On the NP-hardness of computing stabilized Betti numbers

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Multiparameter Persistent Homology

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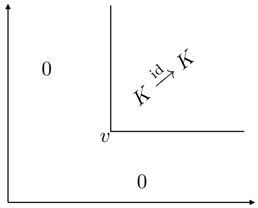

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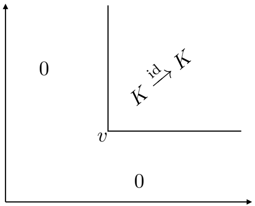

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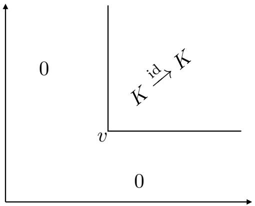

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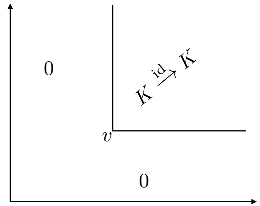

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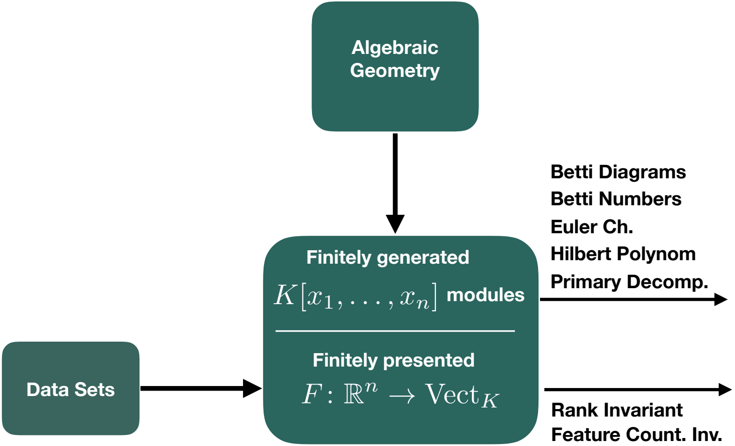
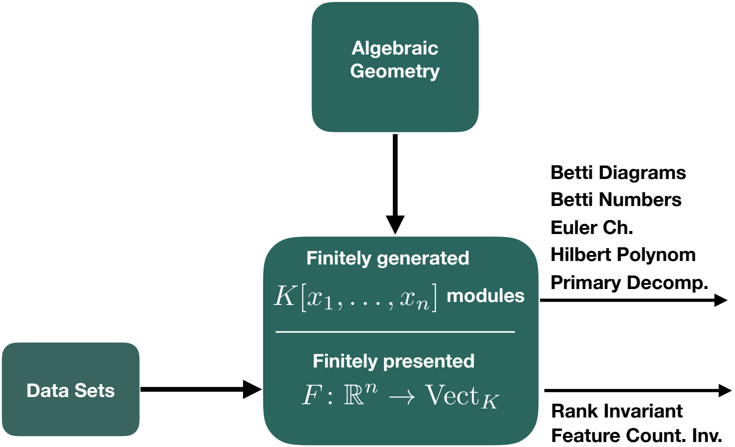
Slogan

We can identify a finitely presented persistence module with an n-graded $K[x_1, \ldots, x_n]$-module by restricting to a small enough grid $\mathbb{N}^n \subset \mathbb{R}^n$.

Invariants


Algebraic Geometry
Finitely generated
$K[x_1, \ldots, x_n]$ modules
Finitely presented
$F: \mathbb{R}^n \to \mathrm{Vect}_K$
Data Sets
Betti Diagrams
Betti Numbers
Euler Ch.
Hilbert Polynom
Primary Decomp.
Rank Invariant
Feature Count. Inv.

Invariants



Need for stable invariants!

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- $\widehat{\beta_0}(F)(\tau) = \min\{\text{rank}(G) \mid d(F, G) \leq \tau\}$
- Can be seen as a stabilization of a classical invariant, namely the zeroth Betti number.

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$$\beta_i \colon T \longrightarrow \mathbb{N}$$

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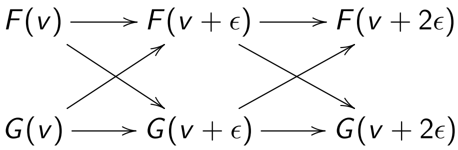

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For $n \geq 2$, computing $\widehat{\beta}_0(F)$ is NP-hard.

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- Computing the *feature counting invariant* of [5] is NP-hard.
- Extracting persistent information is hard.
- Indication that extracting persistent information from the rank invariant might be hard.

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  - For $K = \mathbb{R}$ you can convexify the problem.
  - For $K = \mathbb{Z}/2\mathbb{Z}$ you could use SAT solvers (which are very efficient).

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A *persistence contour* is a functor $C\colon \mathbb{R}^n_\infty \times \mathbb{R} \to \mathbb{R}^n_\infty$ s.t, for any $v \in \mathbb{R}^n_\infty$ and any $\epsilon, \tau \in \mathbb{R}$:

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- The *ϵ-shift*,

$$F[\epsilon] = \langle \{ F(v_i \leq C(v_i, \epsilon))(g_i) \in F(C(v_i, \epsilon)) \} \rangle$$

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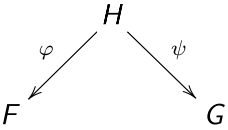
- 

  F and G are *ϵ-close* if $\ker \varphi[a] = \operatorname{coker} \varphi[b] = \ker \psi[c] =$ $= \operatorname{coker} \psi[d] = 0$ and $a + b + c + d \leq \epsilon$.

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- One known type of contour, coming from *domain noise* in [5], for which we can compute $\widehat{\beta}_0(F)$.

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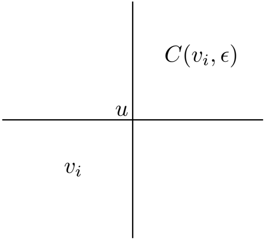
For ϵ large enough, there is u s.t:



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Can be phrased as the following *matrix rank minimization problem with an affine constraint set*:

Input Vectors $x_1, \ldots, x_m \in \mathbb{R}^n$ and subspaces $L_1, \ldots, L_m \subseteq \mathbb{R}^n$
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This is NP-hard, by reduction from a graph colouring problem [3].

Example

Calculating $\widehat{\beta}_0(F)(2)$ for the following functor requires you to solve such a rank minimization problem:

5	0	0	0	0	0	0
4	K	K^2	K^3/L_2	0	0	0
3	K	K^2	K^3	K^3/L_1	0	0
2	K	K^2	K^3	K^3	K^3/L_0	0
1	0	K	K^2	K^2	K^2	0
0	0	0	K	K	K	0
	0	1	2	3	4	5

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$$\min_{A \in \mathcal{C}} \quad \text{rank}(A) \quad \sim \quad \min_{A \in \mathcal{C}} \quad ||A||_*$$

where $||A||_* = \sum \sigma_i(A)$.

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$$\min_{A \in \mathcal{C} \subseteq \mathbb{R}^{m \times m}} \quad \text{rank}(A)$$

Case $K = \mathbb{R}$: The problem can be efficiently approximated by the nuclear norm [4]:

$$\min_{A \in \mathcal{C}} \quad \text{rank}(A) \quad \sim \quad \min_{A \in \mathcal{C}} \quad ||A||_*$$

where $||A||_* = \sum \sigma_i(A)$.

Case $K = \mathbb{Z}/2\mathbb{Z}$: For each k, we can write the determinants of the $k \times k$ minors as Boolean clauses. Can be solved using a modern SAT solver, which can handle millions of variables and clauses.

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  - Could learn a contour to solve classification/regression problems (contour learning).

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