To be handed in no later than **February 25**. *You may cooperate but you must write your solutions by yourselves. Please write full proofs!*

(1) Let M_1 and M_2 be matroids on a set E. Give an example to show that $(E, \mathcal{I}(M_1) \cap \mathcal{I}(M_2))$ need not be a matroid.

(2) Recall that if B is a basis of a matroid M on E, and $e \in E \setminus B$, then $B \cup \{e\}$ contains a *unique* circuit, $C(e, B)$.

   (a) Prove that if C is any circuit of M and $f \in C$, then M has a basis B such that $C = C(f, B)$.

   (b) Suppose B is a basis of a matroid M on E, and $f \in E$ while $e \in E \setminus B$. Prove that $(B \cup \{e\}) \setminus \{f\}$ is a basis if and only if $f \in C(e, B)$.

(3) (a) Prove that a matroid M is uniform if and only if it has no circuits of size less than $r(M) + 1$.

   (b) Let $X \subseteq Y$ be two flats of a matroid M such that $r(Y) = r(X) + 1$. Prove that M has a hyperplane H such that $X = H \cap Y$.

(4) Let L be a geometric lattice and $x \leq y$ in L. Prove that the interval $[x, y]$ is a geometric lattice.

(5) Let A and B be bases of a matroid M, $a \in A \setminus B$. Prove that there is an element $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\}$ and $B \setminus \{b\} \cup \{a\}$ are bases of M.

(6) Let $S = \{s_1 < s_2 < \cdots < s_k = n\}$ be a set of positive integers and consider the family, $\mathcal{B}$, of subsets of $E = \{1, \ldots, n\}$ defined by

$$\mathcal{B} = \big\{\{b_1 < b_2 < \cdots < b_k\} : b_1 \leq s_1, b_2 \leq s_2, \ldots, b_k \leq s_k\big\}.$$

   (a) Prove that $\mathcal{B}$ is the set of bases of a matroid on E.

   (b) Prove the the matroid defined in (a) is representable over some field.