

Green's Hyperplane Restriction Theorem: an extension to modules

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Abstract

In this paper, we prove a generalization of Green's Hyperplane Restriction Theorem to the case of modules over the polynomial ring, providing in particular an upper bound for the Hilbert function of the general linear restriction of a module M in a degree d by the corresponding Hilbert function of a lexicographic module.

Keywords: Hilbert function, General linear restriction, Lexicographic modules

1. Introduction

The extremal properties of the Hilbert function constitute a relevant topic in commutative algebra and, within this topic, an important role is played by lexicographic ideals and binomial representation of integers.

Many results about extremal behavior of Hilbert functions have been proved. In particular, Macaulay's Theorem (see [1, 2, 3]), which characterizes the possible Hilbert functions of homogeneous \mathbb{K} -algebras; or, the analogous result for exterior algebra, proved by Kruskal and Katona (see [4, 5]), which also provides a characterization of f -vectors of simplicial complexes.

More recently, in [6], Hulett proved an extension of Macaulay's Theorem to modules over the symmetric algebra.

Another classical, but slightly recent, result in this topic is the Hyperplane Restriction Theorem (HRT), given by Green in [7], which gives a bound for the codimension of the generic linear restriction of a vector space generated in a certain degree by the codimension of a *lex*-segment space with same degree and dimension. This result was also used by Green to give another,

less technical, proof of Macaulay's Theorem.

It has been also been applied by several authors (see for instance [8, 9, 10]) to get some results about level and Gorenstein algebras, with focus on the weak Lefschetz property.

In this paper, we will focus on Green's theorem, trying to follow Hulett's path in generalizing these results to modules over the polynomial ring.

One generalization in this direction has already been done by Gasharov ([11]), but we will give an extension of Green's theorem that provides a bound which is actually achieved by lexicographic modules.

We will first give the tools to state Green's Hyperplane Restriction Theorem: namely we introduce the binomial expansion of a positive integer, also called Macaulay representation, then, the concept of generic linear forms. Afterwards, we will state Green's theorem.

In the next section, we will extend some of these definitions to the submodule of a finitely generated free module, in particular we will define a monomial order induced by the *deglex* ordering, and will provide the concept of lexicographic module, which is a particular class of monomial modules.

We first try to extend Green's theorem in the simplest case of a rank 2 module, by proving an inequality of Macaulay representations. Later we prove by induction a new inequality which imply the main theorem, namely the extension of the Hyperplane Restriction Theorem to the case of modules over the polynomial ring.

As a consequence of our theorem, we will derive a result which constitute another version of the theorem "after scaling", which means that we divide by the respective Hilbert function of the polynomial ring.

Finally, we apply our main theorem to level algebras, by giving some conditions in which the bound given by the extension to modules of the Green's Hyperplane Restriction Theorem is more useful than the one obtained trough the Green's Hyperplane Restriction Theorem.

2. Green's Theorem

Let $S = \mathbb{K}[x_1, \dots, x_n]$, where \mathbb{K} is an infinite field, and let S be standard graded. Let us fix the *deglex* monomial ordering on S with $x_1 > x_2 > \dots >$

x_n , namely

$$\mathbf{x}^{\mathbf{a}} >_{deglex} \mathbf{x}^{\mathbf{b}} \Leftrightarrow |\mathbf{a}| > |\mathbf{b}| \text{ or } |\mathbf{a}| = |\mathbf{b}| \text{ and } \mathbf{a} >_{lex} \mathbf{b},$$

(see [12]). Notice that in a homogeneous component S_d the monomial orderings, *deglex* and *lex*, coincide.

Definition 2.1. Let $a, d \in \mathbb{N}$, then the d -th Macaulay representation of a (also called the binomial representation of a in base d) is the unique way to write

$$a = \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_\delta}{\delta},$$

where $a_d > a_{d-1} > \cdots > a_\delta \geq \delta$ and $\delta = \min\{i \mid a_i \geq i\}$.

Let us set $\binom{c}{d} = 0$ whenever $c < d$, then, given the d -th Macaulay representation of a , let us define the integer

$$a_{\langle d \rangle} = \binom{a_d - 1}{d} + \binom{a_{d-1} - 1}{d-1} + \cdots + \binom{a_\delta - 1}{\delta}.$$

Sometimes it will be convenient to use the extended d -th Macaulay representation, i.e. $\binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_1}{1}$, where $a_d > a_{d-1} > \cdots > a_1 \geq 0$.

Moreover, since $\binom{a}{b} = 0$ if $a < b$, we will set the numerators of such binomial coefficients in the Macaulay representation equal to zero.

Observation 2.2. If $a = \binom{a_d}{d} + \binom{a_{d-1}}{d-1} + \cdots + \binom{a_1}{1}$ and $b = \binom{b_d}{d} + \binom{b_{d-1}}{d-1} + \cdots + \binom{b_1}{1}$, $a \geq b$ if and only if $(a_d, \dots, a_1) \geq_{lex} (b_d, \dots, b_1)$.

Definition 2.3. Let $d \in \mathbb{N}$, a *lex-segment* is a set constituted by all monomials in S_d lexicographically larger or equal than f , for some $f \in \text{Mon}(S_d)$. A \mathbb{K} -vector subspace V of S_d is a *lex-segment space* if $\text{Mon}(S_d) \cap V$ is a \mathbb{K} -basis of V and a *lex-segment*.

Definition 2.4. A graded monomial ideal I in S is said to be *lexicographic* if, for all degrees d , the homogeneous component I_d is a *lex-segment space*.

Definition 2.5. We say that a property \mathcal{P} holds for a generic linear form ℓ if there is a non-empty Zariski open set $\mathcal{U} \subseteq S_1$ such that \mathcal{P} holds for all $\ell \in \mathcal{U}$.

If $V \subseteq S_d$ a *lex-segment space*, we denote by V_ℓ the image of V in $S/(\ell)$.

Proposition 2.6. [2, Proposition 5.5.23] *Let \mathbb{K} be an infinite field, $d \in \mathbb{N}$, $V \subseteq S_d$ a lex-segment space. Then:*

1. *For a generic linear form $\ell \in S_1$, there is a homogeneous linear change of coordinates $\phi : S \rightarrow S$ such that $\phi(V) = V$ and $\phi(\ell) = x_n$;*
2. *For a generic linear form $\ell \in S_1$, we have*

$$\text{codim}_{\mathbb{K}}(V_{\ell}) = \text{codim}_{\mathbb{K}}(V_{x_n}) = \text{codim}_{\mathbb{K}}(V)_{\langle d \rangle}.$$

So, after a general change of coordinates, we may assume that the generic linear form is x_n .

From now on, we will denote by $HF(M, d)$ the Hilbert function of the S -module M in degree d , i.e. $\dim_{\mathbb{K}}(M_d)$. Moreover, if $\ell \in S_1$ is a linear form and $R = S/I$, we denote by R_{ℓ} the ring $S/[I + (\ell)]$.

Theorem 2.7 (Green's Hyperplane Restriction Theorem). *Let I be a homogeneous ideal in S , $d \in \mathbb{N}$, then*

$$HF((S/I)_{\ell}, d) \leq HF(S/I, d)_{\langle d \rangle},$$

where ℓ is a generic linear form. Moreover, equality holds when I_d is a lex-segment space.

3. Extension to modules

In the case of the restriction of an S -module M to a generic hyperplane ℓ we expect not only an upper bound on the Hilbert function $HF(M_{\ell}, d)$, but also that this upper bound is fully described by lexicographic modules, in complete analogy with Green's theorem.

In this section, we will be able to prove some numerical properties of the Macaulay's representations, that will lead to a version of Green's HRT for modules valid in any characteristic of the infinite field \mathbb{K} .

Let F be a finitely generated graded free S -module, let us fix a homogeneous basis $\{e_1, e_2, \dots, e_r\}$ and let $\deg(e_i) = f_i$, where, without loss of generality, we may assume that $f_1 \leq f_2 \leq \dots \leq f_r$.

We now define the monomial modules and we induce a monomial ordering on F in such a way that the concept of lexicographic module may be defined. We also define another monomial ordering on F , a particular *revlex*, that will let us use the tool of generic initial modules in our circumstances.

Definition 3.1. A monomial in F is an element of the form me_i where $m \in \text{Mon}(S)$. A submodule $M \subseteq F$ is *monomial* if it is generated by monomials, in this case it can be written as $I_1e_1 \oplus I_2e_2 \oplus \cdots \oplus I_re_r$, where I_i is a monomial ideal for $i = 1, 2, \dots, r$.

One can extend a monomial ordering, defined in the polynomial ring S , to a finitely generated free S -module in a really natural way, in particular here we define the *deglex* ordering in F .

Definition 3.2. Given two monomials in F , me_i and ne_j , we say that $me_i >_{\text{deglex}} ne_j$ if either $i = j$ and $m >_{\text{deglex}} n$ in S or $i < j$. In particular, we have that $e_1 > e_2 > \cdots > e_r$.

Again, as in the polynomial ring, when we restrict to a precise homogeneous component, the two induced monomial orderings, *deglex* and *lex*, coincide.

Definition 3.3. A monomial graded submodule L is a *lexicographic module* if, for every degree d , L_d is spanned by the largest, with respect to the lexicographic order (or *deglex*), $HF(L, d)$ monomials.

Let us now recall the definition of reverse lexicographic order on F as given in ([13, pg. 339]), some easy results about it.

Definition 3.4. The *reverse lexicographic order* on F is defined by choosing an order on the basis of F , say $e_1 > \cdots > e_r$ and by setting $me_i >_{\text{revlex}} ne_j$ iff either $\deg(me_i) > \deg(ne_j)$ or the degrees are the same and $m >_{\text{revlex}} n$ in S or $m = n$ and $i < j$.

Remark 3.5. Notice that the *deglex* is a POT (Position over Term) monomial order, on the contrary the *revlex* is TOP (Term over Position) monomial order.

Definition 3.6. The initial module of a submodule M , denoted by $\text{in}(M)$, is the submodule of F generated by the set $\{\text{in}(m) \mid m \in M\}$, i.e. by all the leading terms (according to the lexicographic order or *deglex* or *revlex*) of elements in M .

Proposition 3.7. [13, Proposition 15.12] Suppose that F is a free S -module with basis $\{e_1, \dots, e_r\}$ and reverse lexicographic order. Let M be a graded submodule.

Then $\text{in}(M + x_n F) = \text{in}(M) + x_n F$ and $(\text{in}(M) :_F x_n) = \text{in}(M :_F x_n)$.

Let us define the concept of *generic initial module* as done in ([14, Chapters 1 and 2]).

Let $\mathrm{GL}(n)$ be the general linear group. It acts on S by $(a_{ij})x_j = \sum_{i=1}^n a_{ij}x_i$. Let $\mathrm{GL}(F)$ be the group of the S -module automorphisms of F . An element ϕ in $\mathrm{GL}(F)$ is a homogeneous automorphism and can be represented by a matrix (t_{ij}) , with $t_{ij} \in S_{f_i - f_j}$, where $\phi(e_i) = \sum_{j=1}^r t_{ij}e_j$. We also have an action of $\mathrm{GL}(n)$ on $\mathrm{GL}(F)$ given by $a \cdot \phi = a\phi a^{-1}$, for all $a \in \mathrm{GL}(n)$ and $\phi \in \mathrm{GL}(F)$. The two groups together act on F by mean of their semi-direct product $G = \mathrm{GL}(n) \rtimes \mathrm{GL}(F)$.

Let us consider $B(n)$ the subgroup of $\mathrm{GL}(n)$ of upper triangular invertible matrices: from the action given above, a matrix in $B(n)$ will send x_1 to a multiple of itself, and x_n to the linear combination of all variables x_i 's with coefficients the last column of the matrix. Let $B(F)$ be the subgroup of all automorphisms in $\mathrm{GL}(F)$ represented by lower triangular matrices: they send each e_l to an S -linear combination of e_1, \dots, e_l . Let $B = B(n) \rtimes B(F)$. Pardue, in [14], proved the generalization to modules of the Galligo's theorem, in a more general setting than ours, and in particular he proved the following result:

Proposition 3.8. *Let $M \subset F$ be a graded module, and let $<$ be a monomial order on F , then there exists a Zariski open set $U \subseteq G$ such that $\mathrm{in}_{<}(\phi M)$ is constant for every $\phi \in U$. Moreover, $\mathrm{in}_{<}(\phi M)$ is fixed by the action of group B .*

Definition 3.9. The monomial submodule $\mathrm{in}_{<}(\phi M)$, $\phi \in U$, is called the *generic initial module* of M , and denoted by $\mathrm{gin}_{<}(M)$.

The generic initial module has really nice properties in case we use, as monomial order on F , the *revlex* order defined above.

Proposition 3.10. *For a generic linear form ℓ and a graded submodule M of F , we have that*

$$\mathrm{gin}(M_\ell) = \mathrm{gin}(M)_{x_n}.$$

The *revlex* $\mathrm{gin}(M)$ has the properties in 3.7, and moreover, since it is B -fixed, we can work in generic coordinates, and choose the coordinates in such a way that $\ell = x_n$.

If $\ell \in S_1$ is a linear form, and M a graded submodule of F , let us denote by $(F/M)_\ell$ the restriction of the module F/M to ℓ , which is equal to

$F/(M + \ell F)$. Notice that F_ℓ is a module over S_ℓ and that S_ℓ can be thought as polynomial ring in $n - 1$ variables.

Before the statement of the main theorem of the section, we are going to prove two propositions, regarding the behavior of the function $*_{\langle d \rangle} : \mathbb{N} \rightarrow \mathbb{N}$ (in our convention \mathbb{N} includes 0): in particular we need to provide an upper bound to the function $a_{\langle d_1 \rangle} + b_{\langle d_2 \rangle}$, where $d_1 \geq d_2$, generalizing [11, Lemma 4.4]; and then, we will extend the latter inequality to the sum of $r = \text{rank}(F)$ integers.

These two proposition will be just the numerical translations of the extension of Green's theorem, respectively in the $\text{rank}(F) = 2$ case, and in the $\text{rank}(F) = r$ case.

Lemma 3.11. [11, Lemma 4.4, Lemma 4.5] *For any $a, b, d \in \mathbb{N}$, the two following inequalities hold: $a_{\langle d \rangle} + b_{\langle d \rangle} \leq (a + b)_{\langle d \rangle}$; $a_{\langle d+1 \rangle} \leq a_{\langle d \rangle}$.*

Remark 3.12. If $a, d \in \mathbb{N}$ and $a = \binom{a_d}{d} + \dots + \binom{a_\delta}{\delta}$ ($a_\delta \geq \delta$) is the d -th Macaulay representation of a , then

$$(a - 1)_{\langle d \rangle} = a_{\langle d \rangle} \quad \text{if and only if} \quad a_\delta = \delta.$$

Proposition 3.13. *Given $a, b \in \mathbb{N}$, $a \leq N_1 = \binom{n+d_1-1}{d_1}$, $b \leq N_2 = \binom{n+d_2-1}{d_2}$ and $d_1 \geq d_2$, $d_1, d_2 \in \mathbb{N}$, then*

$$a_{\langle d_1 \rangle} + b_{\langle d_2 \rangle} \leq \begin{cases} (a + b)_{\langle d_2 \rangle} & \text{if } a + b \leq N_2, \\ (a + b - N_2)_{\langle d_1 \rangle} + (N_2)_{\langle d_2 \rangle} & \text{if } a + b \geq N_2. \end{cases}$$

Proof We will now prove the claim by induction on $i = a + b$.

The case $i = 0$ is trivial. Thus, let the claim be true for all a', b' such that $0 < a' + b' < i$.

Let $a = \binom{a_{d_1}}{d_1} + \dots + \binom{a_{d_2}}{d_2} + \dots + \binom{a_1}{1}$, $b = \binom{b_{d_2}}{d_2} + \dots + \binom{b_1}{1}$, where some of the last top coefficients in the two Macaulay representations may be zero, and let us denote these Macaulay representations as vectors $a = (a_{d_1}, \dots, a_1)$ and $b = (b_{d_2}, \dots, b_1)$.

Let us now distinguish between two cases.

First Case If the d_1 -Macaulay representation of a ends after degree d_2 , i.e. $\min\{i \mid a_i \geq i\} \leq d_2$, we write $a = (a_{d_1}, \dots, a_{d_2}, \dots, a_1)$ and $b = (b_{d_2}, \dots, b_1)$, where in both vectors the last entries, after degree

d_2 , may be zero.

Then, we redefine a and b as follows, sorting them entry by entry:

$$\begin{aligned} a &= (a_{d_1}, \dots, a_{d_2+1}, \min\{a_{d_2}, b_{d_2}\}, \dots, \min\{a_1, b_1\}), \\ b &= (\max\{a_{d_2}, b_{d_2}\}, \dots, \max\{a_1, b_1\}). \end{aligned}$$

Observe that, in this process, we are decreasing a of the same amount we increase b (we are not modifying $a + b$) and we are not changing $a_{\langle d_1 \rangle} + b_{\langle d_2 \rangle}$, since the sum of the two binomial expansions will still contain the same terms. Moreover, in this way, we obtain that the Macaulay representation of a ends sooner than that of b .

The idea is to find a', b' such that $a' + b' = a + b$ and $a \geq a', b' \geq b$ and such that the following are satisfied:

1. $a'_{\langle d_1 \rangle} + b'_{\langle d_2 \rangle} \geq a_{\langle d_1 \rangle} + b_{\langle d_2 \rangle}$;
2. if $i < N_2$, either $a' = 0$ and $b' = i$; if $i \geq N_2$, either $a' = i - N_2$ and $b' = N_2$; or at least one between a' and b' has Macaulay representation ending with a binomial coefficient of the form $\binom{t}{i}$ (since, in this case, we can use Remark 3.12 and apply induction hypothesis).

So, we can suppose that

$$\begin{aligned} a &= (a_{d_1}, \dots, a_{d_2}, a_{d_2-1}, \dots, a_s, 0, \dots, 0) \\ b &= (b_{d_2}, b_{d_2-1}, \dots, b_t, 0, \dots, 0), \end{aligned}$$

where $a_s > s$ and $b_t > t$ (otherwise we can use Remark 3.12 and apply induction hypothesis), and $s \geq t$.

We need to show how to find a' and b' , by decreasing a and increasing b of the same amount in three different cases.

In the case in which $s = t = 1$, we have $a_1 > 1$ and $b_1 > 1$, we can consider $(a_{d_1}, \dots, a_{d_2}, \dots, a_1 - 1)$ and $(b_{d_2}, \dots, b_2, b_1 + 1)$, without changing $a_{\langle d_1 \rangle} + b_{\langle d_2 \rangle}$. Applying this several times we will get to the point in which $a_1 = 1$, so we have reached the desired a' and b' .

The second case is when $t = 1$ and $s > 1$, here we have to write $\binom{a_s}{s} = \sum_{j=0}^s \binom{a_s - (j+1)}{s-j}$, and then consider $(a_{d_1}, \dots, a_{d_2}, \dots, a_{s+1}, a_s - 1, \dots, a_s - s)$ and $(b_d, \dots, b_2, b_1 + 1)$ (we are decreasing a and increasing b by 1). After this step, we are again in the first case.

Notice that in all the three cases, whenever it occurs that, after

the changes, two consecutive entries are the same (so the vector obtained does not represent a Macaulay representation), say the tale is $(\dots, c(\text{position } i), c, 0, \dots, 0)$, we should transform it into $(\dots, c + 1(\text{position } i), 0, \dots, 0)$; after this, we may compare and sort the two vectors: namely, it can happen that $b_1 + 1 = b_2$, then we have to consider $(b_{d_2}, \dots, b_2 + 1, 0)$ and then we sort again by taking the $\min\{a_i, b_i\}$ and the $\max\{a_i, b_i\}$.

The last case is when $s > t > 1$: here we write $\binom{a_s}{s} = \sum_{j=0}^{s-t} \binom{a_s - (j+1)}{s-j} + \binom{a_s - (s-t+1)}{t-1}$; so we consider:

$$\begin{aligned} & (a_{d_1}, \dots, a_{d_2}, \dots, a_{s+1}, a_s - 1, \dots, a_s - (s - t + 1) \text{ (position } t), \\ & a_s - (s - t + 1) \text{ (position } t - 1), 0, \dots, 0) \\ & (b_{d_2}, b_{d_2-1}, \dots, b_t, 0, \dots, 0), \end{aligned}$$

the first vector does not correspond to a Macaulay representation since the last two non zero entries are the same. Nevertheless, when we resort the two vectors by taking the min and the max in each modified degree, we will, in particular, exchange the number $a_s - (s - t + 1)$ in the first vector in position $t - 1$ with the 0 in the same position in the second vector; thus, after this exchange, the two vectors will correspond to Macaulay representations, and we shall proceed, depending on the case the belong.

Second Case When the Macaulay representation of a ends before the degree d_2 , say in degree j , say k is the smallest degree for which $b_k \geq k$. If $d_2 > 1$, then let us consider the decomposition:

$$\binom{a_j}{j} = \sum_{i=k+1}^j \binom{a_j - (j - i + 1)}{i} + \binom{a_j - (j - k)}{k};$$

so we write

$$\begin{aligned} a = & (a_{d_1}, \dots, a_{j+1}, a_j - 1, \dots, a_j - (j - d_2 + 1), \dots, \\ & a_j - (j - k), a_j - (j - k), 0, \dots, 0), \end{aligned}$$

again this is not a Macaulay representation, then we compare, entry by entry, with b moving the smaller entries in the new vector a and the larger entries in b .

The only case in which this cannot work is in the trivial case $b = (\beta)$, $a = (a_{d_1}, \dots, \alpha, \alpha)$ (after having used the previous decomposition) and $\alpha = \beta$, in this situation we just consider $(a_{d_1}, \dots, \alpha, \alpha - 1)$ and $(\beta + 1)$.

So, for all the possible cases we have found a way to subtract to a and add to b the same number in such a way that the function $f(x, y) = x_{\langle d_1 \rangle} + y_{\langle d_2 \rangle}$ does not decrease: we will continue, recursively, to do this until we get the extremal cases, or until one of the two numbers has Macaulay representation ending with $\binom{t}{t}$.

□

With the following two examples, we aim to clarify the idea of the previous proof in both cases.

Example

Let us consider $a = 9 = \binom{4}{2} + \binom{3}{1} \cong (4, 3)$ and $b = 14 = \binom{5}{2} + \binom{4}{1} \cong (5, 4)$. This choice belongs to the first case, and since $s = t = 1$, the first thing to do is:

$$a \cong (4, 3) \rightarrow (4, 2), \quad b \cong (5, 4) \rightarrow (5, 5),$$

since $(5, 5)$ is not a Macaulay representation, we should write it as $(6, 0)$. Now we can compare the two obtained vectors, $(4, 2)$ and $(6, 0)$, so that we have

$$(4, 2) \rightarrow (4, 0), \quad (6, 0) \rightarrow (6, 2).$$

Now we are in the first case, in the subcase where $2 = s > t = 1$. Thus by applying the instructions:

$$(4, 0) \rightarrow (3, 2), \quad (6, 2) \rightarrow (6, 3).$$

Now, applying again the instructions for the first case when $s = t = 1$, we get:

$$(3, 2) \rightarrow (3, 1) \cong a', \quad (6, 3) \rightarrow (6, 4) \cong b',$$

and we can stop since we have, as last entry, $\binom{1}{1}$. Notice that $a + b = a' + b' = 23$, and that $a'_{\langle 2 \rangle} + b'_{\langle 2 \rangle} = 15 > 14 = a_{\langle 2 \rangle} + b_{\langle 2 \rangle}$ □

Example

As an example of the second case, let us consider $a = 35 = \binom{7}{4} \cong (7, 0, 0, 0)$ and $b = 21 = \binom{6}{4} + \binom{4}{2} \cong (6, 4, 0)$. Then by applying the argument:

$$(7, 0, 0, 0) \rightarrow (6, 5, 5, 0),$$

and by comparing entry by entry we get

$$(6, 5, 5, 0) \rightarrow (6, 5, 4, 0), \quad (6, 4, 0) \rightarrow (6, 5, 0),$$

thus now we are in the first case. \square

Observation 3.14. Notice that the previous proposition is just the numerical translation of the generalization of Green's HRT to the case in which we have $F = Se_1 \oplus Se_2$, with $e_1 >_{deglex} e_2$, $deg(e_1) = f_1$, $deg(e_2) = f_2$ and $f_1 \leq f_2$. In fact, for any degree $d \in \mathbb{N}$, we set $d_1 = d - f_1$, $d_2 = d - f_2$, and $N_1 = \binom{n+d_1-1}{d_1} = \dim_k(S_{d_1})$, $N_2 = \binom{n+d_2-1}{d_2} = \dim_k(S_{d_2})$.

Remark 3.15. The previous proposition implies the first part of Lemma 3.11: in fact, when we have $f_1 = f_2$, and so $N_1 = N_2 = N$, we can repeat the same argument of the proof of Proposition 3.13, with the only difference that we continue decreasing a and increasing b of the same amount until $a = 0$ and $b = i$, i.e. without distinguish between the cases in which $i \leq N$ and $i \geq N$.

The following proposition just extends the previous inequality to the case of more summands.

Proposition 3.16. *Let $d_1, \dots, d_r \in \mathbb{N}$ a non-increasing sequence, and set $N_i = \binom{n+d_i-1}{d_i}$. Let a_1, \dots, a_r be non-negative integers such that, for each i , $a_i \leq N_i$. Let j be the integer such that $\sum_{i=j+1}^r N_i < \sum_{i=1}^r a_i \leq \sum_{i=j}^r N_i$, then the following inequality holds:*

$$\sum_{i=1}^r a_{i\langle d_i \rangle} \leq \left(\sum_{i=1}^r a_i - \sum_{i=j+1}^r N_i \right)_{\langle d_j \rangle} + \sum_{i=j+1}^r N_{i\langle d_i \rangle}.$$

Proof Let us start with the case $j = r$, i.e. $0 < \sum_{i=1}^r a_i \leq N_r$, then to prove the inequality, i.e. $\sum_{i=1}^r a_{i\langle d_i \rangle} \leq (\sum_{i=1}^r a_i)_{\langle d_r \rangle}$, it is enough to use Lemma 3.11.

Let us prove the statement by induction on r . The case $r = 2$ is given by the previous result, let us now suppose that the inequality holds for $s < r$. Let $\sum_{i=j+1}^r N_i < \sum_{i=1}^r a_i \leq \sum_{i=j}^r N_i$. We then distinguish between two cases.

In case $a_j + a_{j+1} \leq N_{j+1}$, then $a_{j\langle d_j \rangle} + a_{j+1\langle d_{j+1} \rangle} \leq (a_j + a_{j+1})_{\langle d_{j+1} \rangle}$.

Since $\sum_{i=j+1}^r N_i < \sum_{i=1}^r a_i \leq \sum_{i=j}^r N_i \leq N_{j-1} + \sum_{i=j+1}^r N_i$, we can apply

the induction to have the following:

$$\begin{aligned} \sum_{i=1}^r a_{i\langle d_i \rangle} &\leq \sum_{i=1, i \neq j, j+1}^r a_{i\langle d_i \rangle} + (a_j + a_{j+1})_{\langle d_{j+1} \rangle} \leq \left(\sum_{i=1}^r a_i - \sum_{i=j+1}^r N_i \right)_{\langle d_{j-1} \rangle} + \\ &+ \sum_{i=j+1}^r N_{i\langle d_i \rangle} \leq \left(\sum_{i=1}^r a_i - \sum_{i=j+1}^r N_i \right)_{\langle d_j \rangle} + \sum_{i=j+1}^r N_{i\langle d_i \rangle}. \end{aligned}$$

In the case in which $a_j + a_{j+1} > N_{j+1}$, then, by the base of induction, $a_{j\langle d_j \rangle} + a_{j+1\langle d_{j+1} \rangle} \leq (a_j + a_{j+1} - N_{j+1})_{\langle d_j \rangle} + N_{j+1\langle d_{j+1} \rangle}$. We have that $a_j + a_{j+1} - N_{j+1} \leq N_j$ and $\sum_{i=j+2}^r N_i < \sum_{i=1}^r a_i - N_{j+1} \leq N_j + \sum_{i=j+2}^r N_i$, thus by induction:

$$\sum_{i=1, i \neq j, j+1}^r a_{i\langle d_i \rangle} + (a_j + a_{j+1} - N_{j+1})_{\langle d_j \rangle} \leq \left(\sum_{i=1}^r a_i - \sum_{i=j+1}^r N_i \right)_{\langle d_j \rangle} + \sum_{i=j+2}^r N_{i\langle d_i \rangle},$$

By adding $N_{j+1\langle d_{j+1} \rangle}$ to both sides, we get

$$\sum_{i=1}^r a_{i\langle d_i \rangle} \leq \left(\sum_{i=1}^r a_i - \sum_{i=j+1}^r N_i \right)_{\langle d_j \rangle} + \sum_{i=j+1}^r N_{i\langle d_i \rangle}.$$

□

The idea for the extension of the Green's theorem to modules is to reduce to the monomial case: a monomial submodule is direct sum of monomial components, and so we may just apply Theorem 2.7 to each of these components. Afterwards we need to bound the Hilbert function of the quotient by the monomial submodule by the Hilbert function of the quotient by lexicographic module.

Let us now give a definition that will be used inside the statement of our generalization of Green's theorem, and then we are ready to prove our main result.

Definition 3.17. Let M be a submodule in F , where F be a finitely generated graded free S -module, and $m \in \mathbb{N}$. Set $d_i = m - f_i$ ($\{d_1, d_2, \dots, d_r\}$

is a non-increasing sequence) and $N_i = \binom{n+d_i-1}{d_i} = \dim_{\mathbb{K}} S_{d_i}$. Then, if $\sum_{i=j+1}^r N_i \leq HF(F/M, m) \leq \sum_{i=j}^r N_i$, for some j , we define

$$HF(F/M, m)_{\{m,r\}} = \left(HF(F/M, m) - \sum_{i=j+1}^r N_i \right)_{\langle d_j \rangle} + \sum_{i=j+1}^r N_{i\langle d_i \rangle}.$$

Theorem 3.18. *Let $F = Se_1 \oplus \cdots \oplus Se_r$ where $\deg(e_i) = f_i$ for all i . Let M be a submodule in F , then*

$$HF((F/M)_{\ell}, m) \leq HF((F/L)_{\ell}, m)$$

where ℓ is generic linear form, $m \in \mathbb{N}$, and L is a submodule that in degree m is generated by a lex-segment of length $HF(M, m)$.

Moreover,

$$HF((F/L)_{\ell}, m) = HF(F/M, m)_{\{m,r\}}.$$

Proof Assume that M is a monomial submodule of F . Otherwise, let us take the generic initial module of M , $\text{gin}(M)$, with respect to the monomial order *revlex*. We get a new submodule which has the same Hilbert function of M and moreover is a monomial submodule of F . We have that $\text{gin}(M)_{x_n} = \text{gin}(M_{\ell})$. If we work in generic coordinates, we can choose $\ell = x_n$ and so we get that $HF((F/M)_{\ell}, d) = HF((F/\text{gin}(M))_{\ell}, d)$. So we may substitute M by $\text{gin}(M)$ in order to get a monomial submodule. Thus, we can write M as $I_1 e_1 \oplus \cdots \oplus I_r e_r$, where each I_i is a monomial ideal, hence a homogeneous ideal in R .

Let $d \in \mathbb{N}$ and ℓ be a generic linear form, then

$$\begin{aligned} (F/M)_{\ell} &= F/(M + \ell F) \cong F/[(I_1 + (\ell))e_1 \oplus \cdots \oplus (I_r + (\ell))] \cong \\ &(S/I_1)_{\ell} e_1 \oplus \cdots \oplus (S/I_r)_{\ell} e_r. \end{aligned}$$

So we have that, by applying Green's HRT to each $(S/I_i)_{\ell}$,

$$HF((F/M)_{\ell}, m) = \sum_{i=1}^r HF((S/I_i)_{\ell}, m - f_i) \leq \sum_{i=1}^r HF(S/I_i, m - f_i)_{\langle m-f_i \rangle}. \quad (1)$$

Finally, by Proposition 3.16, the last sum in (1) is bounded by

$$\left(HF(F/M, m) - \sum_{i=j+1}^r N_i \right)_{\langle m-f_j \rangle} + \sum_{i=j+1}^r N_{i\langle m-f_i \rangle}$$

for some j . The last term is exactly $HF((F/L)_\ell, m)$ where L is a submodule such that L_m is generated by a *lex*-segment of length $HF(M, m)$ (according to the monomial ordering induced to the module). □

3.1. Green's HRT after scaling for modules

We now derive from last theorem a result, that will give the possibility to apply Green's HRT in case it is difficult to have the Macaulay binomial representations. The idea is to consider the Hilbert function up to a multiplication by an integer. In this way, we will obtain a linear inequality, that will hold for all modules.

Corollary 3.19. *Let T be a module over $S = \mathbb{K}[x_1, \dots, x_n]$, generated in degree 0, and let ℓ be a generic linear form, $d \in \mathbb{N}$, then*

$$HF(T_\ell, d) \leq \frac{n-1}{n+d-1} HF(T, d).$$

Proof For any integer δ , we have that $HF\left(\bigoplus_{i=1}^{\delta} T_\ell, d\right) = \delta HF(T_\ell, d)$, so we can choose δ such that there is $\beta \in \mathbb{N}$ satisfying the following equality

$$\delta HF(T, d) = \beta \dim_{\mathbb{K}}(S_d) = \beta \binom{n+d-1}{d},$$

so by Theorem 3.18 we have

$$\begin{aligned} HF\left(\bigoplus_{i=1}^{\delta} T_\ell, d\right) &\leq \beta \binom{n+d-1}{d} \binom{n+d-2}{d} = \\ &\beta \binom{n+d-1}{d} \frac{n-1}{n+d-1} = \frac{n-1}{n+d-1} HF\left(\bigoplus_{i=1}^{\delta} T, d\right), \end{aligned}$$

then, dividing by δ , we get the claim. □

Observation 3.20. This corollary can be described by the expression "after scaling" because the last fraction satisfies the following equality

$$\frac{n-1}{n+d-1} = \frac{\dim_{\mathbb{K}}(R_d)}{\dim_{\mathbb{K}}(S_d)},$$

where $S = \mathbb{K}[x_1, x_2, \dots, x_n]$ and $R = \mathbb{K}[x_1, x_2, \dots, x_{n-1}] \cong S_\ell$.

Therefore, the claim of the Corollary may be written as:

$$\frac{HF(T_\ell, d)}{HF(S_\ell, d)} \leq \frac{HF(T, d)}{HF(S, d)},$$

so we have scaled by respective value of the Hilbert function of the rings over which we have the two modules.

4. Application to level algebras

In this section we will apply the extension of the Green's HRT to level algebras.

Definition 4.1. Let $A = S/I = A_0 \oplus \dots \oplus A_c$ be an artinian \mathbb{K} -algebra. Let us consider the *socle* of A , which is defined by $\text{soc}(A) = \text{ann}_{\mathbf{m}}(A)$, where $\mathbf{m} = \bigoplus_{i>0} A_i$. If $\text{soc}(A) = \mathcal{U}_0 \oplus \dots \oplus \mathcal{U}_c$, we say that A is *level of type b* if $\text{soc}(A) = \mathcal{U}_c = A_c$ and $\dim_{\mathbb{K}}(A_c) = b$. We say that A is *Gorenstein* when it is level of type 1.

Let $A = A_0 \oplus \dots \oplus A_c$ be a level algebra with Hilbert function (h_0, h_1, \dots, h_c) . By considering the dual of A , $A^\vee = \text{Hom}_{\mathbb{K}}(A, \mathbb{K})$, we get a *level module*, since is an artinian S -module with generators in a single degree, and with one-dimensional socle, i.e. also concentrated in a single degree.

We want to consider the multiplication by a generic linear form $\ell \in A_1$.

We first apply Theorem 2.7. In this way we get the following bound on the Hilbert function of the ideal (ℓ) .

$$\dim_{\mathbb{K}}(\ell)_i \geq h_i - (h_i)_{\langle i \rangle}, \quad 0 \leq i \leq c.$$

We can also apply Theorem 3.18 to the level module $M = A^\vee(-c)$. Notice that M has Hilbert function $(h_c, h_{c-1}, \dots, h_0)$. We get in this case the following bound for the module ℓM :

$$\dim_{\mathbb{K}}(\ell M)_i \geq h_{c-i} - (h_{c-i})_{\{i\}}, \quad 0 \leq i \leq c.$$

In the right hand side of this inequality, $a_{\{i\}}$ denotes $q(s_i)_{\langle i \rangle} + r_{\langle i \rangle}$, where $a = qs_i + r$ and $s_i = \binom{i+n-1}{i}$.

We observe that $(\ell) \cong (\ell M)^\vee(-c-1)$. This implies that we can give two lower bounds for the Hilbert function of (ℓ) , namely h_i^{GM} and h_i^G , where we set $h_i^{GM} = h_i - (h_i)_{\{c-i\}}$ and $h_i^G = h_{i+1} - (h_{i+1})_{\langle i+1 \rangle}$.

We would like to find conditions in which $h_i^{GM} > h_i^G$, so when the bound given by the HRT for modules is better than the one given by the Green's HRT.

In [15], Geramita, Harima, Migliore and Shin, in case the polynomial ring S has three variables, provided a list of all the possible Hilbert functions for level algebras of socle in degree 3, 4, 5 and in degree 6 and type 2. Among these, the condition $h_i^{GM} > h_i^G$ is true only for the ones contained in Table 1 in page 16.

Position	h	h^{GM}	h^G
2	1, 3, 3, 3, 2	1, 3, 2, 1	1, 2, 3, 2
2	1, 3, 3, 3, 3	1, 3, 2, 1	1, 2, 3, 3
2	1, 3, 3, 3, 3, 2	1, 3, 3, 2, 1	1, 2, 3, 3, 2
4	1, 3, 6, 8, 5, 2	1, 3, 5, 5, 2	1, 3, 6, 4, 2
4	1, 3, 6, 9, 5, 2	1, 3, 5, 5, 2	1, 3, 6, 4, 2
4	1, 3, 6, 10, 6, 2	1, 3, 5, 6, 2	1, 3, 6, 5, 2
2	1, 3, 3, 3, 3, 3	1, 3, 3, 2, 1	1, 2, 3, 3, 3
4	1, 3, 6, 8, 5, 3	1, 3, 5, 5, 2	1, 3, 6, 4, 3
4	1, 3, 6, 10, 6, 3	1, 3, 5, 6, 2	1, 3, 6, 5, 3
2	1, 3, 3, 3, 3, 3, 2	1, 3, 3, 3, 2, 1	1, 2, 3, 3, 3, 2
3	1, 3, 4, 4, 4, 3, 2	1, 3, 4, 3, 3, 1	1, 3, 3, 4, 3, 2
3	1, 3, 4, 4, 4, 4, 2	1, 3, 4, 3, 3, 2	1, 3, 3, 4, 4, 2
4	1, 3, 5, 6, 5, 4, 2	1, 3, 4, 5, 3, 2	1, 3, 5, 4, 4, 2
4	1, 3, 6, 8, 6, 4, 2	1, 3, 5, 6, 3, 2	1, 3, 6, 5, 4, 2
5	1, 3, 6, 8, 10, 6, 2	1, 3, 5, 6, 6, 2	1, 3, 6, 8, 5, 2
5	1, 3, 6, 9, 10, 6, 2	1, 3, 5, 6, 6, 2	1, 3, 6, 8, 5, 2
5	1, 3, 6, 9, 11, 6, 2	1, 3, 5, 6, 6, 2	1, 3, 6, 9, 5, 2
5	1, 3, 6, 9, 12, 6, 2	1, 3, 5, 6, 6, 2	1, 3, 6, 9, 5, 2
5	1, 3, 6, 10, 10, 6, 2	1, 3, 5, 6, 6, 2	1, 3, 6, 8, 5, 2
5	1, 3, 6, 10, 11, 6, 2	1, 3, 5, 6, 6, 2	1, 3, 6, 9, 5, 2
5	1, 3, 6, 10, 12, 6, 2	1, 3, 5, 6, 6, 2	1, 3, 6, 9, 5, 2

Table 1: Hilbert functions of artinian level algebras

Proposition 4.2. *For i such that the following conditions are all verified,*

we have that $h_i^{GM} \geq h_i^G$:

$$\begin{cases} i + 1 \leq c - i; \\ h_i = h_{i+1}; \\ s_{c-i} = \binom{(c-i)+n-1}{c-i} > h_{i+1}. \end{cases}$$

Proof We have that $h_i^{GM} = h_i - (h_i)_{\{c-i\}} = h_{i+1} - (h_{i+1})_{\{c-i\}}$. Since $s_{c-i} > h_{i+1}$, $(h_{i+1})_{\{c-i\}} = (h_{i+1})_{\langle c-i \rangle}$. We may then compare $(h_{i+1})_{\langle c-i \rangle}$ and $(h_{i+1})_{\langle i+1 \rangle}$, using Lemma 3.11, and the fact that $i + 1 \leq c - i$, so we get that $h_i^{GM} \geq h_i^G$. \square

The previous condition does not characterize all cases in which $h_i^{GM} > h_i^G$, but only the ones in bold font in the table. Notice that if in the first column we have a certain number i this means, according to our notation, that $h_{i-1}^{GM} > h_{i-1}^G$.

Question. What are the sufficient conditions for which the bound given by the HRT for modules is better than the bound given by the classical HRT?

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