rayleigh measures

**Theorem**[B.]. Let $f \in \mathbb{R}[z_1, \ldots, z_n]$ be a square-free polynomial. Then $f$ is stable iff

$$\frac{\partial f}{\partial z_i} \cdot \frac{\partial f}{\partial z_j} - \frac{\partial f}{\partial z_i \partial z_j} \cdot f \geq 0, \quad \forall 1 \leq i, j \leq n \text{ and } z \in \mathbb{R}^n$$

Strongly Rayleigh measures

$\mathbb{P}$ is strongly Rayleigh if its generating polynomial is stable

Includes e.g. uniform spanning tree measures, product measures, balls-and-bins measures, determinantal measures
Closure properties and CNA+

Strongly Rayleigh g.p.’s are closed under

(1). taking limits

(2). partial differentiation (conditioning on $X_j = 1$)

(3). scaling of variables with positive numbers (external fields)

(4). setting variables equal to nonnegative numbers (projections and conditioning on $X_j = 0$)

$$\text{CNA}^+ := \text{NA} + (2) + (3) + (4)$$

**Theorem** [Borcea, B., Liggett].
Strongly Rayleigh measures are negatively associated, and hence CNA+
Example: balls and bins

$m$ balls and $n$ bins

Ball $i$ goes into bin $j$ with probability $p_{ij}$

$$g_P(z) = \prod_{i=1}^{m} (p_{i1}z_1 + \cdots + p_{in}z_n)$$

$$P'[X = \alpha] = \text{Prob}[\alpha_j \text{ balls in bin } j], \quad \alpha \in \mathbb{N}^n$$

$$g_{P'}(z) = \prod_{i=1}^{m} (p_{i1}z_1 + \cdots + p_{in}z_n)$$

Theorem [Dubhashi, Ranjan].

$P$ and $P'$ are negatively associated

Observation: $P$ and $P'$ are strongly Rayleigh
Example: determinantal measures

Let $A$ be an $n \times n$ matrix with $A(S) \geq 0, \ S \subseteq [n]$

$A(S) := \text{principal minor of } A \text{ obtained by deleting rows and columns indexed by } S$

<table>
<thead>
<tr>
<th>Determinantal measures</th>
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<tr>
<td>$P_A[X = S] = A(S)/\det(I + A), \ S \subseteq [n]$</td>
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Occurences:

**Number theory**: Montgomery, Conrey, ...

**Rep. theory and random permutations**: Johansson, Borodin-Okounkov-Olshanski-Reshetikhin, ...

**Probability theory**: Lyons-Steif, Peres, ...

**Mathematical physics**: Daley, Vere-Jones, ...
Hadamard-Fischer-Kotelyansky inequalities

Let \( A = (a_{ij}) \) be positive semidefinite

\[
\text{Hadamard}
\]
\[
det(A) \leq a_{11} \cdots a_{nn}
\]

\[
\text{Fischer}
\]
\[
det(A) \leq A(S) \cdot A(S^c)
\]

\[
\text{Kotelyansky}
\]
\[
A(S \cup T) \cdot A(S \cap T') \leq A(S') \cdot A(T')
\]

\[\uparrow\]

Theorem[R. Lyons].
If \( A \) is positive semidefinite then \( P_A \) is CNA+
Pencils of matrices

**Proposition.** Let $A_1, \ldots, A_m$ be positive semidefinite and $B$ Hermitian. Then

$$f(z_1, \ldots, z_n) = \det(z_1 A_1 + \ldots + z_n A_n + B)$$

is stable.

The generating polynomial of $P_A$ is

$$g_{P_A}(z) = C \sum_{S \subseteq [n]} A(S) z^S = C \cdot \det(Z + A),$$

where $Z = \text{diag}(z_1, \ldots, z_n)$, $C^{-1} = \det(I + A)$

**Corollary.** Let $A$ be positive semidefinite. Then $P_A$ is strongly Rayleigh.

This strengthens Lyons’ theorem
Proof of Strongly Rayleigh $\Rightarrow$ CNA+

May assume that $\mathbb{P}$ is defined on $\{0, 1\}^E$

**Theorem**[Feder-Mihail]. Let $S$ be a class of probability measures s.t.

- $S$ is closed under conditioning,

- $\mathbb{P}[i, j \in X] \leq \mathbb{P}[i \in X] \cdot \mathbb{P}[j \in X]$ for all $i \neq j$,

- the generating polynomial is homogeneous.

Then each $\mathbb{P} \in S$ is negatively associated

**Observation 1**: (Strongly) Rayleigh measures with homogeneous g.p.’s meet the requirements

**Observation 2**: Negative association is closed under projections

**Question.** Is every strongly Rayleigh measure the projection of a strongly Rayleigh measure with homogeneous generating polynomial?
Symmetric homogenization

Suppose that $P$ is a probability measure on $\{0, 1\}^n$. $P_H$ is the probability measure on $\{0, 1\}^{2n}$ defined by

$$P_H[X = S] := \begin{cases} \frac{P[X = S \cap [n]]}{\binom{n}{|S \cap [n]|}} & \text{if } |S| = n, \\ 0 & \text{otherwise} \end{cases}$$

$$g_{P_H} = \sum_{S \subseteq [n]} P[X = S] \binom{n}{|S|} z^{|S|} e_{n-|S|}(z_{n+1}, \ldots, z_{2n}),$$

where

$$e_j(z_1, \ldots, z_m) = \sum_{1 \leq i_1 < \cdots < i_j \leq m} z_{i_1} \cdots z_{i_j}$$

is the $j$th elementary symmetric polynomial.
Sketch of proof

**Theorem** [Borcea, B., Liggett]. If $P$ is strongly Rayleigh then so is $P_H$

**Proof.** Set $a_S := P[X = S]$

$$\sum_{S \subseteq [n]} a_S z^S \text{ is stable } \implies$$

\{ Using Gårding’s results on hyperbolicity cones \}

$$\sum_{S \subseteq [n]} a_S z^S y^{n-|S|} \text{ is stable } \implies$$

\{ Grace-Walsh-Szegö Coincidence Theorem \}

$$\sum_{S \subseteq [n]} a_S \binom{n}{|S|}^{-1} z^S e_{n-|S|}(z_{n+1}, \ldots, z_{2n}) \text{ is stable}$$

**Theorem** [Borcea, B., Liggett].
Strongly Rayleigh measures are negatively associated (CNA+)
The Laplacian of a graph

Fact: Let $A_1, \ldots, A_m$ be positive semidefinite and $B$ Hermitian. Then

$$f(z_1, \ldots, z_n) = \det(z_1 A_1 + \ldots + z_n A_n + B)$$

is stable.

Let $G = (V, E)$, $V = [n]$, be a graph, and $e$ an edge connecting $i < j$. Let $A_e$ be the $n \times n$ matrix with nonzero entries

$$(A_e)_{ii} = 1 \quad (A_e)_{ij} = -1$$

$$(A_e)_{ji} = -1 \quad (A_e)_{jj} = 1$$

The Laplacian of $G$ is

$$L(G) = \sum_{e \in E} w_e A_e$$
The matrix-tree theorem

\[ Z = \text{diag}(z_1, \ldots, z_n) \]

**Kirchoff’s Matrix-Tree Theorem**: Let \( G \) be a connected graph. Then

\[ \det(L(G)_{ii}) = \frac{\partial}{\partial z_i} \det(L(G) + Z) \Bigg|_{z=0} = \sum_T w^T \]

where the sum is over all spanning trees \( T \subseteq E \).

**All Minors Matrix-Tree Theorem**:

\[ \det(L(G) + Z) = \sum_F z^{\text{roots}(F)} w^{\text{edges}(F)} \]

where the sum is over rooted spanning forests \( F \subseteq E \).

**Fact**. The class of stable polynomials is closed under

1. Partial differentiation,
2. Specializing variables to real numbers,
3. Setting variables equal.
A positive answer to a conjecture of Wagner

Let $G = (V, E)$. For each spanning forest $\mathcal{F}$ let

$$\rho(\mathcal{F}) = \# \text{ of ways to root } \mathcal{F},$$

and define $\mathbb{P}$ on $2^E$ by (let $R = \sum_{\mathcal{F}} \rho(\mathcal{F})$)

$$\mathbb{P}[[S]] = \begin{cases} \rho(\mathcal{F})/R & \text{if } S \text{ is a spanning forest,} \\ 0 & \text{otherwise} \end{cases}$$

**Conjecture [Wagner].** $\mathbb{P}$ is Rayleigh.

**Proof.** The g. p. of $\mathbb{P}$ is

$$g = \sum_{S \subseteq E} \mathbb{P}[[S]]w^S = R^{-1} \text{det}(L(G) + I),$$

so $g$ is in fact stable.
A generalization

Let \( A \) be an \( m \times n \) matrix over \( \mathbb{C} \),

\[
\det(AW A^* + Z) = \sum_{S,T} |A(S, T)|^2 z^{S'} w^T
\]

\[
AW A^* = \sum_{k=1}^{m} w_k (a_{ik} \overline{a_{jk}})_{1 \leq i,j \leq r}
\]

Let \( A \in \mathbb{C}^{m \times n} \). The measure \( \mu \) on \( 2^{[m+n]} \) defined by

\[
\mu(S) = |A(S_1, S_2)|^2, \quad S_1 \sqcup (S_2 + m) = S
\]

has a stable generating function.
The symmetric exclusion process

Let $\tau = (ij)$ be a transposition and for $S \subseteq [n]$ let $\tau(S) = \{\tau(s) : s \in S\}$. For $0 \leq p \leq 1$ let

$$P^{\tau,p}[X = S] := pP[X = S] + (1-p)P[X = \tau(S)].$$

Theorem [Borcea, B., Liggett].

$P$ is strongly Rayleigh $\implies P^{\tau,p}$ is strongly Rayleigh

This solves the following conjecture to the affirmative

Conjecture [Pemantle, Liggett]. If the initial configuration of a symmetric exclusion process is deterministic, then the distribution at time $t$ is negatively associated for all $t \geq 0$. 
Stochastic domination

Let $\mu$ and $\nu$ be probability measures on the same partially ordered space $\Omega$ then

Stochastic domination

$\nu$ stochastically dominates $\mu$ ($\mu \leq \nu$) if

$$\int f \, d\mu \leq \int f \, d\nu$$

for all increasing functions $f : \Omega \rightarrow \mathbb{R}$

Let $A$ and $B$ be symmetric (hermitian) $n \times n$ matrices

Loewner order

$A \leq B$ if $B - A$ is positive semidefinite
Löwner order

**Theorem** [Lyons]. Suppose $A, B \geq 0$ and $AB = BA$. Then

$$A \geq B \implies P_A \leq P_B$$

**Theorem** [Borcea, B., Liggett]. Suppose $A, B \geq 0$. Then

$$A \geq B \implies P_A \leq P_B$$

Does it go the other way?
Log-concavity and rank-sequences

Let \((a_k)^n_{k=0}\) be a sequence of nonnegative numbers

\[
\begin{align*}
\text{Log- Concordave} & \\
& a_k^2 \geq a_{k-1}a_{k+1}
\end{align*}
\]

\[
\begin{align*}
\text{Ultra-Log-Concave (ULC)} & \\
& \frac{a_k^2}{\binom{n}{k}^2} \geq \frac{a_{k-1}}{\binom{n}{k-1}} \cdot \frac{a_{k+1}}{\binom{n}{k+1}} \\
i < j < k, a_i a_k \neq 0 \implies a_j \neq 0
\end{align*}
\]

The rank sequence of \(P\) on \(\{0, 1\}^n\) is \((a_k)^n_{k=0}\)

\[
a_k = P[|X| = k]
\]
Conjectures of Pemantle and Wagner

"Big Conjecture" [Pemantle, Peres, Wagner, ...].
\[
P \text{ is Rayleigh } \iff \left(a_k\right) \text{ is ultra-log-concave}
\]

Conjecture [Pemantle, Peres, ...].
P is CNA+ \iff P is ultra-log-concave

\[
g(z_1, \ldots, z_n) \text{ is stable } \iff
\]
\[
g(t, \ldots, t) \text{ is stable } \iff g(t, \ldots, t) \text{ is real-rooted}
\]

**Newton’s Inequalities**
Suppose that \( \sum_{k=0}^{n} a_k t^k \) is real-rooted. Then
\[
\frac{a_k^2}{\binom{n}{k}^2} \geq \frac{a_{k-1}}{\binom{n}{k-1}} \frac{a_{k+1}}{\binom{n}{k+1}}
\]

**Fact:** P is Strongly Rayleigh \( \iff \left(a_k\right) \text{ is ultra-log-concave} \)
Mason’s conjecture

Let $\mathcal{I} \subseteq \{0, 1\}^n$ be the independent sets of a matroid and

$$P_{\mathcal{I}}[X = S] := \begin{cases} 1/|\mathcal{I}| & \text{if } S \in \mathcal{I}, \\ 0 & \text{if } S \notin \mathcal{I} \end{cases}$$

**Conjecture**[Mason, -72].
The rank sequence of $P_{\mathcal{I}}$ is ULC

**Rayleigh Matroid**: $P_{\mathcal{I}}$ is Rayleigh

Series-parallell matroids are Rayleigh. Conjectured Rayleigh: graphic and regular matroids

**Wagner’s hope**: Big conjecture would prove Mason’s conjecture for Rayleigh-matroids
Reduction of the big conjecture

We could reduce the "Big Conjecture" to a small problem

**Problem 1.** Let \((a_k)_{k=0}^n\) and \((b_k)_{k=0}^n\) be two positive sequences such that

1. \((b_k + \lambda a_k)_{k=0}^n\) is log-concave for all \(\lambda \geq 0\),
2. \(a_k b_{k+1} \geq a_{k+1} b_k\) for all \(0 \leq k \leq n - 1\).

Is it true that the sequence \((b_k + k a_{k-1})_{k=0}^{n+1}\) is log-concave?

A counterexample to Problem 1 will produce a counterexample to the Big Conjecture with g.p. a scalar multiple of

\[
\sum_{k=0}^n b_k e_k(z_1, \ldots, z_m) + m z_{m+1} \sum_{k=0}^n a_k e_k(z_1, \ldots, z_m)
\]

where \(m\) is sufficiently large
For $0 < t \leq 10^{-4}$ the following is a counterexample to Problem 1.

$$a_0 = 2t^2, \quad a_1 = 2t, \quad a_2 = \frac{4}{9} - t, \quad a_3 = \frac{2}{3},$$

$$a_4 = 1, \quad a_5 = \frac{2}{3}t$$

$$b_0 = 9t^3, \quad b_1 = 9t^2, \quad b_2 = 3t, \quad b_3 = 1,$$

$$b_4 = 3, \quad b_5 = 9 - t$$

**Counterexample:** For any $n \geq 20$ there is a probability measure $\mathbb{P}$ on $\{0, 1\}^n$ for which the "Big Conjecture" and CNA+ $\Rightarrow$ ULC fails
Further directions

**Conjecture** [Pemantle]. Rayleigh = CNA+

Let $G = (V, E)$ be a graph and let $\mathcal{I}$ be the set of independent sets. Define $P_G$ on $\{0, 1\}^E$

$$P_G[X = S] := \begin{cases} 1/|\mathcal{I}| & \text{if } S \text{ is independent} \\ 0 & \text{otherwise} \end{cases}$$

**Conjecture** [Grimmett-Winkler]. $P_G$ is negatively associated