SINGULAR OSCILLATORY INTEGRALS ON $\mathbb{R}^n$

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Abstract. Let $\mathcal{P}_{d,n}$ denote the space of all real polynomials of degree at most $d$ on $\mathbb{R}^n$. We prove a new estimate for the logarithmic measure of the sublevel set of a polynomial $P \in \mathcal{P}_{d,n}$. Using this estimate, we prove that

$$\sup_{P \in \mathcal{P}_{d,n}} \left| \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x)} \frac{\Omega(x/|x|)}{|x|^n} \, dx \right| \leq c \log d (\|\Omega\|_{L\log L(S^{n-1})} + 1),$$

for some absolute positive constant $c$ and every function $\Omega$ with zero mean value on the unit sphere $S^{n-1}$. This improves a result of Stein from [3].

1. Introduction

We denote by $\mathcal{P}_{d,n}$ the vector space of all real polynomials of degree at most $d$ in $\mathbb{R}^n$. Let $K$ be a $-n$ homogeneous function on $\mathbb{R}^n$, that is,

$$(1.1) \quad K(x) = \frac{\Omega(x/|x|)}{|x|^n},$$

where $\Omega$ is some function on the unit sphere $S^{n-1}$. Consider the principal value integral

$$I_n(P) = \left| \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x)} K(x) \, dx \right|.$$ 

Stein has proved in [3] that if $\Omega$ has zero mean value on the unit sphere, then

$$(1.2) \quad |I_n(P)| \leq c_d \|\Omega\|_{L^\infty(S^{n-1})},$$

for some constant $c_d$ depending only on $d$. We wish to obtain sharp estimates of the form (1.2). The one dimensional analogue, namely the estimate

$$(1.3) \quad \left| \text{p.v.} \int_{\mathbb{R}} e^{iP(x)} \frac{dx}{x} \right| \leq c \log d,$$

which was proved in [2], suggests that the constant $c_d$ in (1.2) could be replaced by $c \log d$ for some absolute positive constant $c$. The fact that this is indeed the case is the content of the following theorem.

Theorem 1.1. Suppose that $K(x) = \Omega(x/|x|)/|x|^n$ where $\Omega$ has zero mean value on the unit sphere $S^{n-1}$. There exists an absolute positive constant $c$ such that

$$\sup_{P \in \mathcal{P}_{d,n}} \left| \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x)} K(x) \, dx \right| \leq c \log d (\|\Omega\|_{L\log L(S^{n-1})} + 1).$$

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Remark 1.2. Suppose that $K(x) = \Omega(|x|)/|x|^n$ where the function $\Omega$ is odd on the unit sphere. It is an immediate consequence of the one-dimensional result that

$$\sup_{P \in \mathcal{P}_{d,n}} \left| \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x)} K(x) \, dx \right| \leq c \log d \|\Omega\|_{L^1(S^{n-1})}$$

for some absolute positive constant $c$.

The main ingredient of the proof of Theorem 1.1 is an estimate for the logarithmic measure of the sublevel set of a real polynomial in one dimension. This is a lemma of independent interest which we now state.

**Lemma 1.3 (The logarithmic measure lemma).** Let $P(x) = \sum_{k=0}^{d} b_k x^k$ be a real valued polynomial of degree at most $d$, $\alpha > 0$ and $M = \max\{|b_k|: \frac{d}{2} < k \leq d\}$. If $E = \{x \geq 1: |P(x)| \leq \alpha\}$, then

$$\int_E \frac{dx}{x} \leq c \min\left(\left(\frac{\alpha}{M}\right)\frac{1}{d}, 1 + \frac{1}{d} \log \frac{\alpha}{M}\right),$$

where $c$ is an absolute positive constant.

**Notation.** We will use the letter $c$ to denote an absolute positive constant which might change even in the same line of text.

## 2. Preliminary Results

As is usually the case when one deals with oscillatory integrals, a key Lemma is the classical van der Corput Lemma.

**Lemma 2.1 (van der Corput).** Let $\phi: [a, b] \to \mathbb{R}$ be a $C^1$ function and suppose that $|\phi'(t)| \geq 1$ for all $t \in [a, b]$ and $\phi'$ changes monotonicity $N$ times in $[a, b]$. Then, for every $\lambda \in \mathbb{R}$,

$$\left| \int_a^b e^{i\lambda \phi(x)} \, dx \right| \leq \frac{cN}{|\lambda|},$$

where $c$ is an absolute constant independent of $a, b$ and $\phi$.

The proof of Lemma 2.1 is a simple integration by parts.

We will also need a precise estimate for the Lebesgue measure of the sublevel set of a polynomial on $\mathbb{R}^n$.

**Theorem 2.2 (Carbery,Wright).** Suppose that $K \subset \mathbb{R}^n$ is a convex body of volume 1 and $P \in \mathcal{P}_{d,n}$. Let $1 \leq q \leq \infty$. Then,

$$|\{x \in K : |P(x)| \leq \alpha\}| \leq c \min(qd, n) \alpha^{\frac{d}{2}} \|P\|_{L^\infty(K)}^{-\frac{1}{2}}.$$

This is a consequence of a more general Theorem of Carbery and Wright and can be found in [1].

**Corollary 2.3.** Let $P$ be a real homogeneous polynomial of degree $k$ on $\mathbb{R}^n$. Then

$$\int_{S^{n-1}} \frac{\|P\|_{L^\infty(S^{n-1})}^\frac{1}{k}}{|P(x')|^{\frac{1}{k}}} \, d\sigma_{n-1}(x') \leq c.$$
Proof of Corollary 2.3. Let $B = B(0, \rho)$ be the ball of volume 1 on $\mathbb{R}^n$. For $\epsilon < \frac{1}{n}$ and some $\lambda > 0$ to be defined later, we have
\[
\int_B |P(x)|^{-\epsilon} \, dx = \int_0^\infty |\{x \in B : |P(x)|^{-\epsilon} \geq \alpha\}| \, d\alpha
\]
\[
\leq \lambda + \int_\lambda^\infty |\{x \in B : |P(x)| < \alpha^{-\frac{1}{n}}\}| \, d\alpha
\]
\[
\leq \lambda + cn\|P\|_{L^\infty_\nu(B)}^{-\frac{1}{n}} \lambda^{-\frac{1}{n}+1} \frac{\lambda}{k\epsilon - 1},
\]
using Theorem 2.2. Optimizing in $\lambda$ we get
\[
\int_B |P(x)|^{-\epsilon} \, dx \leq \left( \frac{cn}{1 - k\epsilon} \right)^{k\epsilon} \|P\|_{L^\infty_\nu(B)}^{-\frac{1}{n}}.
\]
Using polar coordinates and setting $\epsilon = \frac{1}{2n} < \frac{1}{n}$, we then get
\[
\|P\|_{L^\infty_\nu(S^{n-1})} \int_{S^{n-1}} |P(x')|^{-\frac{1}{n}} \, d\sigma_{n-1}(x') \leq \frac{c n^2 \pi^{n/2}}{\rho^n} = c \frac{n^2 \pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}
\]
\[
\leq c \frac{n^2 \pi \epsilon^{\frac{n}{2}}}{(\frac{n}{2} + 1)^{\frac{n}{2}}}
\]
which completes the proof. \(\square\)

3. The logarithmic measure lemma

The proof of Lemma 1.3 is motivated by an argument of Vinogradov from [4], used to estimate the Lebesgue measure of the sublevel set of a polynomial in a bounded interval. We fix a polynomial $P(x) = \sum_{k=0}^d b_k x^k$ and look at the set $E = \{ x \geq \lambda : |P(x)| \leq \alpha \}$. Note that by replacing $\alpha$ with $\alpha M$ in the statement of the lemma, it is enough to consider the case $M = 1$. Since $E$ is a closed set we can find points $x_0, x_1, \ldots, x_d \in E$ such that $x_0 < x_1 < \cdots < x_d$ and
\[
\frac{1}{d} \int_E \frac{dx}{x} = \int_{E \cap [x_j, x_{j+1}]} \frac{dx}{x} \leq \log \frac{x_{j+1}}{x_j}, \quad 0 \leq j \leq d - 1.
\]
We set $\mu = \int_E \frac{dx}{x}$ and $t = e^\frac{\mu}{d} > 1$ and we have that $x_{j+1} \geq tx_j$, $0 \leq j \leq d - 1$. The Lagrange interpolation formula is
\[
P(x) = \sum_{j=0}^d P(x_j) \frac{(x-x_0) \cdots (\hat{x}_j \cdots (x-x_d)}{(x_j-x_0) \cdots (x_j-x_j) \cdots (x_j-x_d)}, \quad x \in \mathbb{R},
\]
where $\hat{u}$ means that $u$ is omitted. Thus,
\[
b_k = \sum_{j=0}^d P(x_j)(-1)^{d-k} \frac{\sigma_{d-k}(x_0, \ldots, \hat{x}_j, \ldots, x_d)}{(x_j-x_0) \cdots (\hat{x}_j-x_j) \cdots (x_j-x_d)},
\]
where $\sigma_l$ is the $l$-th elementary symmetric function of its variables. Therefore

$$|b_k| \leq \alpha \sum_{j=0}^{d} \frac{\sigma_{d-k}(x_0, \ldots, x_j, \ldots, x_d)}{|x_j - x_0| \cdots |x_j - x_j| \cdots |x_j - x_d|}$$

$$= \alpha \sum_{j=0}^{d} \frac{\sigma_k(x_0, \ldots, \frac{x_j}{x_j}, \ldots, \frac{x_d}{x_d})}{(\frac{x_j}{x_j} - 1) \cdots (\frac{x_j}{x_{j+1}} - 1)(1 - \frac{x_j}{x_{j+1}}) \cdots (1 - \frac{x_j}{x_d})}$$

$$\leq \alpha \sum_{j=0}^{d} (t^j - 1) \cdots (t - 1)(1 - \frac{1}{t^j}) \cdots (1 - \frac{1}{t^{d-j}}).$$

It is easy to see that there exists precisely one $j$, $0 \leq j \leq \frac{d-1}{2} < d$, for which

$$t^{j-1} \leq \frac{2t^d}{t^d+1} \leq t^j.$$

It is exactly for this $j$ that $(t^j - 1) \cdots (t - 1)(1 - \frac{1}{t^j}) \cdots (1 - \frac{1}{t^{d-j}})$ takes its minimum value as $j$ runs from 0 to $d$. On the other hand we have

$$\sum_{j=0}^{d} \sigma_k\left(1, \ldots, \frac{1}{t^j}, \ldots, \frac{1}{t^d}\right) = (d + 1 - k)\sigma_k\left(1, \ldots, \frac{1}{t^d}\right)$$

and, hence

$$|b_k| \leq \alpha (d + 1 - k)\sigma_k\left(1, \ldots, \frac{1}{t^d}\right) \frac{1}{(t^j - 1) \cdots (t - 1)(1 - \frac{1}{t^j}) \cdots (1 - \frac{1}{t^{d-j}})}$$

$$\leq \alpha \frac{(d + 1 - k)(d+1)}{1 \cdot t \cdots t^d} \frac{1}{(t^j - 1) \cdots (t - 1)(1 - \frac{1}{t^j}) \cdots (1 - \frac{1}{t^{d-j}})}.$$

From (3) we easily see that $t^j < 2$ and, since $\frac{\log(x-1)}{x}$ is increasing in the interval $(1,2)$, we find

$$\log(t^j) + \cdots + \log(t^j - 1) = \frac{t}{t - 1} \left(\frac{\log(t-1)}{t}t - 1 + \cdots + \frac{\log(t^j - 1)}{t^j}(t^j - t^{j-1})\right)$$

$$\geq \frac{t}{t - 1} \int_1^{t^j} \frac{\log(x-1)}{x} dx = \frac{t}{t - 1} \int_0^{1} \frac{\log x}{1 + x} dx.$$

Similarly, since $\frac{\log(1-x)}{x}$ is decreasing in the interval $(0,1)$ we get

$$\log\left(1 - \frac{1}{t^{d-j}}\right) + \cdots + \log\left(1 - \frac{1}{t}\right) = \frac{1}{t - 1} \left(\frac{\log(1 - \frac{1}{t^{d-j}})}{t^{d-j-1}} - \frac{1}{t^{d-j-1}} \cdots + \frac{\log(1 - \frac{1}{t})}{t}\right)$$

$$\geq \frac{1}{t - 1} \int_1^{1 - \frac{1}{t^{d-j}}} \frac{\log(1-x)}{x} dx = \frac{1}{t - 1} \int_0^{1 - \frac{1}{t}} \frac{\log x}{1 - x} dx.$$

We let

$$A = \frac{t^d - 1}{t^d + 1}, \quad B = t^j - 1, \quad \Gamma = 1 - \frac{1}{t^{d-j}},$$
and, obviously, $0 < A, B, \Gamma < 1$. From (3.3) and (3.4) we have

$$\log(t - 1) + \cdots + \log(t^j - 1) + \log\left(1 - \frac{1}{t^{d-j}}\right) + \cdots + \log\left(1 - \frac{1}{t}\right) \geq$$

$$\geq \frac{t}{t-1} \int_0^{t-1} \log x \frac{dx}{1+x} + \frac{1}{t-1} \int_{t-1}^{1-\frac{1}{t^{d-j}}} \log x \frac{dx}{1-x}$$

$$= \frac{t}{t-1} \int_0^{B} \log x \frac{dx}{1+x} + \frac{1}{t-1} \int_0^{\Gamma} \log x \frac{dx}{1-x}$$

$$= -\frac{t}{t-1} B \log \frac{1}{B} - \frac{1}{t-1} \Gamma \log \frac{1}{\Gamma} - O\left(\frac{t}{t-1} B \right) - O\left(1 \right).$$

From (3) we get $B, \Gamma \leq \frac{t^{d+1}-1}{t-1}$ and, since $\frac{t^{d+1}-1}{t-1} \Gamma$ is decreasing in $t \in (1, +\infty)$, we find

$$\frac{t}{t-1} B \leq \frac{t+1}{t-1} \frac{t^{d+1}-1}{t^{d+1}+1} \leq d + 1$$

and, similarly,

$$\frac{1}{t-1} \Gamma \leq \frac{t+1}{t-1} \frac{t^{d+1}-1}{t^{d+1}+1} \leq d + 1.$$

Therefore

$$\log(t - 1) + \cdots + \log(t^j - 1) + \log\left(1 - \frac{1}{t^{d-j}}\right) + \cdots + \log\left(1 - \frac{1}{t}\right) \geq$$

$$\geq -\frac{t}{t-1} B \log \frac{1}{B} - \frac{1}{t-1} \Gamma \log \frac{1}{\Gamma} - cd$$

$$\geq -\frac{t}{t-1} B \log \frac{1}{B} - \frac{1}{t-1} \Gamma \log \frac{1}{\Gamma} - 2A \log \frac{1}{A} - cd.$$

Now

$$B \log \frac{1}{B} + \Gamma \log \frac{1}{\Gamma} - 2A \log \frac{1}{A} = (B + \Gamma - 2A) \log \frac{1}{A} + A \log \frac{B}{A} + \Gamma \log \frac{A}{\Gamma}$$

$$\leq \left(\frac{B + \Gamma - 2A}{A} \right) \log \frac{1}{A} + cA.$$

Using (3)

$$\frac{B + \Gamma}{A} - 1 \leq \frac{2(t-1)}{t^{d+1}+1}$$

and we conclude that

$$\frac{1}{t-1} \left(B \log \frac{1}{B} + \Gamma \log \frac{1}{\Gamma} - 2A \log \frac{1}{A} \right) \leq \frac{2}{t^{d+1}+1} \frac{t^{d+1}+1}{t} \log \frac{1}{A} + \frac{c}{t-1} \frac{1}{A}$$

$$\leq c + \frac{t^{d+1}+1}{t-1} \log \frac{1}{A} \leq cd.$$

Therefore

$$\log(t - 1) + \cdots + \log(t^j - 1) + \log\left(1 - \frac{1}{t^{d-j}}\right) + \cdots + \log\left(1 - \frac{1}{t}\right) \geq$$

$$\geq -\frac{2}{t-1} A \log 1A - cd.$$
and, finally, (3.2) implies that for some \(k > \frac{d}{2}\)

\[
1 \leq \frac{c_\alpha}{t^{k-1}} \left( \frac{1}{A} \right)^{\frac{2A}{d}},
\]

where \(c_\alpha\) is an absolute positive constant.

**case 1:** \(c_\alpha \alpha^{\frac{1}{2}} < \frac{1}{2}\). Then, since \(\frac{2A}{d} \leq \frac{t+1}{t+1} A \leq d\), we get

\[
A^d \leq A^{\frac{2A}{d}} \leq c_\alpha \alpha^{\frac{1}{2}}
\]

which implies

\[
\frac{t^d - 1}{t^d + 1} \leq A \leq c_\alpha \alpha^{\frac{1}{2}}
\]

and, finally,

\[
\mu \leq 2^{d^2} \frac{k(k-1)}{k(k-1)} \left( \log(3c_\alpha) + \log \frac{\alpha}{d} \right) \leq c \left( 1 + \log \frac{\alpha}{d} \right)
\]

since \(k > \frac{d}{2}\).

**case 2:** \(c_\alpha \alpha^{\frac{1}{2}} \geq \frac{1}{2}\), \(t^d < 2\). Then

\[
\mu \leq 2^{d^2} \frac{k(k-1)}{k(k-1)} \left( \log(3c_\alpha) + \log \frac{\alpha}{d} \right) \leq c \left( 1 + \log \frac{\alpha}{d} \right)
\]

We conclude that

\[
\mu \leq \frac{2d^2}{k(k-1)} \left( \log(3c_\alpha) + \log \frac{\alpha}{d} \right) \leq c \left( 1 + \log \frac{\alpha}{d} \right)
\]

since \(k > \frac{d}{2}\).

**4. Proof of Theorem 1.1**

Let \(\Omega\) be a function with zero mean value on the unit sphere \(S^{n-1}\) belonging to the class \(L^{\log L}(S^{n-1})\), that is

\[
\|\Omega\|_{L^{\log L}(S^{n-1})} = \int_{S^{n-1}} |\Omega(x')|(1 + \log^+ |\Omega(x')|) d\sigma_{n-1}(x') < \infty.
\]

Set \(K(x) = \Omega(x/|x|)/|x|^n\) and let \(P \in \mathcal{P}_{d,n}\). We will show the theorem for \(d = 2^m\), for some \(m \geq 0\). The general case is then an immediate consequence.

We set

\[
C_d = \sup_{0 < \epsilon < R} \left| \int_{|x| \leq R} e^{\epsilon P(x)} K(x) dx \right|
\]

where \(C_d\) is a constant depending on \(d, \Omega\) and \(n\). For \(0 < \epsilon < R\) we write

\[
I_{\epsilon,R}(P) = \int_{|x| \leq R} e^{\epsilon P(x)} K(x) dx = \int_{S^{n-1}} \int_{\epsilon}^{R} e^{\epsilon P(x') \frac{dx}{r}} \Omega(x') d\sigma_{n-1}(x').
\]

For \(x' \in S^{n-1}\), we have that \(P(x') = \sum_{j=1}^{d} P_j(x') r^j\) where \(P_j\) is a homogeneous polynomial of degree \(j\). Observe that we can omit the constant term, without loss of generality. Set also \(m_j = \|P_j\|_{L^{\infty}(S^{n-1})}\). Since \(\epsilon\) and \(R\) are arbitrary positive numbers, by a dilation in \(r\) we can assume that \(\max_{\frac{\epsilon}{2} \leq j \leq d} m_j = 1\) and, in particular,
that \( m_{j_0} = 1 \) for some \( \frac{d}{2} < j_0 \leq d \). We also write \( Q(x) = \sum_{j=1}^{\frac{d}{2}} P_j(x) \). We split the integral in two parts as follows

\[
|I_{\epsilon,R}(P)| \leq \left\lvert \int_{S^{n-1}} e^{iP(x')} \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right\rvert + \left\lvert \int_{S^{n-1}} e^{iQ(x')} \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right\rvert = I_1 + I_2.
\]

For \( I_1 \) we have that

\[
I_1 \leq \int_{S^{n-1}} \left\lvert \int_{r \in [1,R]} |e^{iP(x')} - e^{iQ(x')}| \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right\rvert + \int_{S^{n-1}} \left\lvert \int_{r \in [1,R]} e^{iQ(x')} \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right\rvert \leq \sum_{\frac{d}{2} < j \leq d} m_j \|\Omega\|_{L^1(S^{n-1})} + C_2 \leq c\|\Omega\|_{L^1(S^{n-1})} + C_\frac{d}{2}.
\]

For \( I_2 \) we write

\[
I_2 \leq \int_{S^{n-1}} \left\lvert \int_{r \in [1,R]} |e^{iP(x')}| \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right\rvert + \int_{S^{n-1}} \left\lvert \int_{r \in [1,R]} \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right\rvert.
\]

Since \( \{r \in [1,R] : \frac{dr}{r} \geq d \} \) consists of at most \( O(d) \) intervals where \( \frac{dr}{r} \) is monotonic, van der Corput's lemma gives the bound

\[
\int_{S^{n-1}} \left\lvert \int_{r \in [1,R]} |e^{iP(x')}| \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right\rvert \leq c\|\Omega\|_{L^1(S^{n-1})}.
\]

On the other hand, the logarithmic measure lemma implies that

\[
\int_{S^{n-1}} \left\lvert \int_{r \in [1,R]} \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right\rvert \leq c\|\Omega\|_{L^1(S^{n-1})} + c\frac{1}{d} \int_{S^{n-1}} \log \max_{\frac{d}{2} < j \leq d} |P_j(x')| \Omega(x') d\sigma_{n-1}(x').
\]

Combining the estimates we get

\[
C_d \leq c\|\Omega\|_{L^1(S^{n-1})} + C_\frac{d}{2} + c\frac{2j_0}{d} \int_{S^{n-1}} \log \frac{\|P_{j_0}\|_{L^\infty(S^{n-1})}}{|P_{j_0}(x')|_{\mu_\nu}} \Omega(x') d\sigma_{n-1}(x')
\]

and, from Young's inequality,

\[
C_d \leq c\|\Omega\|_{L^1(S^{n-1})} + C_\frac{d}{2} + c \int_{S^{n-1}} |\Omega(x')| |(1 + \log^+ |\Omega(x')|) d\sigma_{n-1}(x') + c \int_{S^{n-1}} |\Omega(x')| (1 + \log^+ |\Omega(x')|) d\sigma_{n-1}(x').
\]

Now, using corollary 2.3 we get

\[
C_d \leq C_\frac{d}{2} + c(\|\Omega\|_{L^\infty} \log L_1(S^{n-1}) + 1).
\]
Since $d = 2^m$, this means that

$$C_{2^m} \leq C_{2^{m-1}} + c\|\Omega\|_{L^1 L(S^{n-1})} + 1.$$  

Using induction on $m$ we get that $C_{2^m} \leq C_1 + cm\|\Omega\|_{L^1 L(S^{n-1})} + 1$. Observe that $C_1$ corresponds to some polynomial $P(x) = b_1x_1 + \cdots + b_nx_n$. We write

$$\left| \int_{|x|<R} e^{iP(x)} K(x) dx \right| = \left| \int_{S^{n-1}} \int_0^R \left\{ \int_0^{b} \left( e^{irP(x')} - e^{ir\|P\|_{L^\infty(S^{n-1})}} \right) \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right\} dx \right|.$$  

Using the simple estimate

$$\left| \int_0^R \left\{ e^{ir\epsilon} - e^{ir\epsilon} \right\} \frac{dr}{r} \right| \leq c + c \log \frac{b}{\epsilon},$$

we get

$$\left| \int_{|x|<R} e^{iP(x)} K(x) dx \right| \leq c\|\Omega\|_{L^1 L(S^{n-1})} + c \int_{S^{n-1}} \log \frac{\|P\|_{L^\infty(S^{n-1})}}{\|P(x')\|_{L^\infty(S^{n-1})}} \Omega(x') d\sigma_{n-1}(x').$$

Hence, $C_1 \leq c\|\Omega\|_{L^1 L(S^{n-1})} + c + \|\Omega\|_{L^1 L(S^{n-1})}$ and

$$C_{2^m} \leq cm\|\Omega\|_{L^1 L(S^{n-1})} + 1.$$  

The case of general $d$ is now trivial. If $2^{m-1} < d \leq 2^m$ then

$$C_d \leq C_{2^m} \leq cm\|\Omega\|_{L^1 L(S^{n-1})} + 1 \leq c\log d\|\Omega\|_{L^1 L(S^{n-1})} + 1.$$  

5. The one dimensional case revisited

We will attempt to give a short proof of the one dimensional analogue of theorem 1.1. This is a slight simplification of the proof in [2], with the aid of the logarithmic measure lemma.

So, fix a real polynomial $P(x) = b_0 + b_1 x + \cdots + b_dx^d$ and consider the quantity

$$C_d = \sup_{0<x<R} \left| \int_{|x|<R} e^{iP(x)} \frac{dx}{x} \right|.$$  

By the same considerations as in the $n$-dimensional case, we can assume that $P$ has no constant term and that it can be decomposed in the form

$$P(x) = \sum_{0<j \leq \frac{d}{2}} b_j x^j + \sum_{\frac{d}{2} < j \leq d} b_j x^j = Q(x) + R(x),$$

where $|b_j| \leq 1$ for every $\frac{d}{2} < j \leq d$. As a result

$$\left| \int_{|x|<R} e^{iP(x)} \frac{dx}{x} \right| \leq C_{\frac{d}{2}} + \int_{0<|x|<R} \frac{|R(x)|}{x} dx + \left| \int_{1<|x|<R} e^{iP(x)} \frac{dx}{x} \right| \leq C_{\frac{d}{2}} + c + I.$$
We split $I$ as follows

$$I \leq \left| \int_{\{x \in [1,R) : |P'(x)| > d\}} e^{iP(x)} \frac{dx}{x} \right| + \int_{\{x \geq 1 : |P'(x)| \leq d\}} \frac{dx}{x}.$$ 

Now, using Proposition 2.1 for the first summand in the above estimate and the logarithmic measure lemma to estimate the second summand, we get that $I \leq c$. But this means that $C_d \leq C_{d/2} + c$ which completes the proof by considering first the case $d = 2^m$ for some $m$, as in the $n$–dimensional case.

References


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