*To be handed in no later than* **March 25**. *You may cooperate but you must write your solutions by yourselves. Please write full proofs!*

(1) Recall that $F_7$ is represented over $\mathbb{Z}_2$ by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix},$$

while $F_7^-$ may be defined as the matroid represented by the same matrix thought of as a matrix over $\mathbb{Q}$. Use Proposition R1 and Lemma R3 of the notes to prove:

(a) $F_7$ is representable over a field $F$ if and only if the characteristic of $F$ is 2.

(b) $F_7^-$ is representable over a field $F$ if and only if the characteristic of $F$ is not 2.

(2) Let $M = (E, \mathcal{I})$ be a matroid, and suppose $e \notin E$. Define a matroid $M + e = (E \cup \{e\}, \mathcal{I}')$, where

$$\mathcal{I}' = \mathcal{I} \cup \{I \cup \{e\} : I \in \mathcal{I} \text{ and } r(I) \leq r(E) - 1\}.$$

(a) Find a matroid $M$ and a field $F$ such that $M$ is representable over $F$, while $M + e$ is not representable over $F$.

(b) Let $p$ be a prime number. Prove that if $M$ is representable over some field of characteristic $p$, then so is $M + e$.

(c) For a matroid $M = (E, \mathcal{I})$, define the polynomial

$$f_M(q) = \sum_{I \in \mathcal{I}} q^{r(M) - |I|}.$$

Prove that

$$(-1)^{r(M \times e)} \chi_{M \times e}(-q) = (1 + q) f_M(q),$$

where $M \times e = (M^* + e)^*$.

(3) Let $A$ be an $r \times m$ matrix and $B$ a $m \times r$ matrix. The Binet–Cauchy theorem states that

$$\det(AB) = \sum_S A[[r], S] \cdot B[S, [r]],$$

where the sum is over all subsets $S$ of $[m]$ of cardinality $r$, and $A[T, S]$ denotes the minor of $A$ indexed by rows in $T$ and columns in $S$.

(a) Let $A$ be an $r \times m$ matrix, and $Z$ a diagonal $m \times m$ matrix with variables $z_1, \dots, z_m$ on the diagonal. Prove that

$$\det(AZA^T) = \sum_{|S| = r} A[[r], S]^2 \prod_{i \in S} z_i,$$

where $A^T$ denotes the transpose of $A$.

(b) Prove that if $M$ is a regular rank $r$ matroid which is representable by a totally unimodular matrix $A$, then

$$\det(AZA^T) = \sum_{B \in \mathcal{B}(M)} \prod_{i \in B} z_i,$$

and hence $|\mathcal{B}(M)| = \det(AA^T)$.

(c) Let $G = (V, E)$ be a simple and connected graph on $V = [n]$, and let $E = \{e_1, \dots, e_m\}$. Let $A$ be the matrix whose $k$th column is $\delta_i - \delta_j$, where $\delta_1, \dots, \delta_n$ is the standard basis of $\mathbb{R}^n$ and $e_k = \{i, j\}$ where $i < j$. Use (b) to prove that for any $i \in [n]$

$$\det(U_i) = \sum_T \prod_{e \in T} z_e,$$

where $U_i$ is the matrix obtained from $AZA^T$ be deleting its $i$th row and column, and where the sum is over all spanning trees of $G$ (here $Z$ is a $m \times m$ diagonal matrix with variables indexed by the edges $\{e_1, \dots, e_m\}$).

(4) Let $G$ be a finite abelian group (additive notation) with $q$ elements, and let $A = (a_{ij})$ be an $r \times n$ matrix over $\mathbb{Z}$. Recall that $\mathbb{Z}$ acts on $G$ (as a $\mathbb{Z}$-module) as $0g = 0$, and

$$mg = \sum_{i=1}^{m} g \quad \text{and } (-m)g = -\sum_{i=1}^{m} g,$$

whenever $m$ is a positive integer. Then $A$ defines a group homomorphism $A^G : G^n \to G^r$:

$$(g_1, \dots, g_n)^T \mapsto \left( \sum_{j=1}^{n} a_{1j} g_j, \dots, \sum_{j=1}^{n} a_{rj} g_j \right)^T.$$

Suppose $A$ is totally unimodular.

(a) Prove

$$|\ker(A^G)| = q^{n-r},$$

where $r$ is the rank of $A$. *Hint.* Use

$$\frac{G^n}{\ker(A^G)} \cong \mathrm{Im}(A^G).$$

(b) For $v = (g_1, \dots, g_n)^T \in G^n$, let

$$z(v) = |\{i \in [n] : g_i = 0\}|.$$

Prove that

$$\sum_{v \in \ker(A^G)} (1+t)^{z(v)} = \sum_{C \subseteq [n]} t^{n-|C|} q^{|C|-r(C)},$$

where $r : [n] \to \mathbb{N}$ is the rank function of the matroid, $M$, represented by $A$.

(c) Deduce that

$$\sum_{v \in \ker(A^G)} t^{z(v)} = (t-1)^{n-r(M)} T_M \left( t, \frac{q+t-1}{t-1} \right), \text{ and}$$

$$\chi_{M^*}(q) = |\{(g_1, \dots, g_n)^T \in \ker(A^G) : g_i \neq 0 \text{ for all } i \in [n]\}|.$$