Example: Let $A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$. Then

we may choose $A^{\perp} = \begin{bmatrix} -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 \end{bmatrix}$

More on representing matroids

- Let A be an $r \times n$ matrix over F representing M, and let B be a basis of M. By performing elementary row operations and change the order of columns we may transform A to a matrix $A' = [I_r \, D]$, where I_r is the $r \times r$ identity matrix and D is a $r \times (n-r)$ matrix.

- It is easy to see that if $A = [I_r \, D]$ represents M, then $A^{\perp} = [-D^T \, I_{n-r}]$ represents M^*

- Prop. R1: Let $[I_r \, D]$ and $[I_r \, D']$ be $r \times n$ matrices over F and F', respectively, and representing the matroids M and M' on $E = \{1, \dots, n\}$. TFAE:
  (i). The identity map on E is an isomorphism between M and M'
  (ii). A minor of D is nonzero iff
  —11— D' —11—

(Recall that a minor of D is the determinant of a square sub-matrix of D).

<u>Proof.</u> Suppose $D(S,T)$ is the square sub-matrix of D indexed by columns in T and rows in S.

$$A = \begin{array}{c c} \left[\begin{array}{ccccc|c} 1&0&0&0&0& \\ 0&1&0&0&0& D \\ 0&0&1&0&0& \\ 0&0&0&1&0& \\ 0&0&0&0&1& \end{array}\right] & S \\ S' \qquad\quad T & \end{array}$$

Then $\det D(S;T) = \det A(([r]\setminus S)\cup T, \emptyset)$

Hence $\det D(S,T) \neq 0$ iff $([r]\setminus S)\cup T$ is independent in M.

$\square$

- <u>Remark</u>: In particular, the non-zero entries of D are determined by the matroid. Hence if $F = \mathbb{Z}_2$, then D is unique!

- A real matrix A is <u>totally unimodular</u> if all its minors are elements of $\{0, \pm 1\}$

- A matroid is <u>regular</u> if it may be represented by a totally unimodular matrix.

- The following characterization due to Tutte (1965) characterizes regular matroids.

<u>Theorem R2</u>: Let M be a matroid. TFAE

(A). M is regular
(B). M is representable over every field
(C). M is binary (represented over $\mathbb{Z}_2$), and M is representable over some field F of characteristic $\neq 2$.

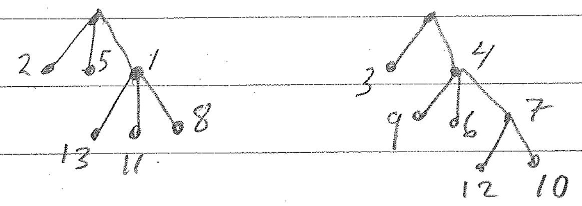
To prove Theorem R2 we need a few lemmas on representability.

• Suppose D is a matrix with rows indexed by R and columns indexed by C. Let $G(D)$ be the bipartite graph on $R \cup C$ with and edge present betwen $r \in R$ and $c \in C$ if $d_{rc} \neq 0$, where $D = (d_{ij})_{\substack{i \in R \\ j \in C}}$.

<u>Lemma R3</u>: Let M be a matroid represented (over F) by $A = [I_r\, D]$. If $\mathcal{B} = \{b_1, \dots, b_k\}$ is a basis of the cycle matroid of $G(D)$ and $\{\xi_b\}_{b \in \mathcal{B}}$ are arbitrary nonzero elements of F, then M has an F-representation $A' = [I_r\, D']$ where the entry corresponding to $b \in \mathcal{B}$ is ξ_b.

<u>Proof</u>: $\mathcal{B}$ is a spanning forest. For each connected component of $\mathcal{B}$ choose a root and then number the non-root vertices of $\mathcal{B}$ injectively with integers so that we get

an increasing forest.



Each vertex represents a column or row of D, and the edges represents entries of D corresponding to $\mathcal{B}$.

Go through the vertices $1, 2, 3, \ldots$ and for each i multiply the corresponding row/column so that the entry corresponding to the edge, b, immediately above i becomes ξ_b.

This will result in a matrix D' of the desired form. Note that I_r will be trasformed to a diagonal matrix, but we can restore this by multiplying these columns by appropriate numbers.
$\square$

Lemma R4: Let M be a binary matroid, and suppose $[I_r D]$ represents M over a field F. Let $\mathcal{B}$ be a basis of the cycle matroid of D, and suppose that each element of D that corresponds to an element of $\mathcal{B}$ is in $\{\pm 1\}$. Then all elements of D are in $\{\pm 1, 0\}$.

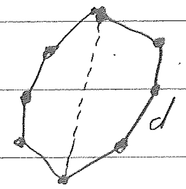
Proof: For any entry $d \neq 0$ of D let e_d be the corresponding edge in $G(D)$.

If $e_d \notin \mathcal{B}$, there is a cycle C containing e_d s.t. all other edges in C correspond to

± 1-entries. Indeed $\mathcal{B} \cup \{e_d\}$ is dependent, so it contains a cycle.

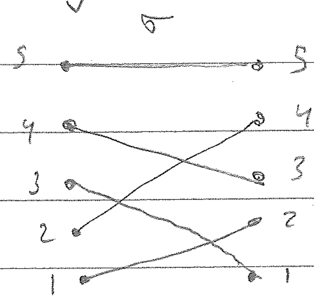
If the lemma fails, let C_0 be a cycle of minimal length and which contains exactly one edge e_d with $d \neq \pm 1$.

If R and C are the rows and columns used in C_0, we claim that the sub-matrix $D' = D(R,C)$ of D has non-zero entries exactly those corresponding to C_0. Indeed, otherwise there would be a smaller such cycle:

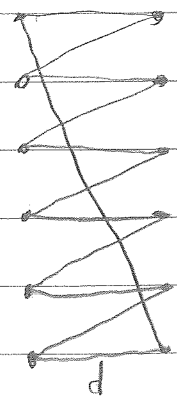


Now let us compute $\det(D')$ using

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)}$$



Each term in $\det(D')$ corresponds to a perfect matching of C_0. There are only two such; of which exactly one contains e_d:



Hence $\det(D') = \pm 1 \pm d$.
Since M is binary, $\det(D') = 0$ by Prop R1. Hence $d = \pm 1$, which is a contradiction. $\square$