- Let P be a graded poset with $\hat{0}$ and $\hat{1}$.
- The order complex is the simplicial complex of all chains $x_1 < x_2 < \dots < x_h$ in $\overline{P} = P \setminus \{\hat{0}, \hat{1}\}$, denoted $\Delta(\overline{P})$.

- We will prove that for any geometric lattice L, $\Delta(\overline{L})$ is shellable.

- Let $\mathcal{E}(P)$ denote the set of covering relations in P.

- Suppose we have a map $\lambda : \mathcal{E}(P) \to \mathbb{Z}$. Then if $c : x_0 < x_1 < \dots < x_k$ is an unrefinable chain, we let $\lambda(c) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \dots, \lambda(x_{k-1}, x_k)) \in \mathbb{Z}^k$.

- $\lambda : \mathcal{E}(P) \to \mathbb{Z}$ is called an L-labelling if for each pair $x < y$:
  (1). The exists a unique unrefinable chain $C_{xy} : x = x_0 < x_1 < \dots < x_h = y$ with increasing labels
  (2). $\lambda(C_{xy})$ is smaller than $\lambda(d)$ for all unrefinable chains $d : x = x_0' < x_1' < \dots < x_h' = y$. (in the lexicographical order).

<u>Theorem</u> (Björner 1980). If P admits an L-labelling, then $\Delta(\bar{P})$ is shellable.

<u>Proof</u>: Let λ be an L-labelling. Let Ω be a linear order on the maximal chains in P s.t.

$\lambda(c) < \lambda(d) \Rightarrow c <_\Omega d$.

We shall prove that if $c <_\Omega d$, then there exists $c' <_\Omega d$ s.t.

(*) $c \cap d \subseteq c' \cap d = d - \{x\}$ for some x.

If $d: \hat{0} = x_0 < x_1 < \dots < x_r = \hat{1}$ and

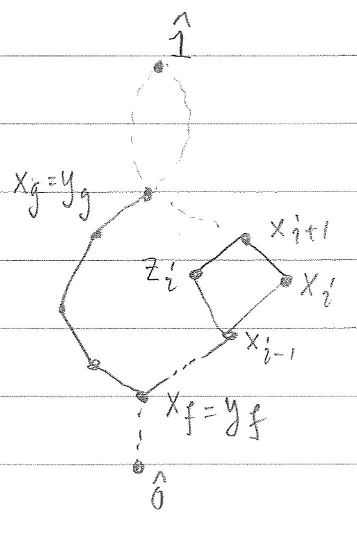
$c: \hat{0} = y_0 < y_1 < \dots < y_r = \hat{1}$,

let $f < g$ be determined by $x_i = y_i$ for $i = 0, \dots, f$, $x_{f+1} \neq y_{f+1}$, and g is the smallest integer greater than f s.t. $x_g = y_g$.

Since $c <_\Omega d$ $x_f < x_{f+1} < \dots < x_g$ is not the unique increasing unrefinable chain in $[x_f, x_g]$. Hence $\lambda(x_{i-1}, x_i) > \lambda(x_i, x_{i+1})$ for some $f < i < g$. By definition there is a $z_i \in P$ s.t. $x_{i-1} < z_i < x_{i+1}$ and $\lambda(x_{i-1}, z_i) < \lambda(z_i, x_{i+1})$.

Let $c': \hat{0} = x_0 < \dots < x_{i-1} < z_i < x_{i+1} < \dots < x_r = \hat{1}$.

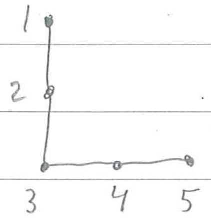
Then (*) holds with $x = x_i$.

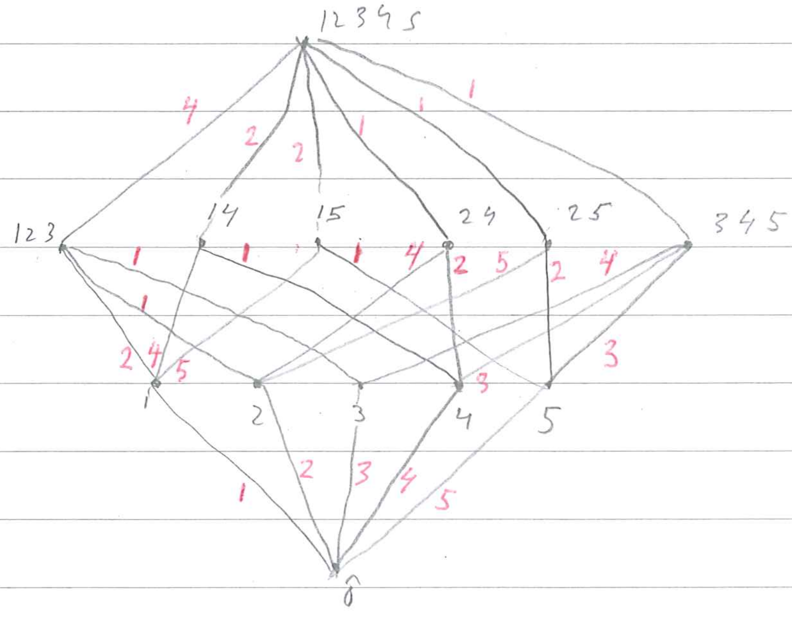


- Note that $\mathcal{R}(c) = \{x_i \in c : \lambda(x_{i-1}, x_i) > \lambda(x_i, x_{i+1})\}$ where $c: \hat{0} = x_0 < x_1 < \dots < x_r = \hat{1}$.

- Let L be a geometric lattice with atoms A. We assume $A \subseteq \mathbb{Z}$.

- Let $A(x) = \{a \in A : a \leq x\}$. If $(x,y) \in \mathcal{E}(L)$, let $\lambda(x,y) = \min A(y) \setminus A(x)$





- Theorem: λ is an L-labelling.

Proof: Let $x < y$. We need to prove (1) and (2). For (1), construct an unrefinable chain

$c: x = x_0 < x_1 < \dots < x_k = y$, recursively by $x_i = x_{i-1} \vee a_i$, where $a_i = \min A(y) \setminus A(x_{i-1})$. Clearly c is increasing.

Let $d: x = y_0 < y_1 < \dots < y_k = y$ be another unrefinable chain and let e be the index for which $x_i = y_i$ for $0 \leq i \leq e$, while $x_{e+1} \neq y_{e+1}$.

Suppose $\lambda(c) = (p_1, p_2, \dots, p_k)$ and
$\lambda(d) = (q_1, q_2, \dots, q_k)$.

Then $p_i = q_i$ for $i = 1, \dots, e$ and by construction $p_{e+1} < q_{e+1}$. Moreover since $p_{e+1} \in A(y_k)$ we have $p_{e+1} = q_f$ for some $f > e$. Hence $\lambda(c) < \lambda(d)$ and $\lambda(d)$ is not increasing. This establishes both (1) and (2).

Corollary: Let L be a geometric lattice.
Then
$$\chi(\Delta(\bar{L})) = \mu(\hat{0}, \hat{1}) =$$
$$= (-1)^r \left| \{ \text{maximal chains s.t. } \lambda(c) \text{ is decreasing} \} \right|$$