To be handed in no later than **May 6**. *You may cooperate but you must write your solutions by yourselves. Please write full proofs!*

(1) Let $\mathcal{A} = \{A_j\}_{j \in J}$ be a family of subsets of E, and let m be a positive integer. Use Theorem T1 of the notes to prove:

(a) $\mathcal{A}$ has m disjoint transversals if and only if

$$|A(K)| \geq m|K|, \quad \text{for all } K \subseteq J;$$

(b) $\mathcal{A}$ has a system of representatives in which no element occurs more than m times if and only if

$$|A(K)| \geq \frac{|K|}{m}, \quad \text{for all } K \subseteq J.$$

(2) Let $C_1, \dots, C_n$ be distinct circuits of $M = (E, \mathcal{I})$ such that

$$C_k \nsubseteq \bigcup_{j \in [n] \setminus \{k\}} C_j, \quad \text{for all } 1 \leq k \leq n.$$

Prove that if $D \subseteq E$ and $|D| < n$, then there exists a circuit C with

$$C \subseteq (C_1 \cup \cdots \cup C_n) \setminus D.$$

(3) Let B_1 and B_2 be two bases of a matroid M. Prove that there is a bijection $\sigma : B_1 \to B_2$ such that

$$(B_2 \setminus \{\sigma(e)\}) \cup \{e\}$$

is a basis for all $e \in B_1$.

Hint: It suffices to find an injection $\sigma : B_1 \setminus B_2 \to B_2 \setminus B_1$ satisfying the above. For $e \in B_1 \setminus B_2$, let $C_e = C(e, B_2)$ (see Problem 2 of Problem set 1), and let $C'_e = C_e \cap (B_2 \setminus B_1)$. We want to find a transversal of $\mathcal{A} = \{C'_e\}_{e \in B_1 \setminus B_2}$. Use Hall's theorem, Problem (2) above and Problem 2 of Problem set 1.

(4) Let Δ be a simplicial complex on vertex set E (here we don't assume that every $e \in E$ is actually used as a vertex of Δ). The concept of *vertex-decomposability* for a simplicial complex Δ on vertex set E is defined recursively: both the complex $\Delta = \emptyset$ having no faces at all (not even the empty face) and any complex Δ consisting of a single vertex are defined to be vertex-decomposable, and then Δ is said to be vertex-decomposable if it is pure, and there exists a vertex $e \in E$ for which both its deletion and link

$$\mathrm{del}_\Delta(e) := \{F \in \Delta : e \notin F\}, \qquad \mathrm{link}_\Delta(e) := \{F \setminus \{e\} : e \in F \in \Delta\}$$

are vertex-decomposable complexes.

(a) Show that vertex-decomposable complexes are shellable. (*Hint:* Induction on $|E|$. Let the facets in the deletion come first in the shelling order, and then continue with the facets $F \ni e$ such that $F \setminus \{e\} \in \mathrm{link}_\Delta(e)$.)

(b) Let M be a matroid. Show that $\Delta = \mathcal{I}(M)$ is vertex-decomposable.