(*) If $F_1, \ldots, F_m$ is a shelling of Δ, let $\mathcal{R}(F_k) = \{x \in F_k : F_k - x \in \Delta_{k-1}\}$ be the restriction of F_k. Hence $F_k \setminus x$, where $x \in \mathcal{R}(F_k)$ are the facets of $\{G \cap F_k : G \in \Delta_{k-1}\}$.

<u>Proposition 52</u>: Let $F_1, \ldots, F_m$ be a shelling of Δ. Then Δ is the disjoint union:

$$\Delta = \dot{\bigcup_{i=1}^{m}} [\mathcal{R}(F_i), F_i]$$

<u>Proof</u>: We need to prove that $\Delta_k \setminus \Delta_{k-1} = [\mathcal{R}(F_k), F_k]$. By the observation (*) we have for $G \subseteq F_k$:

$$G \notin \Delta_k \setminus \Delta_{k-1} \iff G \subseteq F_k \setminus x \text{ for some } x \in \mathcal{R}(F_k)$$

Hence

$$G \in \Delta_k \setminus \Delta_{k-1} \iff G \nsubseteq F_k \setminus x \text{ for all } x \in \mathcal{R}(F_k)$$
$$\iff x \in G \text{ for all } x \in \mathcal{R}(F_k).$$

$\square$

- If Δ is a simplicial complex of $\dim = d-1$, then the <u>h-polynomial</u> of Δ is

$$h_\Delta(x) = \sum_{i=0}^{d} h_i(\Delta) x^i = (1-x)^d f_\Delta\left(\frac{x}{1-x}\right), \text{ or equivalently}$$

$$f_\Delta(x) = (1+x)^d h_\Delta\left(\frac{x}{1+x}\right) = \sum_{i=0}^{d} h_i(\Delta) x^i (1+x)^{d-i}$$

Note that $\tilde{\chi}(\Delta) = -f_\Delta(-1) = (-1)^{d-1} h_d(\Delta)$

<u>Corollary 52</u>. If Δ is shellable of dim $= d-1$, then

$$h_\Delta(x) = \sum_{i=1}^{m} x^{|\mathcal{R}(F_i)|} = \sum_{i=0}^{d} h_i(\Delta)\, x^i$$

Moreover $(-1)^{d-1}\tilde{\chi}(\Delta) = |\{F_i : \mathcal{R}(F_i) = F_i\}|$.

<u>Proof</u>: Note that

$$\sum_{A \subseteq [\mathcal{R}(F_i), F_i]} x^{|A|} = \sum_{B \subseteq F_i \setminus \mathcal{R}(F_i)} x^{|\mathcal{R}(F_i)|} x^{|B|} = x^{|\mathcal{R}(F_i)|}(1+x)^{d-|\mathcal{R}(F_i)|},$$

so by Prop. 51,

$$f_\Delta(x) = \sum_{i=1}^{m} x^{|\mathcal{R}(F_i)|}(1+x)^{d-|\mathcal{R}(F_i)|}$$

$$= (1+x)^d \sum_{i=1}^{m} \left(\frac{x}{1+x}\right)^{|\mathcal{R}(F_i)|},$$

from which the corollary follows. $\square$

- If $A = \{a_1 < a_2 < \dots < a_k\}$ and $B = \{b_1 < b_2 < \dots < b_k\}$ are two subsets of the totally ordered set E, then we say that A is <u>lexicographically</u> smaller than B, written $A < B$ if $a_1 < b_1$ or there is a number $m \geq 1$ s.t
  $a_i = b_i$ for all $1 \leq i \leq m$ and
  $a_{i+1} < b_{i+1}$.

· Note that the lexicographic order induces a total order on the set of basis of an ordered matroid M.

<u>Theorem</u>[53] (Björner, 1979).
The lexicographic order on the set of bases of M is a shelling order s.t.

$$\mathcal{R}(B) = B \setminus I(B)$$

for all $B \in \mathcal{B}(M)$.

<u>Proof</u>: Let $B = \{b_1 < \dots < b_r\} < C = \{c_1 < \dots < c_r\}$, $B, C \in \mathcal{B}(M)$ and let $k+1$ be the first index s.t.

$$b_i = c_i \quad \text{for all} \quad 1 \le i \le k$$

$$b_{k+1} < c_{k+1}.$$

Hence $b_{k+1} < c_i$ for all $k+1 \le i \le n$ and $b_{k+1} \in B \setminus C$. By the basis exchange property there is a $c_j \in C \setminus B$ s.t. $A = (C - c_j) \cup b_{k+1} \in \mathcal{B}(M)$. Thus

$$B \cap C \subseteq A \cap C = C \setminus c_j \quad \text{and} \quad A < C.$$

Hence $<$ is a shelling order. Now

$$\mathcal{R}(C) = \{ c_j \in C : \exists\, p \notin C \text{ s.t. } (C - c_j) \cup p < C \}$$

Clearly $(C - c_j) \cup p < C$ iff $p < c_j$, which precisely means

$$\mathcal{R}(C) = C \setminus I(C). \quad \square$$

<u>Corollary</u>: The h-polynomial of $\mathcal{I}(M)$ is

$$\sum_{B \in \mathcal{B}(M)} x^{r - |I(B)|} = x^r \, T_M\left(\frac{1}{x}, 1\right)$$

- If C is a circuit and $c = \min C$, then $C \setminus \{c\}$ is called a <u>broken circuit</u>.

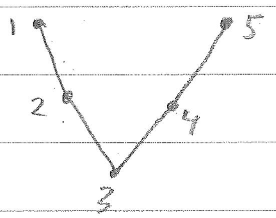
- Recall that if B is a basis and $e \notin B$, then $C(e, B)$ is the unique circuit contained in $B \cup \{e\}$.

- Recall also from HW 1, that if $e \in E \setminus B$ and $f \in B$:

$$(B - f) \cup e \text{ is a basis} \iff f \in C(e, B) \qquad (*)$$

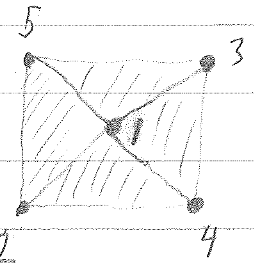
- The <u>broken circuit complex</u>, $BC(M)$, is the simplicial complex of all (independent) sets in M that contain <u>no</u> broken circuits.

<u>Example</u>:



Circuits: $123, 345, 1245$

broken circuits: $23, 45, 245$



$= BC(M)$

Lemma $BC(M)$ is pure and the facets are the bases of M that contain no broken circuits.

Proof: Let $X \in BC(M)$. We need to prove that there is a basis $B \supseteq X$ such that B contains no broken circuit. Let B_0 be the smallest bases w.r.t the lex. order s.t. $X \subseteq B_0$. Suppose B_0 contains a broken circuit $C \setminus c$, where $c = \min C$. Then $C = C(B_0, c)$. Since X contains no broken circuit we have $C \setminus c \not\subseteq X$. Let $z \in (C \setminus c) \setminus X$. Then $c < z$, and by $(*)$, $B_1 := (B_0 - z) \cup c$ is a basis s.t. $X \subseteq B_1$ and $B_1 < B_0$. Contradiction. $\square$

Theorem 54. Let $B_1 < B_2 < \ldots < B_m$ be bases of M that contain no broken circuit. Then $B_1, \ldots, B_m$ is a shelling of $BC(M)$. Moreover

$$\mathcal{R}(B_i) = B_i \setminus I(B_i).$$

Proof: If $B_i < B_j$, then by Theorem 53 there is a basis $B < B_j$ (which could contain a broken circuit) such that $B_i \cap B_j \subseteq B \cap B_j = B_j \setminus b$, where $b \in B_j$. We need to prove that there is an element $p \notin B_j$ s.t $p < b$ and $(B_j \setminus b) \cup p$ is a basis which contains no broken circuit.

Because then $B_1, \dots, B_m$ is a shelling and the restriction function will be the same as the one for the shelling of $\mathcal{I}(M)$, which we have proved equals $B_j \setminus I(B_j)$.

Let $p_0 = \min\{p : (B_j - b) \cup p \text{ is a basis}\}$. Then $B' := (B_j - b) \cup p_0 < B_j$. If B' contains a broken circuit $C \setminus c$, where $c = \min C$, then $C = C(B', c)$. Now $p_0 \in C \setminus c$, since otherwise $C \setminus c \subseteq B_j - b \subseteq B_j$, and then B_j would contain a broken circuit. By $(*)$ $(B' - p_0) \cup c = (B_j - b) \cup c$ is a basis, and since $c < p_0$ this is a contradiction. $\square$

Lemma 55. Let B be a basis of M. Then B contains no broken circuit iff $E(B) = \emptyset$.

Proof: Recall that $E(B) = \{p \notin B : p = \min C(B, p)\}$. Now B contains a broken circuit $C \setminus c$ where $c = \min C$ iff $C = C(B, c)$ where $c = \min C(B, c)$. The lemma follows.

Corollary: The f-polynomial and h-polynomial of $\Delta = BC(M)$ are

$$f_\Delta(x) = x^r T_M(1 + \tfrac{1}{x}, 0) = (-x)^r \chi_M(-x^{-1})$$

$$h_\Delta(x) = x^r T_M(\tfrac{1}{x}, 0)$$

Proof: Apply Corollary 44, Lemma 55, Theorem 54 and Corollary 52.