Infinite-Dimensional Lie Algebras

Part III Essay

by

Nima Amini

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Department of Pure Mathematics and Mathematical Statistics

University of Cambridge

Contents

“ *It is a well kept secret that the theory of Kac-Moody algebras has been a disaster. True, it is a generalization of a very important object, the simple finite-dimensional Lie algebras, but a generalization too straightforward to expect anything interesting from it. True, it is remarkable how far one can go with all these* e_i*'s,* f_i*'s and* h_i*'s. Practically all, even most difficult results of finite-dimensional theory, such as the theory of characters, Schubert calculus and cohomology theory, have been extended to the general set-up of Kac-Moody algebras. But the answer to the most important question is missing: what are these algebras good for? Even the most sophisticated results, like the connections to the theory of quivers, seem to be just scratching the surface. However, there are [...] notable exceptions. The best known one is, of course, the theory of affine Kac-Moody algebras. This part of the Kac-Moody theory has deeply penetrated many branches of mathematics and physics. The most important single reason for this success is undoubtedly the isomorphism of affine algebras and central extensions of loop algebras, often called current algebras.* ” [Ka1]

- Victor G. Kac

Introduction

In this essay we consider infinite-dimensional generalizations of complex finite-dimensional semisimple Lie algebras. The first of these (based purely on the Chevalley-Serre presentation) was obtained independently by V.Kac [Ka2] and R.Moody [Moo] in the late 1960's through a straightforward generalization of the Cartan matrix. The study of these algebras however turned out much richer and intricate than was initially anticipated. Several basic questions about general Kac-Moody algebras still remain open. A particular well-understood class of infinite-dimensional Kac-Moody algebras, the affine Lie algebras, have found numerous applications throughout mathematics and physics. Aside from connections to number theory (e.g modular forms), algebra (e.g tame quivers) and geometry (e.g geometric Langlands program) they have been curiously related to sets of combinatorial identities, the so called Macdonald identities [Mac] through Weyl character formula [Ka3]. In physics, affine Lie algebras are associated with conformal field theories. Since finite-dimensional semisimple Lie algebras are classified by discrete data they are inherently rigid objects within their own category. However in the mid 1980's, V.Drinfeld [Dri] and M.Jimbo [Ji1] independently showed that the universal enveloping algebra of a semisimple Lie algebra admits a deformation inside the category of Hopf-algebras. The resulting algebra, coined 'quantum group', derives from earlier works in mathematical physics where similar algebras had been developed as solutions to the quantum Yang-Baxter equation [KR].

This work can be summarized as the study of below big-picture diagram; the objects involved, their finite-dimensional representation theories and their pairwise analogies.

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\text{Affinization}"] \arrow[d, "\text{Quantization}"'] & \hat{\mathfrak{g}} \arrow[d, "\text{Quantization}"] \\
U_q(\mathfrak{g}) \arrow[r, "\text{Affinization}"'] & U_q(\hat{\mathfrak{g}})
\end{tikzcd}$$

$\mathfrak{g}$ - finite-dimensional simple Lie algebra

$\hat{\mathfrak{g}}$ - affine Lie algebra

$U_q(\mathfrak{g})$ - quantized enveloping algebra (or quantum group)

$U_q(\hat{\mathfrak{g}})$ - quantum affine algebra

Underlying field in this essay is $\mathbb{C}$ unless otherwise stated.

About this essay

A very thin introductory section §1.1 has been provided in order to summarize background about finite-dimensional semisimple Lie algebras. The reader is referred to [Hum] and [FH] for further details. In the first chapter we treat fundamental aspects of Kac-Moody algebras. Main references for this chapter are [Kac] and [Car]. The second chapter collects facts about affine Lie algebras. Results are used for comparison in the third chapter where quantum affine algebras are considered. References for the second chapter are [Kac], [CP2], [CFS] and [Rao]. The third chapter begins by stating facts about quantized enveloping algebras. These are drawn from [CP1],[CP3] and [BG]. Subsequent sections develop the theory of affine Lie algebras and are additionally based on [CP4] and [CP5]. Reference for the final section on q-characters is given by [FR].

Despite the essay-premise the text contains pieces of novel material. In chapter 2 we prove that the center of an affine Lie algebra (without derivation) acts trivially on unfaithful representations. From this we deduce the well-known fact that the finite-dimensional representation theory of affine Lie algebras reduces to the finite-dimensional representation theory of the loop algebra (Proposition 2.2.1). In the same chapter we also give an alternative proof of the result that finite-dimensional representations of affine Lie algebras decompose into tensor products of evaluation representations. In Chapter 3 we give missing details (of the sketch proof in [CP3]) of a straightforward extension (to arbitrary finite-dimensional simple Lie algebras $\mathfrak{g}$) of the theorem by Chari and Pressley stating that finite-dimensional irreducible type 1 representations of quantum affine algebras admit bijective parametrization by Drinfeld polynomials. Finally the drawn parallels between representation theories, the reorganization and detail expansion of the exposition (including the numerous calculations and checks throughout the essay) are personal contributions, so the blame for any mistakes found therein should be laid at the author.

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Chapter 1

Kac-Moody Algebras

This chapter introduces the basic notions associated with the underlying object of this essay - the Kac-Moody algebra. Our study begins where the classical finite-dimensional theory left off, namely with the Cartan-Killing classification of complex finite-dimensional semisimple Lie algebras. After recalling familiar concepts from classical theory we generalize to a much wider class of Lie algebras known as Kac-Moody algebras. One of the key objects from finite-dimensional theory is the Cartan matrix which encodes everything necessary to recover the Lie algebra from the Chevalley-Serre presentation. By relaxing its axioms we will see in detail how the Kac-Moody algebra is constructed and characterize its dimensionality through familiar notions. Subsequent sections will be spent generalizing further notions from standard theory. We will finally describe how the (indecomposable) generalized Cartan matrices split into a trichotomy of finite, affine and indefinite type. Very little is known about Kac-Moody algebras of indefinite type. The reason is largely due to the difficulty of finding good concrete realizations for them. This stands in contrast to the affine situation where a nice realization exists and enables a fruitful study of its representation theory (the following chapter will make the concrete affine realization more precise).

We shall begin the chapter with a brief summary of the classification programme for finite-dimensional semisimple Lie algebras.

1.1 Summary of finite-dimensional semisimple Lie algebras

In finite-dimensional theory, the role of finite dimension manifests itself most prominently via the presence of inductive dimension-arguments. Along with the underlying field assumptions (algebraic closure and characteristic 0) this enables the proofs of several major results, including classical theorems by Engel, Lie and Cartan (see [Hum]). Yet a classical result, of general interest for this essay, concerns the complete reducibility of finite-dimensional representations of finite-dimensional semisimple Lie algebras, due to Weyl:

Theorem 1.1.1. *(Weyl's Theorem).*
Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and V a finite-dimensional $\mathfrak{g}$-module. Then $V = V_1 \oplus \cdots \oplus V_n$ for some irreducible $\mathfrak{g}$-submodules V_i, $1 \leq i \leq n$.

Weyl's theorem implies it is sufficient to understand the finite-dimensional irreducible representations in order to understand all finite-dimensional representations.

The Lie algebra central to the theory is $\mathfrak{sl}_2$, the Lie algebra generated by E, F, H with relations

$$[E, F] = H, \qquad [H, E] = 2E, \qquad [H, F] = -2F$$

As is noticed from the relations, H acts diagonally on E and F through the Lie bracket. With the aim of generalizing the role of H in $\mathfrak{sl}_2$ to arbitrary semisimple Lie algebras one makes the following important definition:

Definition 1.1.2. (Cartan subalgebra).
Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $\mathfrak{h} \leq \mathfrak{g}$ is a **Cartan subalgebra** if it is maximal with respect to being abelian and consisting only of semisimple (i.e ad-diagonalizable) elements.

We remark that Lie homomorphisms preserve semisimple elements (indeed they preserve the whole Jordan decomposition). Thus the elements of the Cartan subalgebra $\mathfrak{h}$ act diagonalizably on every finite-dimensional $\mathfrak{g}$-module. By standard linear algebra we have that any commuting family of diagonalizable linear endomorphisms is simultaneously diagonalizable. We may hence decompose any $\mathfrak{g}$-module V into simultaneous $\mathfrak{h}$-eigenspaces

$$V = \bigoplus_{\alpha \in W} V_\alpha$$

where $V_\alpha = \{v \in V : Hv = \alpha(H)v \ \forall H \in \mathfrak{h}\}$ and $W = \{\alpha \in \mathfrak{h}^* : \ V_\alpha \neq 0\}$. The generalized eigenvalues $\alpha \in \mathfrak{h}^*$ are called **weights** and V_α the corresponding **weight spaces**. By considering the adjoint representation

$$\begin{aligned} ad : \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g}) \\ (ad\, X)(Y) &\longmapsto [X, Y] \end{aligned}$$

we get a special set of *non-zero* weights Φ called **roots**. The corresponding weight space decomposition has a name:

Definition 1.1.3. (Cartan decomposition).
Let $\mathfrak{g}$ be a semisimple Lie algebra.
Then

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

is the **Cartan decomposition** of $\mathfrak{g}$ where $\mathfrak{g}_\alpha$ are the weight spaces of the adjoint representation.

We state a few important properties related to this decomposition.

Proposition 1.1.4.

i) $\mathfrak{g}_0 = \mathfrak{h}$

ii) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \;\; \forall \alpha, \beta \in \Phi$

iii) $dim \mathfrak{g}_\alpha = 1 \;\; \forall \alpha \in \Phi$

iv) $\alpha \in \Phi \Leftrightarrow -\alpha \in \Phi$

v) $X_\alpha \in \mathfrak{g}_\alpha \Rightarrow \exists Y_\alpha \in \mathfrak{g}_{-\alpha} \;\; s.t \;\; \mathfrak{s}_\alpha := \langle X_\alpha, Y_\alpha, H_\alpha \rangle \cong \mathfrak{sl}(2),\; H_\alpha = [X_\alpha, Y_\alpha]$

vi) $\beta(H_\alpha) \in \mathbb{Z} \;\; \forall \alpha \in \Phi$ *and for all weights* β *of a finite-dimensional* $\mathfrak{g}$*-module* V*.*

Remark 1.1.5.
i) says that the zero weight space is precisely the Cartan subalgebra. In other words $\mathfrak{h}$ is self-centralizing.
ii) says that $\mathfrak{g}$ is graded by the root lattice (indeed more generally $\mathfrak{g}_\alpha V_\beta \subseteq V_{\alpha+\beta}$).
iii) will no longer hold in the infinite-dimensional case as can be seen in §1.3.
v) suggests that every semisimple Lie algebra is built up from copies of $\mathfrak{sl}_2$. We will see how this happens more explicitly in the Chevalley-Serre presentation at the end of this section.
vi) is not difficult to see if one notices that every weight β of a $\mathfrak{g}$-module V induces a weight of an $\mathfrak{sl}_2$-module, namely $\bigoplus_{k \in \mathbb{Z}} V_{\beta+k\alpha}$, via restriction of the action of $\mathfrak{g}$ to $\mathfrak{s}_\alpha$. It is a standard fact that $\mathfrak{sl}_2$-modules have integral weights. It therefore follows that β must take integral values on H_α.

We can also think about roots more geometrically which is the motivation behind the next definition.

Definition 1.1.6. (Weight lattice).
The **weight lattice** is defined by $\bigwedge_W := \{\beta \in \mathfrak{h}^* : \; \beta(H_\alpha) \in \mathbb{Z} \text{ for all roots } \alpha\}$

Proposition 1.1.5(vi) justifies the name of this lattice since it contains all weights and hence all integral linear combination of them. The fact that every representation may be sliced into $\mathfrak{sl}_2$-representations (as implied by Remark 1.1.5 (vi)) yields plenty of information about the symmetry of the weights in $\bigwedge_W$. This symmetry arises from the fact that weights of $\mathfrak{sl}_2$-representations are symmetric about 0 in $\mathbb{Z}$. In other words it follows that weights are symmetric with respect to reflections in hyperplanes Ω_α perpendicular to each root α where

$$\Omega_\alpha = \{\beta \in \mathfrak{h}^* : \beta(H_\alpha) = 0 \; \}.$$

We also have a set of associated root-preserving involutions acting on the weights via reflections in Ω_α

$$\begin{aligned} W_\alpha : \mathfrak{h}^* &\longrightarrow \mathfrak{h}^* \\ \beta &\longmapsto \beta - \frac{2\beta(H_\alpha)}{\alpha(H_\alpha)}\alpha = \beta - \beta(H_\alpha)\alpha \end{aligned}$$

These reflections form more than a set, they also generate a (finite) group.

Definition 1.1.7. (Weyl group).
Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and let W be the group generated by the reflections W_α, $\alpha \in \Phi$. We call W the **Weyl group** of $\mathfrak{g}$.

We can now begin to summarize the classification of finite-dimensional semisimple Lie algebras. A convenient approach to the classification is through the definition of a root system.

Definition 1.1.8. (Root system).
A subset Φ of a Euclidean space $\mathbb{E}$ endowed with a positive definite symmetric bilinear form $(\, , \,)$ is called a **root system** in $\mathbb{E}$ if the following axioms are satisfied:
(R1) Φ spans $\mathbb{E}$ and does not contain 0
(R2) $\alpha \in \Phi \Leftrightarrow -\alpha \in \Phi$
(R3) $\alpha, \beta \in \Phi \Rightarrow \langle \beta, \alpha \rangle := 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$
(R4) $\alpha \in \Phi \Rightarrow W_\alpha(\Phi) \subseteq \Phi$ where $W_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$

As we have noted throughout the section the roots of every finite-dimensional semisimple Lie algebra satisfy the axioms of a root system by setting

$$\langle \beta, \alpha \rangle := \beta(H_\alpha) \text{ and } W_\alpha(\beta) := \beta - \beta(H_\alpha)\alpha = \beta - \langle \beta, \alpha \rangle \alpha$$

Since one can attach a root system to every finite-dimensional semisimple Lie algebra, focus may be shifted to understanding the possible (abstract) root systems. Moreover every finite-dimensional semisimple Lie algebra is a direct sum of simple Lie algebras as a consequence of Cartan's Criterion [Hum Thm 5.2]. It is hence enough to do the classification for finite-dimensional simple Lie algebras. Therefore it is helpful to work with a corresponding notion of irreducibility for root systems.

Definition 1.1.9. (Irreducible root system).
A root system Φ is **irreducible** if it cannot be written as a disjoint union $\Phi = \Phi_1 \sqcup \Phi_2$ of two non-empty subsets $\Phi_1, \Phi_2 \subseteq \Phi$ such that $(\alpha, \beta) = 0 \ \forall \alpha \in \Phi_1, \beta \in \Phi_2$

Below facts ensure the notion is well-defined.

Proposition 1.1.10. *Let $\mathfrak{g}$, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be finite-dimensional semisimple Lie algebras. Then*

i) $\mathfrak{g}_1 \cong \mathfrak{g}_2 \Leftrightarrow \Phi_{\mathfrak{g}_1} \cong \Phi_{\mathfrak{g}_2}$

ii) $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \Rightarrow \Phi_{\mathfrak{g}_1} \sqcup \Phi_{\mathfrak{g}_2}$

iii) $\mathfrak{g}$ simple $\Leftrightarrow$ $\Phi_{\mathfrak{g}}$ irreducible

The proposition implies we may cut down our investigation to irreducible root systems. Since $\langle \alpha, \beta \rangle = 2\frac{||\beta||}{||\alpha||} cos\, \theta$ we have by axiom (**R3**) that

$$\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 4cos^2\theta \in \{0, 1, 2, 3\} \text{ whenever } \beta \neq \pm\alpha$$

This immediately puts severe finite restrictions on the possible angles and length ratios between roots. It is moreover not necessary to keep track of all elements of the root system.

Definition 1.1.11. (Base).
A subset Δ of a root system Φ is a **base** if:
(**B1**) Δ is a basis of $\mathbb{E}$,
(**B2**) $\forall \beta \in \Phi$, $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ where $k_\alpha \in \mathbb{Z}$ and all k_α have the same sign.

The elements of Δ are called **simple roots**.
It is enough to maintain pairwise angle and length relationships between the simple roots. This information can be recorded more compactly as a graph.

Definition 1.1.12. (Dynkin Diagram).
Let $\Delta = \{\alpha_1, ..., \alpha_n\}$ be a base for a root system Φ. Then the **Coxeter graph** with respect to Φ is the graph $G(\Phi)$ with vertices Δ where the number of edges between nodes labelled by roots α_i and α_j is given by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$.
The **Dynkin diagram** with respect to Φ is the Coxeter graph with respect to Φ where nodes connected by 2 or 3 edges have an arrow pointing from the shorter to the longer root.

The class of Dynkin diagrams corresponding to irreducible root systems are subject to certain constraints:

Proposition 1.1.13.
Let $\Delta = \{\alpha_1, ..., \alpha_n\}$ be a base for an irreducible root system Φ of a finite-dimensional simple Lie algebra. Then

i) The Dynkin diagram is connected.

ii) Any subdiagram of a Dynkin diagram is a Dynkin diagram.

iii) There are at most $n - 1$ pairs of connected nodes, in particular there are no cycles.

iv) The maximum degree of any node is 3.

v) Contracting any degree 2 node leaves a Dynkin diagram.

These restrictions leave only a handful number of cases for valid Dynkin diagrams of irreducible root systems. These consist of four infinite families of diagrams labelled A_n, B_n, C_n, D_n and 5 exceptional diagrams labelled E_6, E_7, E_8, F_4, G_2. It is then possible to work backwards and explicitly identify complex finite-dimensional simple Lie algebras having Dynkin diagrams covering remaining possibilities. This identification completes the classification. As an example, the families A_n, B_n, C_n, D_n correspond to the familiar simple Lie algebras $\mathfrak{sl}(n+1), \mathfrak{so}(2n+1), \mathfrak{sp}(2n), \mathfrak{so}(2n)$ respectively.

A yet more compressed way of encoding the information in the Dynkin diagram is by way of a matrix.

Definition 1.1.14. (Cartan matrix).
Let Δ be a base of a root system Φ.
Fix an ordering $(\alpha_1, \ldots, \alpha_n)$ of the simple roots.
The **Cartan matrix** of Φ is defined to be the $n \times n$ matrix $A = (A_{ij})$ where $A_{ij} = \langle \alpha_j, \alpha_i \rangle$.

Proposition 1.1.15. (Cartan matrix classification).
Cartan matrices $A = (A_{ij})$ corresponding to simple finite-dimensional Lie algebras satisfy:
(**C1**) $A_{ii} = 2$,
(**C2**) $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$,
(**C3**) $A_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$,
(**C4**) $det(A) > 0$,
(**C5**) A indecomposable.

Note that axiom (**C5**) is a consequence of the Lie algebra being simple. Dropping it gives the matrices for finite-dimensional semisimple Lie algebras. Worth keeping in mind is the following correspondence between irreducible objects:

$$\mathfrak{g} \text{ simple} \leftrightarrow \Phi \text{ irreducible} \leftrightarrow G(\Phi) \text{ connected} \leftrightarrow A \text{ indecomposable.}$$

The following remarkable theorem by Serre allows us to recover a semisimple Lie algebra from its Cartan matrix.

Theorem 1.1.16. *(Serre's Theorem).*
Let $A = (A_{ij})$ be the Cartan matrix of a root system.
Let $\mathfrak{g}$ be the Lie algebra generated by e_i, f_i, h_i for $1 \leq i \leq n$ subject to relations:
(**S1**) $[h_i, h_j] = 0$
(**S2**) $[h_i, e_j] = A_{ij} e_j$, $[h_i, f_j] = -A_{ij} f_j$
(**S3**) $[e_i, f_j] = \delta_{ij} h_i$
(**S4**) $(ad\, e_i)^{1-A_{ij}}(e_j) = 0$, $(ad\, f_i)^{1-A_{ij}}(f_j) = 0 \quad (i \neq j)$
where $1 \leq i, j \leq n$. Then $\mathfrak{g}$ is finite-dimensional and semisimple with Cartan subalgebra $\mathfrak{h}$ spanned by $\{h_1, \ldots, h_n\}$. Moreover the root system corresponding to $\mathfrak{g}$ has Cartan matrix A.

Remark 1.1.17.
Relations (**S1**), (**S2**), (**S3**) are called Chevalley relations and (**S4**) Serre relations. The presentation given by Serre's theorem is sometimes referred to as the **Chevalley-Serre presentation**. Note in particular the subalgebras

$$\langle e_i, f_i, h_i \rangle \cong \mathfrak{sl}_2 \quad (i = 1, \ldots, n).$$

This shows rather explicitly how the Lie algebras generated by this presentation is a result of stringing together copies of $\mathfrak{sl}_2$ according to the coefficients in the Cartan matrix. This is one of the reasons $\mathfrak{sl}_2$ occasionally earns the nickname 'The mother of all Lie algebras'. The Serre relations restrict the "freeness" of the bracket. It is therefore plausible that a large exponent will make it difficult for the presentation to remain finite-dimensional. Indeed axiom (**C3**) can be strengthened to $A_{ij} \in \{0, -1, -2, -3\}$.

1.2 Definitions

Since the Cartan matrix can be characterized axiomatically this opens up possibilities for straightforward generalizations. By dropping axiom (**C4**) we retrieve a wider class of matrices from which a wider class of Lie algebras may be obtained, most of which are no longer finite-dimensional. It is this seemingly innocent generalization that is the starting point for the rich theory of Kac-Moody algebras. Let us begin by defining explicitly the notion of a generalized Cartan matrix.

Definition 1.2.1. (Generalized Cartan Matrix (GCM)).
A **Generalized Cartan Matrix** (GCM) is an $n \times n$ matrix $A = (A_{ij})$ satisfying:
(**GCM1**) $A_{ii} = 2$,
(**GCM2**) $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$,
(**GCM3**) $A_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$.

Note that this relaxation implies the GCM may hold arbitrary rank.
In finite-dimensional theory we defined roots to be non-zero functionals on the dual space of the Cartan subalgebra. Flexing the size of the Cartan subalgebra presents consequences for the domain on which the roots are defined. Since the Cartan matrix was obtained from a set of simple roots, the linear independence of the roots was baked into the definition. When $det(A) = 0$ we cannot simply plug the matrix back into the Chevalley-Serre presentation as defined in Serre's Theorem and hope to get linearly independent roots on the domain generated by $\{h_1, \ldots, h_n\}$. In fact this will always be false due to the presence of zero eigenvalues. More explicitly a zero eigenvalue guarantees the existence of a non-zero vector $\mathbf{v} = (\lambda_i) \in \mathbb{C}^n$ such that

$$A\mathbf{v} = 0 \Rightarrow \sum_{i=0}^{n} \lambda_i A_{ij} = 0 \text{ for } 1 \leq j \leq n \Rightarrow \sum_{i=0}^{n} \lambda_i \alpha_i(h_j) = 0 \text{ for } 1 \leq j \leq n.$$

Since the roots are determined by the values they take on $\{h_1, \ldots, h_n\}$ we have a non-trivial relation

$$\sum_{i=0}^{n} \lambda_i \alpha_i \equiv 0$$

Hence the roots are linearly dependent.

To get around this minor technical issue we may enlargen our Cartan subalgebra until we get linear independence between the roots. This is the main motivation behind the definition of a *realization* of a GCM. Although a realization can be defined for any complex matrix we will throughout restrict ourselves to GCMs in order to avoid any confusion.

Definition 1.2.2. (Realization).
Let A be an $n \times n$ GCM.
A **realization** of A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ such that:
$\mathfrak{h}$ is a finite-dimensional vector space over $\mathbb{C}$,
$\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is a linearly independent subset of $\mathfrak{h}^*$ called *simple roots*,
$\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ is a linearly independent subset of $\mathfrak{h}$ called *simple coroots*,
$\alpha_j(\alpha_i^\vee) = A_{ij}$ for all i, j

This way we end up with linearly independent simple roots by construction. The question that remains is how large the vector space $\mathfrak{h}$ needs to be for this to work?

Proposition 1.2.3. *(Lower bound on dim* $\mathfrak{h}$*).*
Let A *be an* $n \times n$ *GCM and* $(\mathfrak{h}, \Pi, \Pi^\vee)$ *a realization of* A*.*
Then $dim\, \mathfrak{h} \geq 2n - rank(A)$*.*

Proof.
This is a rather straightforward piece of linear algebra. Recall that $A_{ij} = \alpha_j(\alpha_i^\vee)$ and extend Π, $\Pi^\vee$ to bases of $\mathfrak{h}^*$, $\mathfrak{h}$ respectively such that the corresponding matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of rank $dim\, \mathfrak{h}$ is non-singular. Then the submatrix $(A\ B)$ on the first n rows has rank n since its rows are linearly independent. B must therefore have rank at least $n - rank(A)$. The rank of $\begin{pmatrix} B \\ D \end{pmatrix}$ is $dim\, \mathfrak{h} - n$ and is moreover at least the rank of B. Hence $dim\, \mathfrak{h} \geq 2n - rank(A)$. $\square$

Since we have lower bounds on the dimension it makes sense to talk about minimal realizations.

Definition 1.2.4. (Minimal realization).
A realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ of a GCM A is **minimal** if $dim\, \mathfrak{h} = 2n - rank(A)$

By [Car Prop 14.2, 14.3] every GCM admits a minimal realization and every minimal realization is unique up to isomorphism. We may therefore construct an auxillary Lie algebra $\tilde{\mathfrak{g}}(A)$ that depends only on A and defined via something that closely resembles the Chevalley-relations in Theorem 1.1.16. We must however enlargen our set of generators to maintain the linear independence properties of the realization.

Definition 1.2.5. (Auxillary Lie algebra).
Let A be an $n \times n$ GCM and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a minimal realization of A.
Define the **auxillary Lie algebra** $\tilde{\mathfrak{g}}(A)$ to be the Lie algebra generated by e_i, f_i $(i = 1, \ldots, n)$ and the vector space $\mathfrak{h}$ subject to relations
$[e_i, f_j] = \delta_{ij}\alpha_i^\vee$ for $i, j = 1, \ldots, j$
$[h, h'] = 0 \ \ \forall\, h, h' \in \mathfrak{h}$
$[h, e_i] = \alpha_i(h)e_i$ for $i = 1, \ldots, n$; $h \in \mathfrak{h}$
$[h, f_i] = -\alpha_i(h)f_i$ for $i = 1, \ldots, n$; $h \in \mathfrak{h}$

Definition 1.2.6. (Root lattice).
Let A be an $n \times n$ GCM and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A.
Define

$$\bigwedge_\Pi = \sum_{i=1}^{n} \mathbb{Z}\alpha_i, \quad \bigwedge_\Pi^+ = \sum_{i=1}^{n} \mathbb{Z}^+\alpha_i$$

The lattice $\bigwedge_\Pi$ is called the **root lattice**.

Definition 1.2.7. (Height).
For $\alpha = \sum_{i=1}^{n} k_i \alpha_i$ we call the number $ht\,\alpha = \sum_{i=1}^{n} k_i$ the **height** of α.

We now proceed to give a detailed proof of the main result of this section concerning the structure decomposition of $\tilde{\mathfrak{g}}(A)$.

Theorem 1.2.8. *(Auxillary decomposition).*
Let A be an $n \times n$ GCM.
Denote by $\tilde{\mathfrak{n}}_+$ the subalgebra of $\tilde{\mathfrak{g}}(A)$ generated by $e_1, \dots, e_n$.
Denote by $\tilde{\mathfrak{n}}_-$ the subalgebra of $\tilde{\mathfrak{g}}(A)$ generated by $f_1, \dots, f_n$.
Then

i) $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_-$ *as a vector space direct sum.*

ii) $\tilde{\mathfrak{n}}_+$ *(resp $\tilde{\mathfrak{n}}_-$) is freely generated by $e_1, \dots, e_n$ (resp $f_1, \dots, f_n$).*

iii) With respect to $\mathfrak{h}$ one has the root space decomposition

$$\tilde{\mathfrak{g}}(A) = \Big(\bigoplus_{\substack{\alpha \in \bigwedge_{\Pi}^{+} \\ \alpha \neq 0}} \tilde{\mathfrak{g}}_{-\alpha}\Big) \oplus \mathfrak{h} \oplus \Big(\bigoplus_{\substack{\alpha \in \bigwedge_{\Pi}^{+} \\ \alpha \neq 0}} \tilde{\mathfrak{g}}_{\alpha}\Big)$$

where $\tilde{\mathfrak{g}}_\alpha = \{x \in \tilde{\mathfrak{g}}(A) : [h, x] = \alpha(h)x\}$
and $\tilde{\mathfrak{g}}_{\pm\alpha} \subseteq \tilde{\mathfrak{n}}_{\pm}$ for $\alpha \in \bigwedge_{\Pi}^{+}$, $\alpha \neq 0$

iv) $\dim \tilde{\mathfrak{g}}_\alpha < \infty$

v) The set $\{\mathfrak{i} \trianglelefteq \tilde{\mathfrak{g}}(A) : \mathfrak{i} \cap \mathfrak{h} = \{0\}\}$ has a unique maximal element $\mathfrak{m}$. Moreover

$$\mathfrak{m} = (\mathfrak{m} \cap \tilde{\mathfrak{n}}_-) \oplus (\mathfrak{m} \cap \tilde{\mathfrak{n}}_+)$$

Lemma 1.2.9.
Let V be an n-dimensional complex vector space with basis $v_1, \dots, v_n$ and let $T(V)$ denote the tensor algebra of V.
Let $\lambda \in \mathfrak{h}^*$ and define a linear map $\rho : \tilde{\mathfrak{g}}(A) \longrightarrow End(T(V))$ via

(a) $\rho(f_i)(w) = v_i \otimes w$ *for $w \in T(V)$,*

(b) $\rho(h)(1) = \lambda(h)1$,

(c) $\rho(h)(v_j \otimes w) = -\alpha_j(h) v_j \otimes w + v_j \otimes \rho(h)(w)$ *inductively on s, for $w \in T^{s-1}(V)$, $j = 1, \dots, n$,*

(d) $\rho(e_i)(1) = 0$,

(e) $\rho(e_i)(v_j \otimes w) = \delta_{ij} \rho(\alpha_i^\vee)(w) + v_j \otimes \rho(e_i)(w)$ *inductively on s for $w \in T^{s-1}(V)$, $j = 1, \dots, n$.*

Then ρ is a representation of $\tilde{\mathfrak{g}}(A)$.

Proof.
We need to check that ρ is a well-defined Lie algebra homomorphism i.e that

$$\rho([x, y]_{\tilde{\mathfrak{g}}(A)}) = [\rho(x), \rho(y)]_{End(T(V))} = \rho(x)\rho(y) - \rho(y)\rho(x) \quad \forall x, y \in \tilde{\mathfrak{g}}(A)$$

It is enough to verify that ρ preserves the defining bracket relations of $\tilde{\mathfrak{g}}(A)$ (see Definition 1.2.5):

$$\begin{aligned}(\rho(e_i)\rho(f_i)-\rho(f_i)\rho(e_i))(w) &\overset{(a)}{=} \rho(e_i)(v_j\otimes w)-v_j\otimes\rho(e_i)(w)\\ &\overset{(e)}{=} \delta_{ij}\rho(\alpha_i^\vee)(w)+v_j\otimes\rho(e_i)(w)-v_j\otimes\rho(e_i)(w)\\ &=\delta_{ij}\rho(\alpha_i^\vee)(w)\\ &=\rho([e_i,f_j])(w)\end{aligned}$$

It follows by (b) and (c) that $\mathfrak{h}$ defines a diagonal action on $T(V)$.
Therefore given $h,h'\in\mathfrak{h}$ and $w\in T(V)$ there exists $k,k'\in\mathbb{C}$ such that

$$\rho(h)(w)=kw,\ \rho(h')(w)=k'w.$$

Hence

$$(\rho(h)\rho(h')-\rho(h')\rho(h))(w)=k'kw-kk'w=0=\rho(0)(w)=\rho([h,h'])(w)$$

The preservation of the third relation ($[h,e_i]=\alpha_i(h)e_i$) is proved by induction on s. For $s=0$ we have

$$(\rho(h)\rho(e_i)-\rho(e_i)\rho(h))(1)\overset{(d)(b)}{=}\rho(h)(0)-\lambda(1)\rho(e_i)(1)\overset{(d)}{=}0\overset{(d)}{=}\alpha_i(h)\rho(e_i)(1)=\rho([h,e_i])(1)$$

For $s>0$ let $w=v_k\otimes w'$ where $w'\in T^{s-1}(V)$, $k\in\{1,\dots,n\}$.
Then

$$\begin{aligned}(\rho(h)\rho(e_i)-\rho(e_i)\rho(h))(v_k\otimes w') &\overset{(c)(e)}{=}\rho(h)(\delta_{ik}\rho(\alpha_i^\vee)(w'))+\rho(h)(v_k\otimes\rho(e_i)(w'))\\ &\quad-\rho(e_i)(-\alpha_k(h)v_k\otimes w'+v_k\otimes\rho(h)(w'))\\ &\overset{(c)(e)}{=}\delta_{ik}\rho(\alpha_i^\vee)(\rho(h)(w'))\\ &\quad-\alpha_k(h)v_k\otimes\rho(e_i)(w')+v_k\otimes\rho(h)(\rho(e_i)(w'))\\ &\quad+\alpha_k(h)\delta_{ik}\rho(\alpha_i^\vee)(w')+\alpha_k(h)v_k\otimes\rho(e_i)(w')\\ &\quad-\delta_{ik}\rho(\alpha_i^\vee)(\rho(h)(w'))-v_k\otimes\rho(e_i)(\rho(h)(w'))\\ &=\alpha_k(h)\delta_{ik}\rho(\alpha_i^\vee)(w')+v_k\otimes(\rho(h)\rho(e_i)-\rho(e_i)\rho(h))(w')\\ &\overset{I.H}{=}\alpha_k(h)\delta_{ik}\rho(\alpha_i^\vee)(w')+v_k\otimes\rho([h,e_i])(w')\\ &=\alpha_i(h)(\delta_{ik}\rho(\alpha_i^\vee)(w')+v_k\otimes\rho(e_i)(w'))\\ &=\alpha_i(h)\rho(e_i)(v_k\otimes w')\\ &=\rho(\alpha_i(h)e_i)(v_k\otimes w')\\ &=\rho([h,e_i])(v_k\otimes w')\end{aligned}$$

Hence the bracket is preserved by induction. Finally for the fourth relation ($[h,f_i]=-\alpha_i(h)f_i$) we have

$$\begin{aligned}(\rho(h)\rho(f_i)-\rho(f_i)\rho(h))(w) &=\rho(h)(v_i\otimes w)-v_i\otimes\rho(h)(w)\\ &=-\alpha_i(h)v_i\otimes w+v_i\otimes\rho(h)(w)-v_i\otimes\rho(h)(w)\\ &=-\alpha_i(h)\rho(f_i)(w)\\ &=\rho([h,f_i])(w)\end{aligned}$$

Thus ρ is a well-defined homomorphism and hence a representation of $\tilde{\mathfrak{g}}(A)$. $\square$

Lemma 1.2.10.
$\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+$

Proof.
We first check

$$[\mathfrak{h}, \tilde{\mathfrak{n}}_\pm] \subseteq \tilde{\mathfrak{n}}_\pm \text{ and } [\tilde{\mathfrak{n}}_-, \tilde{\mathfrak{n}}_+] \subseteq \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+$$

Let w_s denote an arbitrary Lie word on s (not necessarily distinct) letters.
Then $\exists k \in \mathbb{N}$ such that $w_s = [w_k, w_{s-k}]$ for some Lie words w_k, w_{s-k}.
Now argue by induction on s. For $s = 1$ we have by definition

$$[h, e_i] = \alpha_i(h)e_i \in \tilde{\mathfrak{n}}_+ \ \ \forall h \in \mathfrak{h} \text{ and } [e_i, f_j] = \delta_{ij}\alpha_i^\vee \in \mathfrak{h} \subseteq \mathfrak{h} + \tilde{\mathfrak{n}}_+$$

For $s > 1$ let w_s be a Lie word of length s in $\{e_1, \ldots, e_n\}$.
Then by Jacobi identity

$$[\mathfrak{h}, w_s] = [\mathfrak{h}, [w_k, w_{s-k}]] = [\underbrace{[\mathfrak{h}, w_k]}_{\in \tilde{\mathfrak{n}}_+ \text{ by I.H}}, w_{s-k}] - [w_k, \underbrace{[\mathfrak{h}, w_{s-k}]}_{\in \tilde{\mathfrak{n}}_+ \text{ by I.H}}] \in \tilde{\mathfrak{n}}_+$$

$$[w_s, f_j] = [[w_k, w_{s-k}], f_j] = [w_k, \underbrace{[w_{s-k}, f_j]}_{\in \mathfrak{h} + \tilde{\mathfrak{n}}_+ \text{ by I.H}}] + [w_{s-k}, \underbrace{[w_k, f_j]}_{\in \mathfrak{h} + \tilde{\mathfrak{n}}_+ \text{ by I.H}}] \in \mathfrak{h} + \tilde{\mathfrak{n}}_+$$

Hence by induction and a similar argument we have

$$[\tilde{\mathfrak{n}}_+, f_j] \subseteq \mathfrak{h} + \tilde{\mathfrak{n}}_+, \ j = 1, \ldots, n \tag{1.1}$$

$$[\mathfrak{h}, \tilde{\mathfrak{n}}_\pm] \subseteq \tilde{\mathfrak{n}}_\pm \tag{1.2}$$

To prove $[\tilde{\mathfrak{n}}_-, \tilde{\mathfrak{n}}_+] \subseteq \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+$ we do yet another induction.
By (1.1) we get the base case

$$[\tilde{\mathfrak{n}}_+, f_j] \subseteq \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+$$

For $s > 0$ let w_s be a Lie word of length s in $\{f_1, \ldots, f_n\}$.
A similar calculation using Jacobi identity, (1.2) and inductive hypothesis yields

$$\begin{aligned}
[\tilde{\mathfrak{n}}_+, [w_k, w_{s-k}]] &= [[\tilde{\mathfrak{n}}_+, w_k], w_{s-k}] - [w_k, [\tilde{\mathfrak{n}}_+, w_{s-k}]] \\
&\subseteq [\tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+, w_{s-k}] - [w_k, \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+] \\
&\subseteq \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+
\end{aligned}$$

Hence by induction

$$[\tilde{\mathfrak{n}}_-, \tilde{\mathfrak{n}}_+] \subseteq \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+ \tag{1.3}$$

The final induction on the length of a general Lie word will give us the lemma.
There is nothing to do for the base case.
Suppose $s > 1$ and let w_s be a Lie word in $\{e_i, f_i, h : h \in \mathfrak{h}, i = 1, \ldots, n\}$.
Using inductive hypothesis, (1.2) and (1.3) we have

$$w_s = [w_k, w_{s-k}] \in [\tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+ \, , \, \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+] \subseteq \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+$$

Thus $\tilde{\mathfrak{g}}(A) \subseteq \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+$ by induction.
Hence $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+$ □

Lemma 1.2.11.
Let $\mathfrak{h}$ be a finite-dimensional commutative Lie algebra and V an $\mathfrak{h}$-module such that

$$V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha \quad \textit{where } V_\alpha = \{v \in V : h(v) = \alpha(h)v \,\forall h \in \mathfrak{h}\}$$

Then for any submodule $U \leq V$ we have $U = \bigoplus_{\alpha \in \mathfrak{h}^*} (U \cap V_\alpha)$.

Proof.

Let $u \in U$ and write $u = \sum_{i=1}^{n} v_i$ where $v_i \in V_{\alpha_i}$.
We want to show $v_i \in U$ for all $i = 1, \ldots, n$.
Let $\alpha_1, \ldots, \alpha_n$ be a basis of $\mathfrak{h}^*$.
The subspaces $\mathfrak{h}_{ij} = \{h \in \mathfrak{h} : \alpha_i(h) = \alpha_j(h)\}$, $i \neq j$ must be proper.
If not then $\alpha_i \equiv \alpha_j$ for some $i \neq j$ so $\alpha_1, \ldots, \alpha_n$ are not linearly independent contrary to basis assumption. Moreover from linear algebra we know $\mathfrak{h}$ cannot be a union of finitely many proper subspaces and hence $\exists h \in \mathfrak{h} \setminus \bigcup_{i,j} \mathfrak{h}_{ij}$ such that $\alpha_1(h), \ldots, \alpha_n(h)$ are pairwise distinct. Since U is a submodule we have

$$\sum_{i=1}^{n} \alpha_i^k(h) v_i = \sum_{i=1}^{n} h^k v_i = h^k u \in U \text{ for } k = 0, \ldots, n-1$$

This may be expressed as a matrix equation $A\mathbf{v} = \mathbf{u}$ where A is the Vandermonde matrix, $\mathbf{v} = (v_i)_{1 \leq i \leq n}$ and $\mathbf{u} = (h^k u)_{0 \leq k \leq n-1}$.
It is well-known that the Vandermonde matrix has determinant $\prod_{i<j} (\alpha_i(h) - \alpha_j(h))$ which is non-zero as $\alpha_1(h), \ldots, \alpha_n(h)$ are pairwise distinct.
Therefore A is invertible and so each of $v_1, \ldots, v_n$ is expressible as a linear combination in $u, hu, h^2u, \ldots, h^{n-1}u \in U$ finishing the proof. $\square$

Proof. (Auxillary Decomposition)
Let $\rho_\lambda : \tilde{\mathfrak{g}}(A) \longrightarrow End(T(V))$ be the family of representations in Lemma 1.2.9 where $\lambda \in \mathfrak{h}^*$. By Lemma 1.2.10 we have $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+$. To prove (i) it remains to show that the sum is direct. Suppose $n_- + h + n_+ = 0$ where $n_\pm \in \tilde{\mathfrak{n}}_\pm$ and $h \in \mathfrak{h}$. By definition of the action of e_i on 1 it follows $\rho(n_+)(1) = 0$ and so

$$0 = \rho_\lambda(n_- + h + n_+)(1) = \rho_\lambda(n_-)(1) + \lambda(h)$$

Since $\tilde{\mathfrak{n}}_-$ acts by appending basis elements of V to every tensor monomial it follows that $\rho(n_-)(1)$ cannot contain a non-zero scalar term.
Since $\lambda(h)$ is the lone scalar on the right hand side we must have

$$\lambda(h) = 0 \;\; \forall \lambda \in \mathfrak{h}^* \Longrightarrow h = 0.$$

Therefore $\rho_\lambda(n_-)(1) = 0$.
We want to show $n_- = 0$.
Define the Lie algebra homomorphism

$$\begin{aligned} \phi : \tilde{\mathfrak{n}}_- &\longrightarrow T(V) \\ f_j &\longmapsto \rho(f_j)(1) \end{aligned}$$

where $T(V)$ is equipped with the canonical Lie bracket. All Lie words on $X := \{\rho(f_1)(1), \ldots, \rho_\lambda(f_n)(1)\}$ are contained in $T(V)$. So the image of ϕ is the free Lie algebra $F(X)$ on X. On the other hand by definition of free Lie algebra the map

$$\begin{aligned} \psi : X &\longrightarrow \tilde{\mathfrak{n}}_- \\ \rho_\lambda(f_j)(1) &\longmapsto f_j \end{aligned}$$

extends uniquely to a Lie algebra homomorphism

$$\begin{aligned} \tilde{\psi} : F(X) &\longrightarrow \tilde{\mathfrak{n}}_- \\ \rho_\lambda(f_j)(1) &\longmapsto f_j. \end{aligned}$$

Clearly ϕ and $\tilde{\psi}$ are mutually inverse on their images.
Hence

$$\tilde{\mathfrak{n}}_- \cong \phi(\tilde{\mathfrak{n}}_-) = F(X) \cong F(f_1, \ldots, f_n).$$

This shows half of (ii) and that $n_- = 0$ since $\phi(n_-) = \rho_\lambda(n_-)(1) = 0$.
Therefore $n_+ = 0$ since $n_- + h + n_+ = 0$. Hence the sum is direct proving (i).
The second half of (ii) is obtained by invoking the automorphism given by

$$e_i \longmapsto -f_i \ , \ f_i \longmapsto -e_i \ , \ h \longmapsto -h \ (h \in \mathfrak{h}). \tag{1.4}$$

This map is an involution sending $\tilde{\mathfrak{n}}_+$ to $\tilde{\mathfrak{n}}_-$ (and vice versa) so (ii) follows.
Since $ad(\mathfrak{h})$ acts diagonally on $\tilde{\mathfrak{g}}(A)$ (by definition of $\tilde{\mathfrak{g}}(A)$) we have an eigenspace decomposition

$$\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{g}}_0 \oplus \bigoplus_{\substack{\alpha \in \mathfrak{h}^* \\ \alpha \neq 0}} \tilde{\mathfrak{g}}_\alpha$$

where $\tilde{\mathfrak{g}}_\alpha = \{x \in \tilde{\mathfrak{g}}(A) : [h, x] = \alpha(h)x \ \ \forall h \in \mathfrak{h}\}$.
The e_i (resp f_i) lie in positive (resp negative) root spaces by definition of $\tilde{\mathfrak{g}}(A)$.
Since $[\tilde{\mathfrak{g}}_\alpha, \tilde{\mathfrak{g}}_\beta] \subseteq \tilde{\mathfrak{g}}_{\alpha+\beta}$ and e_i, f_i generate $\tilde{\mathfrak{n}}_+$, $\tilde{\mathfrak{n}}_-$ respectively we have that

$$\tilde{\mathfrak{n}}_+ \subseteq \bigoplus_{\substack{\alpha \in \bigwedge_\Pi^+ \\ \alpha \neq 0}} \tilde{\mathfrak{g}}_\alpha \ \text{ and } \ \tilde{\mathfrak{n}}_- \subseteq \bigoplus_{\substack{\alpha \in \bigwedge_\Pi^+ \\ \alpha \neq 0}} \tilde{\mathfrak{g}}_{-\alpha}.$$

From the decomposition in part (i) we infer equality in above inclusions since $\mathfrak{h} \subseteq \tilde{\mathfrak{g}}_0$. The decomposition in part (iii) follows. Each positive (resp negative) root space $\tilde{\mathfrak{g}}_\alpha$ is certainly in the span of all Lie words of length $|ht\,\alpha|$ on the generators e_i (resp f_i). Since there are only finitely many general non-commutative words of length $|ht\,\alpha|$ on n letters, and only finitely many ways to bracket each $|ht\,\alpha|$-length word, there are only finitely many Lie words of length $|ht\,\alpha|$ in the e_i (resp f_i). Hence dim $\tilde{\mathfrak{g}}_\alpha < \infty$. This finishes (iv).

For part (v):
Let $\mathfrak{i} \trianglelefteq \tilde{\mathfrak{g}}(A)$ such that $\mathfrak{i} \cap \mathfrak{h} = \{0\}$.
View $\tilde{\mathfrak{g}}(A) = \bigoplus_{\alpha \in \mathfrak{h}^*} \tilde{\mathfrak{g}}_\alpha$ as an $\mathfrak{h}$-module.
By Lemma 1.2.11 we have $\mathfrak{i} = \bigoplus_{\alpha \in \mathfrak{h}^*} (\mathfrak{i} \cap \tilde{\mathfrak{g}}_\alpha)$.

By (iii) we know that $\tilde{\mathfrak{g}}_{\pm\alpha} \subseteq \tilde{\mathfrak{n}}_{\pm}, \;\; \alpha \neq 0$.
Therefore $\mathfrak{i} = (\mathfrak{i} \cap \tilde{\mathfrak{n}}_{-}) \oplus (\mathfrak{i} \cap \tilde{\mathfrak{n}}_{+})$.
In particular $\mathfrak{i} \subseteq \tilde{\mathfrak{n}}_{-} \oplus \tilde{\mathfrak{n}}_{+}$.
So $\mathfrak{m} = \sum_{\mathfrak{i} \in \{\mathfrak{i} \trianglelefteq \tilde{\mathfrak{g}}(A) \,:\, \mathfrak{i} \cap \mathfrak{h} = 0\}} \mathfrak{i} \subseteq \tilde{\mathfrak{n}}_{-} \oplus \tilde{\mathfrak{n}}_{+}$ is the unique max ideal not meeting $\mathfrak{h}$.

□

Proposition 1.2.12. *(Non-trivial $\mathfrak{h}$ intersection).*
Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and let $\mathfrak{i} \trianglelefteq \mathfrak{g}$.
Then $\mathfrak{i} \cap \mathfrak{h} \neq \{0\}$

Proof.
Consider the usual Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}.$$

Applying Lemma 1.2.11 we see that

$$\mathfrak{i} = (\mathfrak{h} \cap \mathfrak{i}) \oplus \bigoplus_{\alpha \in \Phi} (\mathfrak{i} \cap \mathfrak{g}_{\alpha}).$$

It follows that $\mathfrak{i}$ must intersect at least one of the spaces non-trivially.
If $\mathfrak{i} \cap \mathfrak{h} \neq \{0\}$ then we are done. Otherwise $\mathfrak{i} \cap \mathfrak{g}_{\alpha} \neq \{0\}$ for some $\alpha \in \Phi$. Since $dim\, \mathfrak{g}_{\alpha} = 1$ we have that $\mathfrak{g}_{\alpha} \subseteq \mathfrak{i}$. By a standard result $dim\, [\mathfrak{g}_{-\alpha}, \mathfrak{g}_{\alpha}] = 1$ [Hum Prop 8.3]. Finally since $\mathfrak{i}$ is an ideal and $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_{\alpha}] \subseteq \mathfrak{g}_0 = \mathfrak{h}$ we have $[\mathfrak{g}_{-\alpha}, \mathfrak{g}_{\alpha}] \subseteq \mathfrak{i} \cap \mathfrak{h}$. □

Proposition 1.2.12 tells us that it is sensible to throw away the ideals which intersect $\mathfrak{h}$ trivially in our endeavour to make $\tilde{\mathfrak{g}}(A)$ reasonable generalization. We finally arrive at the main definition.

Definition 1.2.13. (Kac-Moody algebra).
Let A be a GCM with minimal realization $(\mathfrak{h}, \Pi, \Pi^{\vee})$.
Let $\mathfrak{m}$ be the unique maximal ideal of $\tilde{\mathfrak{g}}(A)$ such that $\mathfrak{m} \cap \mathfrak{h} = \{0\}$
and let

$$\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\mathfrak{m}$$

Then $\mathfrak{g}(A)$ is called the **Kac-Moody algebra** with GCM A.

Remark 1.2.14.
There are two slightly different definitions of a Kac-Moody algebra.
One can also define a Kac-Moody algebra purely in terms of generators and relations as the Chevalley-Serre presentation. The definition we have made will simplify matters when it comes to proving the existence of a non-degenerate bilinear form and has an inbuilt mechanism for testing whether a Lie algebra is a Kac-Moody algebra. Instead of proving the Serre relations it is enough to show there are no non-zero ideals trivially intersecting $\mathfrak{h}$ (the presence of such an ideal would contradict the maximality of $\mathfrak{m}$). This fact serves as a useful criteria in many arguments. Kac proves in [Kac Thm 9.11] that the two definitions are equivalent at least in the case where the GCM is symmetrizable (see Definition 1.4.1). Nevertheless the Serre relations always hold in $\mathfrak{g}(A)$ (see Proposition 1.2.16).

Note that there is a natural epimorphism

$$\theta : \tilde{\mathfrak{g}}(A) \twoheadrightarrow \mathfrak{g}(A)$$

Therefore the analogous decompositions in Theorem 1.2.8 hold for Kac-Moody algebras as well. In particular we have the so called **triangular decomposition**

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

where $\mathfrak{n}_\pm = \theta(\tilde{\mathfrak{n}}_\pm)$ and $\mathfrak{h}$ is identified with its image because $\theta(\mathfrak{h}) \cong \mathfrak{h}$ since $\mathfrak{m} \cap \mathfrak{h} = \{0\}$.

In contrast to the finite-dimensional case, not only do we get infinitely many roots (see Example 1.3.1), the root spaces $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}(A) : [h, x] = \alpha(h)x \; \forall h \in \mathfrak{h}\}$ need no longer be one-dimensional (see Example 1.3.2).
We therefore make the following definition:

Definition 1.2.15. (Multiplicity).
Let α be a root of $\mathfrak{g}(A)$.
Then the number $mult\,\alpha = dim\,\mathfrak{g}_\alpha$ is called the **multiplicity** of α.

Note that the multiplicity is a well-defined number by Theorem 1.2.8 (iv).
By the root space decomposition in Theorem 1.2.8(iii) it follows that every root is either positive or negative. We denote the set of all roots by Φ, the set of positive roots by Φ_+ and the set of negative roots by Φ_-.

We conclude this section with two applications that follow from the triangular decomposition and will be useful in subsequent discussions.

Proposition 1.2.16. *(Serre relations hold in $\mathfrak{g}(A)$).*
The following relations are satisfied between the Chevalley generators in $\mathfrak{g}(A)$:

$$(ad\, e_i)^{1-A_{ij}} e_j = 0; \qquad (ad\, f_i)^{1-A_{ij}} f_j = 0$$

Proof.
We prove the second relation.
The first relation is then deduced by invoking the involution in (1.4).
View $\mathfrak{g}(A)$ as an $\mathfrak{sl}_2$-module by restricting the adjoint action of $\mathfrak{g}(A)$ to each of the subalgebras $\mathfrak{s}_i = \langle e_i, \alpha_i^\vee, f_i \rangle \cong \mathfrak{sl}_2$. By definition of the Chevalley relations we have:

$$[\alpha_i^\vee, f_j] = -A_{ij} f_j; \qquad [e_i, f_j] = 0 \qquad \text{for } i \neq j$$

Via induction on the exponent we have that

$$[e_i, (ad\, f_i)^{1-A_{ij}} f_j] = (1 - A_{ij})(-A_{ij} - (1 - A_{ij}) + 1)(ad\, f_i)^{A_{ij}} f_j = 0.$$

Again by induction using Jacobi identity for $k \neq i, j$ one shows

$$[e_k, (ad\, f_i)^{1-A_{ij}} f_j] = (ad\, f_i)^{1-A_{ij}} [e_k, f_j] = 0.$$

If $k = j$ we have

$$[e_j, (ad\, f_i)^{1-A_{ij}} f_j] = (ad\, f_i)^{1-A_{ij}} [e_j, f_j] = (ad\, f_i)^{1-A_{ij}} \alpha_i^\vee.$$

So if $1 - A_{ij} > 1$ then above expression is 0 since $\alpha_i^\vee$ acts by a scalar on f_i. Otherwise if $1 - A_{ij} = 1$ then $A_{ij} = 0$ so $\alpha_i^\vee$ acts by the zero scalar. Hence $(ad\, f_i)^{1-A_{ij}} f_j$ commutes with each e_i. Thus the ideal of $\mathfrak{g}(A)$ generated by the element $(ad\, f_i)^{1-A_{ij}} f_j$ is contained in $\mathfrak{n}_-$ and hence intersects $\mathfrak{h}$ trivially by the triangular decomposition. This contradicts the definition of $\mathfrak{g}(A)$ unless $(ad\, f_i)^{1-A_{ij}} f_j = 0$. □

Proposition 1.2.17. *(Center of $\mathfrak{g}(A)$).*
$Z(\mathfrak{g}(A)) = \{h \in \mathfrak{h} : \alpha_i(h) = 0\ ,\ i = 1, \ldots, n\} \subseteq \sum_{i=1}^{n} \mathbb{C}\alpha_i^\vee$.
Moreover $dim\, Z(\mathfrak{g}(A)) = corank(A)$.

Proof.
Let $c \in Z(\mathfrak{g}(A))$. Then in particular $[h, c] = 0\ \forall h \in \mathfrak{h}$ so $c \in \mathfrak{g}_0$. By Theorem 1.2.8(iii) $\mathfrak{g}_0 = \mathfrak{h}$ so $c \in \mathfrak{h}$. Finally $0 = [c, e_i] = \alpha_i(c)e_i \implies \alpha_i(c) = 0$ for $i = 1, \ldots, n$. Conversely if $h \in \mathfrak{h}$ and $\alpha_i(h) = 0$ for $i = 1, \ldots, n$ then $[h, e_i] = \alpha_i(h)e_i = 0$ and $[h, f_i] = -\alpha_i(h)f_i = 0$ so h commutes with all Chevalley generators and therefore belongs to the center. Hence

$$Z(\mathfrak{g}(A)) = \{h \in \mathfrak{h} : \alpha_i(h) = 0\ ,\ i = 1, \ldots, n\} \subseteq \mathfrak{h}.$$

Since $\alpha_i(\alpha_j^\vee) = A_{ji}$, each $\alpha_j^\vee$ picks out the j^{th} row of A over $i = 1, \ldots, n$. Therefore any linear combination of the rows induces a linear combination of the coroots. So linear combinations of rows summing to zero are in correspondence with elements of the center, so the dimension of the null space of A (i.e the corank) is in correspondence with the maximal number of linearly independent elements of $\sum_{i=1}^{n} \mathbb{C}\alpha_i^\vee$ also contained in the center.
Thus

$$corank(A) = dim \left(Z(\mathfrak{g}(A)) \cap \sum_{i=1}^{n} \mathbb{C}\alpha_i^\vee \right) \leq dim\, Z(\mathfrak{g}(A)).$$

On the other hand the simple coroots $\alpha_j^\vee$ $(j = 1, \ldots, n)$ are contained in the vector space complement of $Z(\mathfrak{g}(A))$ in $\mathfrak{h}$ since $\alpha_j(\alpha_j^\vee) = A_{jj} = 2 \neq 0$ by axiom (GCM1). They are also linearly independent by definition.
Therefore

$$dim\, Z(\mathfrak{g}(A)) \leq dim\, \mathfrak{h} - n = n - rank(A) = corank(A).$$

Hence

$$dim \left(Z(\mathfrak{g}(A) \cap \sum_{i=1}^{n} \mathbb{C}\alpha_i^\vee) \right) = dim\, Z(\mathfrak{g}(A)) \implies Z(\mathfrak{g}(A) \subseteq \sum_{i=1}^{n} \mathbb{C}\alpha_i^\vee.$$

□

1.3 Examples

Neither the size nor growth of a Kac-Moody algebra is immediately obvious from its definition. For now at least, it is clear from the root space decomposition in Theorem 1.2.8(*iii*) that $\mathfrak{g}(A)$ is infinite-dimensional if (and only if by (*iv*)) it has infinitely many roots. We will look at two small examples, one which highlights infinite-dimensionality and one that produces a root space of multiplicity > 1. The discussion in the latter example is inspired by two exercises from [Kac exercise 1.5-1.6].

Example 1.3.1. (Infinite-dimensionality).
Consider the two GCMs

$$A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad A_1^{(1)} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

A_2 is the familiar Cartan matrix for the 8-dimensional Lie algebra $\mathfrak{sl}_3$ and $A_1^{(1)}$ is a matrix of affine type. We have Chevalley relations:

$$A_2 : \ [h_1, h_2] = 0,\ [h_1, e_2] = -e_2,\ [h_2, e_1] = -e_1,\ [e_i, f_j] = \delta_{ij} h_i.$$

$$A_1^{(1)} : \ [\overline{h}_1, \overline{h}_2] = 0,\ [\overline{h}_1, \overline{e}_2] = -2\overline{e}_2,\ [\overline{h}_2, \overline{e}_1] = -2\overline{e}_1,\ [\overline{e}_i, \overline{f}_j] = \delta_{ij} \overline{h}_i.$$

We can see that the two sets of relations are strikingly similar.
However the presentation for $A_1^{(1)}$ yields an infinite-dimensional Lie algebra, so one could ask how such a small difference can have such a large impact? The answer lies in the Serre relations:

$$A_2 : \ [e_1, [e_1, e_2]] = 0,\ [e_2, [e_2, e_1]] = 0$$

$$A_1^{(1)} : \ [\overline{e}_1, [\overline{e}_1, [\overline{e}_1, \overline{e}_2]]] = 0,\ [\overline{e}_2, [\overline{e}_2, [\overline{e}_2, \overline{e}_1]]] = 0$$

The Serre relations for A_2 kill all Lie words of length greater than 2 in the e_i. Corresponding fact also holds for words in f_i. This gives a basis

$$\{e_1, e_2, [e_1, e_2], f_1, f_2, [f_1, f_2], h_1, h_2\}$$

for the presentation. However $[\overline{e}_1, [\overline{e}_2, \overline{e}_1]] \neq 0$ in $\mathfrak{g}(A_1^{(1)})$. In fact elements of type $[\overline{e}_1, [\overline{e}_2, [\overline{e}_1, \ldots, [\overline{e}_2, \overline{e}_1] \ldots]]]$ form an infinite family of linearly independent elements in $\mathfrak{g}(A_1^{(1)})$, but unfortunately it is not very pleasant to prove using algebraic relations. To prove infinite-dimensionality it is simpler to compute that $A_1^{(1)}$ has an infinite set of associated roots of the form

$$\{m\delta \pm \alpha : m \in \mathbb{Z}\} \cup \{m\delta : m \in \mathbb{Z}\}.$$

The labelling follows from the distinction between real and imaginary roots (terminology explained in §1.5).

Example 1.3.2. (Multiplicity > 1).
The goal of this example is to illustrate that root spaces corresponding to GCMs may have dimension greater than 1 (as opposed to the finite-dimensional case). In our attempt to find such a root space consider an arbitrary GCM A with simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. Clearly $mult(\alpha_i) = 1$ since $e_i \in \mathfrak{g}_{\alpha_i}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. Next we consider multiplicities for roots of the form $\alpha_i + s\alpha_j$, $s \in \mathbb{N}$. It is clear that the only Lie word that can be non-zero in $\mathfrak{g}_{\alpha_i + s\alpha_j}$ is given by

$$\underbrace{[e_j, [e_j, [\ldots, [e_j,}_{s \text{ times}} e_i] \ldots]$$

Any other Lie word on these letters would contain a subword consisting only of e_j's killing the product. Hence $mult(\alpha_i + s\alpha_j) \leq 1$. In fact

$$mult(\alpha_i + s\alpha_j) = 1 \Leftrightarrow |A_{ji}| \leq s$$

by the Serre relations.

What about $mult(2(\alpha_i + \alpha_j))$?
Note first that by induction using the Lie relations any Lie word can be put in the bracket form

$$\pm[a_1, [a_2, [\ldots [a_{k-1}, a_k] \ldots]$$

Thus by inspection there exists a unique Lie word $w(e_i, e_i, e_j, e_j)$ (up to sign). Hence $mult(2(\alpha_i + \alpha_j)) \leq 1$.

We continue the search by considering height 5 roots of the form $2\alpha_i + 3\alpha_j$. By above there are unique Lie words (up to sign) of type $w(e_j, e_j, e_j, e_i)$ and $w(e_j, e_j, e_i, e_i)$. Thus there are only two Lie words to consider:

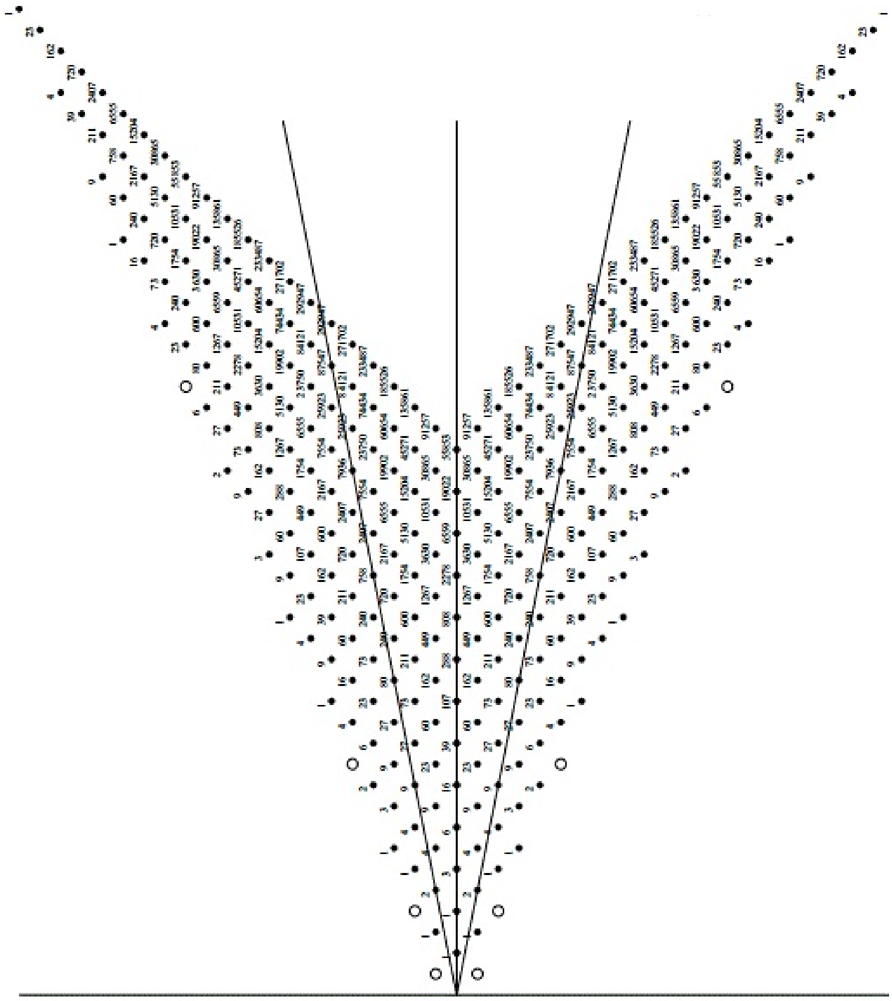
$$[e_i, [e_j, [e_j, [e_j, e_i]]]] \quad \text{and} \quad [e_j, [e_j, [e_i, [e_i, e_j]]]]$$

Hence provided the Serre relations does not kill either of them we will have $mult(2\alpha_i + 3\alpha_j) = 2$.

For example the following indefinite GCM fits the criteria

$$A = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$$

Clearly due to the combinatorial explosion it becomes impractical to continue this analysis much further. The matrix A above leads to the so called Fibonacci algebra, named so because its roots are linear combinations of simple roots with Fibonacci coefficients [FN]. A is also an example of a *hyperbolic* GCM. Hyperbolic GCMs form a small subclass of matrices of indefinite type consisting of all indecomposable symmetrizable matrices such that each proper connected sub-Dynkin diagram is of finite or affine type [Kac Def 5.10]. They were classified in 1988 by Wang Lai [Lai]. Although multiplicities for finite/affine matrices are well-known there is not a single example of a hyperbolic matrix where $mult(\alpha)$ is known for all α in closed form [FN]. Below diagram taken from [FN] depicts the proliferation of multiplicities for the Fibonacci algebra:



The open circles represent *real roots*, and the solid dots *imaginary roots*. Real roots are roots which belong to the Weyl-group orbit of a simple root, imaginary roots are those which do not. We will briefly discuss the significance of this distinction in §1.5.

1.4 Symmetrizable Kac-Moody algebras

For finite-dimensional semisimple Lie algebras the existence of the Killing form enables the complete characterization of semisimplicity by the non-degeneracy of the form (via Cartan's criterion). When non-degenerate it moreover induces an isomorphism $\mathfrak{h} \longrightarrow \mathfrak{h}^*$ which gives rise to an inner product on the weight lattice. This inner product makes up important formulations in the representation theory of semisimple Lie algebras (such as Freudenthal and Weyl formula) for calculating the dimension of any irreducible representation. We may therefore naturally ask (in the very least) whether a similar form exists for Kac-Moody algebras.

It is clearly impossible to reuse the Killing form

$$\kappa(X,Y) = tr(ad(X)ad(Y)) \quad \forall X,Y \in \mathfrak{g}$$

since $\mathfrak{g}(A)$ may be infinite-dimensional and so we can't take traces of endomorphisms. We could instead ask whether there exists a similar form enjoying the same fundamental properties as the Killing form. Such properties include non-degeneracy, ad-invariancy, symmetry and bilinearity. The answer is no in general but it turns out such a form exists whenever the GCM is *symmetrizable*.

Definition 1.4.1. (Symmetrizable Kac-Moody algebra).
An $n\times n$ matrix $A = (A_{ij})$ is called **symmetrizable** if there exists an invertible diagonal matrix $D = diag(d_1,\ldots,d_n)$ and a symmetric matrix $B = (B_{ij})$ such that $A = DB$. $\mathfrak{g}(A)$ is a **symmetrizable Kac-Moody algebra** if A is a symmetrizable GCM.

Let A be a symmetrizable GCM and let $(\mathfrak{h},\Pi,\Pi^\vee)$ be a minimal realization. Let $\mathfrak{h}' = \displaystyle\sum_{i=1}^{n}\mathbb{C}\alpha_i^\vee$ and let $\mathfrak{h}''$ be a vector space complement of $\mathfrak{h}'$ in $\mathfrak{h}$.
We define a bilinear form $(\ ,\ ) : \mathfrak{h}\times\mathfrak{h} \longrightarrow \mathbb{C}$ via

$$(\alpha_i^\vee, x) = (x,\alpha_i^\vee) = d_i\alpha_i(x) \ \ \forall x\in\mathfrak{h} \ \ (i=1,\ldots,n)$$
$$(x,y) = 0 \ \ \forall x,y\in\mathfrak{h}''$$

Then we have

$$(\alpha_i^\vee,\alpha_j^\vee) = d_i\alpha_i(\alpha_j^\vee) = d_iA_{ji} = d_id_jB_{ji} = d_jd_iB_{ij} = d_jA_{ij} = (\alpha_j^\vee,\alpha_i^\vee).$$

Hence the form is symmetric.
Our aim is to extend this form to $\mathfrak{g}(A)$ with the properties we are looking for. The strategy will be to build it inductively using above form as the base case.

First note that the form we have defined on $\mathfrak{h}$ is non-degenerate.
Suppose $x\in\mathfrak{h}$ such that $(x,h) = 0 \ \ \forall h\in\mathfrak{h}$.
Then in particular $0 = (x,\alpha_i^\vee) = d_i\alpha_i(x) \Longrightarrow \alpha_i(x) = 0$ for $i=1,\ldots,n$.
Hence by Proposition 1.2.17 $x\in Z(\mathfrak{g}(A))\subseteq\mathfrak{h}'$.
Therefore $x = \displaystyle\sum_{i=1}^{n} c_i\alpha_i^\vee$ for some $c_i\in\mathbb{C}$.
So for every $h\in\mathfrak{h}$ we have

$$0 = \left(\sum_{i=1}^{n} c_i\alpha_i^\vee, h\right) = \sum_{i=1}^{n} c_i(\alpha_i^\vee,h) = \sum_{i=1}^{n} c_id_i\alpha_i(h)$$

Hence $\displaystyle\sum_{i=1}^{n} c_id_i\alpha_i \equiv 0 \Longrightarrow c_id_i = 0 \ \ (i=1,\ldots,n)$ by linear independence.
Therefore $c_i = 0$ for $i=1,\ldots,n$ since $d_i\neq 0$ (D invertible).
Hence $x=0$ and so the form is non-degenerate.
It is moreover trivial to verify ad-invariance since the Lie bracket is zero because $\mathfrak{h}$ is abelian. We now extend this form to $\mathfrak{g}(A)$ via the next theorem.

Theorem 1.4.2. *(Invariant bilinear form).*
Suppose $\mathfrak{g}(A)$ is a symmetrizable Kac-Moody algebra.
Then $\mathfrak{g}(A)$ has a non-degenerate ad-invariant symmetric bilinear form.

Proof.
Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ we obtain a natural $\mathbb{Z}$-grading

$$\mathfrak{g}(A) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$$

by height where

$$\mathfrak{g}_i = \bigoplus_{\alpha : ht\,\alpha = i} \mathfrak{g}_\alpha.$$

For each $s \in \mathbb{N}$ define

$$\mathfrak{g}(s) = \bigoplus_{i=-s}^{s} \mathfrak{g}_i.$$

Note that this induces a filtration

$$\mathfrak{h} = \mathfrak{g}(0) \subset \mathfrak{g}(1) \subset \mathfrak{g}(2) \subset \dots$$

where $\mathfrak{g}(A) = \bigcup_{s \in \mathbb{N}} \mathfrak{g}(s)$.
We may therefore construct a form on $\mathfrak{g}(A)$ incrementally by induction on s.
The base case $s = 0$ is given by the form we have on $\mathfrak{h} = \mathfrak{g}(0)$.
For $s = 1$ we extend the form from $\mathfrak{h} = \mathfrak{g}(0)$ to $\mathfrak{g}(1)$ via

$$\begin{aligned}
&(e_i, f_j) = (f_j, e_i) = \delta_{ij} d_i \ \ (i, j = 1, \dots, n) \\
&(\mathfrak{g}_i, \mathfrak{g}_j) = 0 \ \ \text{whenever } i + j \neq 0,\ |i|, |j| \leq s \ \ (i, j \in \mathbb{Z}).
\end{aligned}$$

The checks for ad-invariance i.e

$$([x, y], z) = (x, [y, z]) \ \ \forall x, y, z \in \mathfrak{g}(1)$$

either vanishes on both sides or reduces to checking

$$\delta_{ij}(\alpha_i^\vee, h) = \delta_{ij} d_i \alpha_j(h) \ \ \forall h \in \mathfrak{h}$$

which we know holds by definition of the form. Suppose now we have an ad-invariant form on $\mathfrak{g}(s)$ for some $s \geq 1$ with

$$(\mathfrak{g}_i, \mathfrak{g}_j) = 0 \ \ \text{whenever } i + j \neq 0,\ |i|, |j| \leq s.$$

We extend the definition to $\mathfrak{g}(s+1)$:

$$(\mathfrak{g}_i, \mathfrak{g}_j) = 0 \ \ \text{whenever } i + j \neq 0,\ |i|, |j| \leq s + 1.$$

We have then left to define what the form should do on

$$x \in \mathfrak{g}_{s+1} \quad \text{and} \quad y \in \mathfrak{g}_{-(s+1)}.$$

It follows from Theorem 1.2.8(i),(ii) that we may write $y = \sum_{i \in I} [u_i, v_i]$ where $u_i, v_i \in \mathfrak{g}(s)$ are non-zero Lie monomials on $\{f_1, \dots, f_n\}$ of degree $\leq s$. Note that $[x, u_i] \in \mathfrak{g}(s)$ since $u_i \in \mathfrak{g}_i$ for some $i < 0$ because u_i is a non-zero Lie monomial in $\{f_1, \dots, f_n\}$. Given this we define

$$(x, y) := \sum_{i \in I} ([x, u_i], v_i)$$

in terms of the form on $\mathfrak{g}(s)$. Note that care must be taken. There is some potential ambiguity arising here from the fact that the expression for y need not be unique, so there is an issue of well-definedness since we do not want the form to depend in any way on the particular expression chosen for y.

To establish well-definedness we first check the identity

$$([[x_i, x_j], x_k], x_l) = (x_i, [x_j, [x_k, x_l]]) \tag{1.5}$$

for $x_i \in \mathfrak{g}_i, x_j \in \mathfrak{g}_j, x_k \in \mathfrak{g}_k, x_l \in \mathfrak{g}_l$
where $i + j + k + l = 0,\ |i + j| = |k + l| = s + 1,\ |i|, |j|, |k|, |l| \leq s$.

Calculating we have

$$\begin{aligned}
([[x_i, x_j], x_k], x_l) &= ([[x_i, x_k], x_j], x_l) - ([[x_j, x_k], x_i], x_l) && \text{Jacobi + bilinearity} \\
&= ([x_i, x_k], [x_j, x_l]) + (x_i, [[x_j, x_k], x_l]) && \text{I.H} \times 2 \\
&= (x_i, [x_k, [x_j, x_l]] + [[x_j, x_k], x_l]) && \text{I.H + bilinearity} \\
&= (x_i, [x_j, [x_k, x_l]]) && \text{Jacobi}
\end{aligned}$$

Again by Theorem 1.2.8 we may write $x = \sum_{j \in J} [u'_j, v'_j]$ where $u'_j, v'_j \in \mathfrak{g}(s)$ are non-zero Lie monomials on $\{e_1, \ldots, e_n\}$ of degree $\leq s$. By (1.5) it follows that

$$\begin{aligned}
(x, y) &= \sum_{i \in I} ([x, u_i], v_i) \\
&= \sum_{i \in I} \sum_{j \in J} ([[u'_j, v'_j], u_i], v_i) = \sum_{i \in I} \sum_{j \in J} (u'_j, [v'_j, [u_i, v_i]]) = \sum_{j \in J} (u'_j, [v'_j, \sum_{i \in I} [u_i, v_i]]) \\
&= \sum_{j \in J} (u'_j, [v'_j, y])
\end{aligned}$$

Therefore regardless of the expression chosen for y the end result is always $\sum_{j \in J} (u'_j, [v'_j, y])$ so the form is independent of the choice of u_i and v_i. Hence we have a well-defined bilinear form on $\mathfrak{g}(s + 1)$. For ad-invariance we have to check (in particular) that

$$([x, y], h) = (x, [y, h]) \ \ \forall h \in \mathfrak{h}$$

$$\begin{aligned}
([x, y], h) &= \sum_{i \in I} ([x, [u_i, v_i]], h) && \text{bilinearity} \\
&= \sum_{i \in I} ([[x, u_i], v_i], h) - ([[x, v_i], u_i], h) && \text{Jacobi + bilinearity + anti-symmetry} \\
&= \sum_{i \in I} (x, [u_i, [v_i, h]]) - (x, [v_i, [u_i, h]]) && \text{I.H} \times 4 \\
&= \sum_{i \in I} (x, [[u_i, v_i], h]) && \text{bilinearity + Jacobi} \\
&= (x, [y, h]) && \text{bilinearity}
\end{aligned}$$

Remaining checks are verified via calculations following the same (or very similar) sequence of steps. Hence the form is ad-invariant by induction.

Symmetry now follows from ad-invariance and bilinearity. Let $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ and choose $h \in \mathfrak{h}$ such that $\alpha(h) \neq 0$. Then on one hand

$$([x, y], h) = (x, [y, h]) = -(x, [h, y]) = -(x, -\alpha(h)y) = \alpha(h)(x, y).$$

On the other hand

$$([x, y], h) = -([y, x], h) = -(y, [x, h]) = (y, [h, x]) = (y, \alpha(h)x) = \alpha(h)(y, x).$$

Hence $(x, y) = (y, x)$.

It remains to check non-degeneracy. Let $\mathfrak{i} = \{x \in \mathfrak{g}(A) : (x, y) = 0 \ \ \forall y \in \mathfrak{g}(A)\}$. We claim $\mathfrak{i}$ is an ideal of $\mathfrak{g}(A)$. Let $x \in \mathfrak{i}$ and $a, y \in \mathfrak{g}(A)$. Since the form is ad-invariant we have $([a, x], y) = -(x, [a, y]) = 0$. Therefore $\mathfrak{i} \trianglelefteq \mathfrak{g}(A)$. Since the form is non-degenerate on $\mathfrak{h}$ by construction we have $\mathfrak{i} \cap \mathfrak{h} = \{0\}$. But $\mathfrak{g}(A)$ has no non-zero ideals trivially intersecting $\mathfrak{h}$ by definition. Therefore $\mathfrak{i} = \{0\}$. Hence the form is non-degenerate. □

Remark 1.4.3.
The constructed form is often referred to as the **standard form**. The proof of the theorem shows that the form is determined by its restriction to $\mathfrak{h}$. Notice also how our definition of Kac-Moody algebra come into play in the proof of non-degeneracy. If we instead defined the Kac-Moody algebra purely in terms of the Chevalley-Serre presentation it is not clear up front whether the form is non-degenerate. Fortunately the two definitions coincide whenever the GCM is symmetrizable so we get non-degeneracy in both cases.

1.5 The Weyl group

The Weyl group is another important notion from the theory of semisimple Lie algebras. It is most notably involved in the Weyl character formula and is a natural description of the symmetries of the roots. We therefore have motivation for understanding how it generalizes to Kac-Moody algebras. The Weyl group is defined, like previously, as the group W generated by fundamental Weyl reflections

$$w_i(\beta) = \beta - \beta(\alpha_i^\vee)\alpha_i, \quad \beta \in \mathfrak{h}^*, \ i = 1, \ldots, n.$$

There is one major difference between the root orbits in the finite and infinite-dimensional case. While the Weyl group acts transitively on the roots in finite dimension, this no longer true for infinite-dimensional Kac-Moody algebras. This is basis for the dichotomy between *real* and *imaginary* roots i.e the distinction between roots which lie in an orbit of a simple root and roots which do not. Real roots enjoy many of the classical properties from finite-dimensional theory. For instance they have multiplicity 1 and the only scalar multiple of a real root which is again a root is ± 1 [Kac Prop 5.1]. For imaginary roots on the other hand any integer multiple of an imaginary root is again an imaginary root [Kac Prop 5.5] and such roots may have arbitrary multiplicity. It is therefore imaginary roots that are responsible for the rapid dimension growth in Kac-Moody algebras as height increases. When the Kac-Moody algebra is symmetrizable imaginary roots are characterized by having non-positive norm i.e $(\alpha, \alpha) \leq 0$ in the standard bilinear form [Kac Prop 5.2]. This gives some justification for the term 'imaginary'.

If $|\Phi| < \infty$ then W must leave Φ invariant by the axioms of a root system. Therefore $W \hookrightarrow Sym(\Phi)$ and so $|\Phi| < \infty \implies |W| < \infty$. By [Kac Prop 3.12] the converse is true also. In combination with Theorem 1.2.8 we get another characterization of finite-dimensionality for Kac-Moody algebras.

Proposition 1.5.1.
The following are equivalent:

i) $|W| < \infty$

ii) $|\Phi| < \infty$

iii) $dim\, \mathfrak{g}(A) < \infty$

In particular this proposition implies the Weyl group of a Kac-Moody algebra may be infinite, and certainly infinite in the presence of an imaginary root.

1.6 Classification of generalized Cartan matrices

Recall axioms (**GCM1-3**) from §1.2 for a generalized Cartan matrix (GCM). The aim of this section is to prove that indecomposable GCMs fall under one of three mutually exclusive classes: finite, affine or indefinite type. Let us begin by defining what these classes mean.

Notational convention: for a vector $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ we write $u > 0$ if $u_i > 0$ for all $i = 1, \ldots, n$ and $u \geq 0$ if $u_i \geq 0$ for all $i = 1, \ldots, n$. Note in particular that $u \geq 0$ does not in general imply ($u > 0$ or $u = 0$).

Definition 1.6.1. (GCM classes).
Let A be a GCM.
We say A is of **finite type** if
(i) $det\, A \neq 0$
(ii) $\exists u > 0$ with $Au > 0$
(iii) $Au \geq 0 \implies u > 0$ *or* $u = 0$

We say A is of **affine type** if
(i) $corank(A) = 1$ (i.e $rank(A) = n - 1$)
(ii) $\exists u > 0$ such that $Au = 0$
(iii) $Au \geq 0 \implies Au = 0$

We say A is of **indefinite type** if
(i) $\exists u > 0$ such that $Au < 0$
(ii) $Au \geq 0$ *and* $u \geq 0 \implies u = 0$

Definition 1.6.2. (Indecomposability).
A GCM A is **indecomposable** if it cannot be conjugated to a block diagonal matrix

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where A_1 and A_2 are GCMs of smaller size.

We wish to prove the following theorem:

Theorem 1.6.3. *(GCM classification).*
Let A be an indecomposable GCM.
Then A has exactly one of the following types:
(1) *finite*
(2) *affine*
(3) *indefinite*

To prove the main theorem we need a few lemmas.

Lemma 1.6.4. *(Mutual exclusivity).*
The classes finite, affine and indefinite type are mutually exclusive.

Proof.
Condition (i) in finite and affine classes are incompatible.
Condition (ii) in finite and indefinite classes are incompatible.
Condition (ii) in affine and indefinite classes are incompatible. □

Remark 1.6.5.
It is therefore sufficient to prove that every GCM fall under at least one of the three types.

Lemma 1.6.6.
Let $\mathbf{a}_i = (a_{i1}, \ldots, a_{in}) \in \mathbb{R}^n$ for $i = 1, \ldots, m$. Then

$$\sum_{j=1}^{n} a_{ij} x_j < 0 \ , \ i = 1, \ldots, m$$

has a solution if and only if

$$\sum_{i=1}^{m} b_i \mathbf{a}_i = 0 \ , \ b_1, \ldots, b_m \geq 0 \Longrightarrow b_1, \ldots, b_m = 0$$

Proof. (See [Car, Lemma 15.2]). □

Remark 1.6.7.
Note that $\sum_{i=1}^{m} b_i \mathbf{a}_i = 0$ has a non-zero solution if and only if there exists a non-zero solution with $\sum_{i=1}^{m} b_i = 1$. The geometric interpretation of Lemma 1.6.6 can thus be expressed as the existence of a half-space containing the vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ if and only if the convex hull of $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ can be separated from the origin. It is therefore an essentially weaker version of the hyperplane separation theorem in which one of the convex sets is a point.

A closely related statement to Lemma 1.6.6 (in the form we want to use) is:

Lemma 1.6.8.
Let $A \in M_{m\times n}(\mathbb{R})$.
Suppose $(u \geq 0$ *and* $A^T u \geq 0) \implies u = 0$.
Then $\exists v > 0$ *such that* $Av < 0$.

Proof. See [Car Prop 15.3]

□

Lemma 1.6.9.
Let A *be an indecomposable GCM.*
Then $Au \geq 0,\ u \geq 0 \implies u > 0$ *or* $u = 0$

Proof.
Suppose for a contradiction that $Au \geq 0,\ u \geq 0$ but $u \not> 0$ and $u \neq 0$. Without loss of generality we may reorder indices s.t $u_i = 0$ for $i = 1, \ldots, s$ and $u_i > 0$ for $i = s+1, \ldots, n$. Note that $0 < s < n$ due to hypothesis and negated conclusion. Let $u' = (u_{s+1}, \ldots, u_n)^T > 0$. Denoting by P, Q, R, S the submatrices of A with dimensions $s \times s$, $s \times (n-s)$, $(n-s) \times s$ and $(n-s) \times (n-s)$ respectively we get

$$Au = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ u' \end{pmatrix} = \begin{pmatrix} Qu' \\ Su' \end{pmatrix}$$

By (GCM3) Q has all non-positive entries. Therefore unless $Q = 0$, Qu' is a vector containing strictly negative entries contradicting $Au \geq 0$. But if $Q = 0$ then $R = 0$ by (GCM2). Therefore A is block diagonal. P and S both inherit GCM axioms from A being principal submatrices. This finally contradicts indecomposability of A. □

Now let A be a GCM and consider the convex cone

$$K_A = \{u \in \mathbb{R}^n : Au \geq 0\}$$

By Lemma 1.6.9 it follows that

$$K_A \cap \{u \in \mathbb{R}^n : u \geq 0\} \subseteq \{u \in \mathbb{R}^n : u > 0\} \cup \{0\}$$

i.e the only time the non-negative part of the cone intersects a coordinate hyperplane is at 0. Therefore $\{u \in \mathbb{R}^n : u \geq 0$ and $Au \geq 0\} \neq \{0\}$ only in one of below mutually exclusive cases:

$$K_A \subseteq \{u \in \mathbb{R}^n : u > 0\} \cup \{0\} \tag{1.6}$$

(entire cone is positive except at origin)

or

$$K_A = \{u \in \mathbb{R}^n : Au = 0\} \text{ and } K_A \text{ is a 1-dim subspace} \tag{1.7}$$

(cone degenerates into a line through the origin).

It is geometrically clear that convexity is lost if the cone has a non-positive point and is not a line, since we know it only enters non-positive region through the origin (the drawn line between a non-positive point and a positive point on the cone need not pass through the origin and is therefore not fully contained in the cone). So if (1.6) does not hold then K_A is a line spanned by some vector w. By definition $Aw \geq 0$. We also have $-w \in K_A$ since K_A is a line, so $A(-w) \geq 0 \Longrightarrow Aw \leq 0$ and hence $Aw = 0$, giving (1.7). Details for above statements can be found in [Car, Prop. 15.6].

We now characterize GCMs by reconciling above conditions with the definitions we gave at the beginning for finite, affine and indefinite-type matrices. This will complete the classification.

Proof. (GCM Classification).
Suppose

$$\{u \in \mathbb{R}^n : u \geq 0 \text{ and } Au \geq 0\} \neq \{0\} \tag{$*$}$$

Then as observed above there are two mutually exclusive cases (1.6) and (1.7) which we consider separately:

Case: $K_A \subseteq \{u \in \mathbb{R}^n : u > 0\} \cup \{0\}$.
We claim A is of finite type. Axiom (iii) follows immediately by the case assumption. If there exists $u \neq 0$ such that $Au = 0$ then span$\{u\}$ is a 1-dim subspace of K_A. But K_A contains only strictly positive points (apart from the origin) so such a subspace cannot be contained in K_A, contradiction. Therefore A is nonsingular and so $det(A) \neq 0$ asserting axiom (i). By $(*)$ there exists $u \neq 0$ such that $u \geq 0$ and $Au \geq 0$. Lemma 1.6.9 implies that $u > 0$. Since $u > 0$ we may perturb the entries of u with suitable $\epsilon_i > 0$ ensuring $Au > 0$. Therefore axiom (ii) holds. Hence A is of finite type.

Case: $K_A = \{u \in \mathbb{R}^n : Au = 0\}$ and K_A is a 1-dim subspace.
We claim A is of affine type. Axiom (i) follows immediately by the case assumption. By $(*)$ there exists $u \neq 0$ such that $u \geq 0$ and $Au \geq 0$. By case assumption $Au \geq 0 \Longrightarrow Au = 0$ and by Lemma 1.6.9 $u > 0$ asserting axioms (ii) and (iii). Hence A is of affine type.

Now drop $(*)$ and assume: $\{u \in \mathbb{R}^n : u \geq 0 \text{ and } Au \geq 0\} = \{0\}$.
We claim A has indefinite type. Axiom (ii) follows immediately by case assumption. By Lemma 1.6.8 it follows that there exists $v > 0$ such that $A^T v < 0$. Suppose A^T is of finite type. Then

$$A^T v < 0 \Longrightarrow A^T(-v) > 0 \Longrightarrow -v > 0 \text{ or } -v = 0 \Longrightarrow v < 0 \text{ or } v = 0$$

contradicting the fact that $v > 0$. Suppose A^T is of affine type. Then

$$A^T v < 0 \Longrightarrow A^T(-v) > 0 \Longrightarrow A^T(-v) = 0 \Longrightarrow A^T v = 0$$

contradicting the fact that $A^T v < 0$. Since A^T is neither of finite nor affine type we must have $\{u \in \mathbb{R}^n : u \geq 0 \text{ and } A^T u \geq 0\} = \{0\}$ (by what has been shown in preceding cases). Thus invoking Lemma 1.6.8 again we get $u > 0$ such that $Au < 0$. This asserts axiom (i). Hence A is of indefinite type.

Hence every GCMs is of finite, affine or indefinite type.

$\square$

Chapter 2

Affine Lie Algebras

In this chapter we initiate our shift in focus from general Kac-Moody algebras to affine Lie algebras and their representation theory. In the previous chapter we established the well-known trichotomy of Kac-Moody algebras. We identified Kac-Moody algebras of finite type with finite-dimensional simple Lie algebras. We will now look at affine Lie algebras and see how they depend, in a very concrete way, on a finite-dimensional simple Lie algebra. They are therefore some of the most natural infinite-dimensional generalizations one could think of. We only get so far by working with the Chevalley-generators of the algebra presentation. To do representation theory we ideally need something more tangible. Fortunately for affine Lie algebras the close relationship with a finite-dimensional simple Lie algebra can be exploited to yield a concrete realization. Roughly speaking this realization is a tensor between a finite-dimensional simple Lie algebra and the Laurent polynomials, extended by a 1-dimensional center and a derivation element. Due to this, much of the theory for finite-dimensional semisimple Lie algebras can be lifted to affine Lie algebras. While only the full realization can be identified with an affine Kac-Moody algebra one can also consider partial constructions which yield further infinite-dimensional generalizations, each with a distinguished representation theory. Somewhat ambiguously they are all referred to as affine Lie algebras in literature (underlying context being understood). We shall however mainly be concerned with the (untwisted) construction up to central extension which can be identified with the derived subalgebra of an affine Kac-Moody algebra. We will nevertheless describe the full construction for completeness before we enter discussions about finite-dimensional representation theory.

2.1 Realizations of affine Lie algebras

The purpose of this section is to describe the construction of a concrete realization for affine (Kac-Moody) Lie algebras. We begin with a notable observation concerning GCMs of affine type.

Proposition 2.1.1. *(Principal submatrices of affine GCMs are finite type). Let A be an indecomposable GCM of affine type and let A_m be a proper indecomposable principal submatrix of A of size $m \times m$. Then A_m is a GCM of finite type.*

Proof.
Without loss of generality we can consider $A = \begin{pmatrix} P & Q \\ R & A_m \end{pmatrix}$.

Since A is affine $\exists u > 0$ such that $Au = 0$. Moreover

$$Au = \begin{pmatrix} P & Q \\ R & A_m \end{pmatrix} \begin{pmatrix} u_{n-m} \\ u_m \end{pmatrix} = \begin{pmatrix} Pu_{n-m} + Qu_m \\ Ru_{n-m} + A_m u_m \end{pmatrix} = 0.$$

Therefore $Ru_{n-m} + A_m u_m = 0$. Since R has non-positive entries by axiom (GCM3) it follows that $Ru_{n-m} \leq 0$. Therefore $A_m u_m \geq 0$. Suppose for a contradiction that $A_m u_m = 0$. Then $Ru_{n-m} = 0$ and since $u_{n-m} > 0$ we have $R = 0$. But then $Q = 0$ by (GCM2) which contradicts indecomposability of A. We now have $u_m > 0$, $A_m u_m \geq 0$ and $A_m u_m \neq 0$. Note that A_m is certainly a GCM being a principal submatrix of a GCM. Therefore A_m is neither of affine nor indefinite type by definition. Hence A_m is of finite type by trichotomy. □

This proposition illustrates the close relationship between affine and finite type Kac-Moody algebras. By deleting first row and column of an affine matrix we get a matrix for a finite-dimensional simple Lie algebra. Hence given an affine Kac-Moody algebra $\mathfrak{g}(A)$ we may associate a finite-dimensional simple Lie algebra $\mathring{\mathfrak{g}} = \mathfrak{g}(A_{(n-1)})$. It may therefore not seem unreasonable that $\mathfrak{g}(A)$ can be built up as an extension of $\mathring{\mathfrak{g}}$.

What is known about $\mathfrak{g}(A)$?
(1) $\mathfrak{g}(A)$ is infinite-dimensional since A has affine type
(2) $dim\, Z(\mathfrak{g}(A)) = 1$ by Proposition 1.2.17
(3) $dim\, \mathfrak{h} = n + 1$ by minimal realization

In contrast, $\mathring{\mathfrak{g}}$ is finite-dimensional, has trivial center being a simple Lie algebra and has Cartan subalgebra of dimension $n - 1$ coming from a $(n-1) \times (n-1)$ GCM of full rank. These are all discrepancies that needs to be resolved before we can hope to identify our construction with $\mathfrak{g}(A)$.

Let $\mathfrak{L} = \mathbb{C}[t, t^{-1}]$ denote the algebra of Laurent polynomials in variable t. Consider the **loop algebra**

$$\mathfrak{L}(\mathring{\mathfrak{g}}) = \mathfrak{L} \otimes_{\mathbb{C}} \mathring{\mathfrak{g}}.$$

defined with pointwise multiplication

$$[P \otimes x, Q \otimes y]_{\mathfrak{L}(\mathring{\mathfrak{g}})} := PQ \otimes [x, y]_{\mathring{\mathfrak{g}}} \qquad (P, Q \in \mathfrak{L};\ x, y \in \mathring{\mathfrak{g}}).$$

Tensoring with the Laurent polynomials gives an infinite-dimensional Lie algebra along with a $\mathbb{Z}$-grading

$$\mathfrak{L}(\mathring{\mathfrak{g}}) = \bigoplus_{j\in\mathbb{Z}} (t^j \otimes \mathring{\mathfrak{g}}).$$

There are also homological reasons for considering the loop algebra. $\mathfrak{L}(\mathring{\mathfrak{g}})$ has a 1-dimensional second cohomology space $H^2(\mathfrak{L}(\mathring{\mathfrak{g}}), \mathbb{C})$ [Wei, 7.9.6] so there exists a non-zero 2-cocycle. We will see that this in turn gives rise to a non-trivial central extension.

By definition a 1-dim central extension of $\mathfrak{L}(\mathring{\mathfrak{g}})$ is given by a short exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{C}c \longrightarrow \hat{\mathring{\mathfrak{g}}} \longrightarrow \mathfrak{L}(\mathring{\mathfrak{g}}) \longrightarrow 0$$

where the image of c belongs to $Z(\hat{\mathring{\mathfrak{g}}})$.

We may write $\hat{\mathring{\mathfrak{g}}} = \mathfrak{L}(\mathring{\mathfrak{g}}) \oplus \mathbb{C}c$ as a vector space direct sum. To make this an extension of Lie algebras we need a working bracket on the vector space. We may consider a bracket definition of the form

$$[u + \lambda c, v + \mu c]_{\hat{\mathring{\mathfrak{g}}}} := [u, v]_{\mathfrak{L}(\mathring{\mathfrak{g}})} + \psi(u, v)c \qquad (u, v \in \mathfrak{L}(\mathring{\mathfrak{g}});\ \lambda, \mu \in \mathbb{C})$$

where $\psi : \mathfrak{L}(\mathring{\mathfrak{g}}) \times \mathfrak{L}(\mathring{\mathfrak{g}}) \longrightarrow \mathbb{C}$ is a bilinear form. The bracket becomes valid if and only if ψ defines a 2-cocycle i.e

$$\begin{aligned}
&\psi(u, v) = -\psi(v, u), \\
&\psi(u, [v, w]) + \psi([v, w], u) + \psi([w, u], v) = 0
\end{aligned}$$

for all $u, v, w \in \mathfrak{L}(\mathring{\mathfrak{g}})$

Note that c is indeed central with respect to this definition by bilinearity of ψ. To find a concrete ψ a natural starting point is the standard bilinear form $(\ ,\ )_\circ$ on $\mathring{\mathfrak{g}}$ (which is moreover known to be unique up to a constant multiple). This form can be extended linearly to an $\mathfrak{L}$-valued form on the loop algebra

$$\begin{aligned}
(\ ,\ ) : \mathfrak{L}(\mathring{\mathfrak{g}}) \times \mathfrak{L}(\mathring{\mathfrak{g}}) &\longrightarrow \mathfrak{L} \\
(P \otimes x, Q \otimes y) &\longmapsto PQ(x, y)_\circ
\end{aligned}$$

for all $P, Q \in \mathfrak{L},\ \ x, y \in \mathring{\mathfrak{g}}$.
Define the linear functional (the *residue*)

$$\begin{aligned}
Res : \mathfrak{L} &\longrightarrow \mathbb{C} \\
Res\left(\sum_{k\in\mathbb{Z}} c_k t^k\right) &\longmapsto c_{-1}
\end{aligned}$$

and set

$$\psi(P \otimes x, Q \otimes y) = Res\left(\tfrac{dP}{dt} \otimes x, Q \otimes y\right) \qquad (P, Q \in \mathfrak{L};\ x, y \in \mathring{\mathfrak{g}}).$$

It can now be simply checked that this defines a 2-cocycle on $\mathfrak{L}(\mathring{\mathfrak{g}})$ (see [Car Lemma 18.3]).

At this point our extension $\hat{\mathring{\mathfrak{g}}} = \mathfrak{L}(\mathring{\mathfrak{g}}) \oplus \mathbb{C}c$ has a 1-dimensional center and a Cartan subalgebra $(1 \otimes \mathring{\mathfrak{h}}) \oplus \mathbb{C}c$ of dimension n (here $\mathring{\mathfrak{h}}$ denotes a Cartan subalgebra of $\mathring{\mathfrak{g}}$ having dimension $n-1$). Our Cartan subalgebra is therefore still missing a dimension. Furthermore this extension has introduced a new issue. Since affine GCMs are symmetrizable we know from Theorem 1.4.2 that $\mathfrak{g}(A)$ has an ad-invariant bilinear form. For any ad-invariant bilinear form on $\hat{\mathring{\mathfrak{g}}}$ we have $([u,v],c) = (u,[v,c]) = 0$ since $[v,c] = 0 \;\; \forall v \in \hat{\mathring{\mathfrak{g}}}$. Given that $[\mathring{\mathfrak{g}}, \mathring{\mathfrak{g}}] = \mathring{\mathfrak{g}}$ since $\mathring{\mathfrak{g}}$ simple, it follows that $[\mathfrak{L}(\mathring{\mathfrak{g}}), \mathfrak{L}(\mathring{\mathfrak{g}})] = \mathfrak{L}(\mathring{\mathfrak{g}})$ so every element of $\mathfrak{L}(\mathring{\mathfrak{g}})$ is expressible as a sum of elements of the form $[u,v]$. Hence every ad-invariant bilinear form on $\hat{\mathring{\mathfrak{g}}}$ is degenerate.

To fix above issues we must introduce another extension. We need a space which acts non-trivially on the loop algebra to prevent degeneracy, while simultaneously killing the central element to maintain a 1-dimensional center and augments the Cartan subalgebra by one dimension. This can be achieved by adjoining a derivation d which acts on $\mathfrak{L}(\mathring{\mathfrak{g}})$ by multiplying each basis element with its degree with respect to the gradation. In other words d acts by $t\frac{d}{dt}$ on $\mathfrak{L}(\mathring{\mathfrak{g}})$ and trivially on c. This way we ensure it commutes with the abelian subalgebra $(1 \otimes \mathring{\mathfrak{h}}) \oplus \mathbb{C}c$. One then constructs an extension

$$\tilde{\mathring{\mathfrak{g}}} = \mathfrak{L}(\mathring{\mathfrak{g}}) \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with bracket defined via

$$\begin{gathered}
[(t^i \otimes x) + \lambda_1 c + \mu_1 d\,,\, (t^j \otimes x) + \lambda_2 c + \mu_2 d]_{\tilde{\mathring{\mathfrak{g}}}} := \\
[(t^i \otimes x) + \lambda_1 c, (t^j \otimes x) + \lambda_2 c]_{\hat{\mathring{\mathfrak{g}}}} + \lambda d.((t^j \otimes x) + \lambda_2 c) - \mu d.((t^i \otimes x) + \lambda_1 c) = \\
(t^{i+j} \otimes [x,y]_{\mathring{\mathfrak{g}}}) + \mu_1 (j t^j \otimes y) - \mu_2 (i t^i \otimes x) + i\delta_{i,-j}(x,y)c
\end{gathered}$$

and checks that the action of d is a derivation on $\hat{\mathring{\mathfrak{g}}}$ [Car Lemma 18.4]. This gives Jacobi identity for the bracket and anti-symmetry is clear. Hence the extension is valid. We remark also that c remains central with respect to the new bracket.

We now have a construction which resolves $(1), (2)$ and (3).
It turns out this is sufficient to identify $\tilde{\mathring{\mathfrak{g}}}$ with the affine Kac-Moody algebra $\mathfrak{g}(A)$ [See Kac Thm 7.4]. However only a subset of affine Kac-Moody algebras can be related to this construction. Such Kac-Moody algebras are labelled **untwisted**. Remaining affine Kac-Moody algebras can also be related to a finite-dimensional simple Lie algebra, but in a slightly more convoluted way that depends on a finite order automorphism of $\mathring{\mathfrak{g}}$. Such Kac-Moody algebras are labelled **twisted**. Roughly speaking an automorphism $\sigma : \mathring{\mathfrak{g}} \longrightarrow \mathring{\mathfrak{g}}$ such that $\sigma^m = 1$ gives rise to an eigenspace decomposition

$$\mathring{\mathfrak{g}} = \bigoplus_{j \in \mathbb{Z}_m} \mathring{\mathfrak{g}}_j$$

where $\mathring{\mathfrak{g}}_j$ is the eigenspace of σ with eigenvalue ϵ^j where $\epsilon = e^{\frac{2\pi i}{m}}$.
A subalgebra $\mathfrak{L}(\mathring{\mathfrak{g}}, \sigma, m) \leq \mathfrak{L}(\mathring{\mathfrak{g}})$ is then associated to σ given by

$$\mathfrak{L}(\mathring{\mathfrak{g}}, \sigma, m) = \bigoplus_{j \in \mathbb{Z}} \mathfrak{L}(\mathring{\mathfrak{g}}, \sigma, m)_j$$

where $\mathfrak{L}(\mathring{\mathfrak{g}}, \sigma, m)_j = t^j \otimes \mathring{\mathfrak{g}}_j$.
One finally makes extensions similar to those in the untwisted construction to obtain a subalgebra of $\tilde{\mathring{\mathfrak{g}}}$ given by

$$\tilde{\mathring{\mathfrak{g}}}^\sigma = \mathfrak{L}(\mathring{\mathfrak{g}}, \sigma, m) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

We will not go any deeper into this construction (cf [Kac Chapter 8]).
We just note that this is a generalization of the untwisted construction in which $\sigma = id$.

2.2 Finite-dimensional representations

In this section we collect a few important results regarding the finite-dimensional representation theory of affine Lie algebras, in this context being centrally extended loop algebras i.e $\hat{\mathring{\mathfrak{g}}} = \mathfrak{L}(\mathring{\mathfrak{g}}) \oplus \mathbb{C}c$ (cf §2.1). The discussion here will be used as a basis for comparison in the next chapter when we consider quantum affine algebras (which are certain deformations of affine Lie algebras).

We first show that the center is redundant for the study of finite-dimensional representations of $\hat{\mathring{\mathfrak{g}}}$ because its action is trivial. The finite-dimensional representation theory of $\hat{\mathring{\mathfrak{g}}}$ thus reduces to the finite-dimensional representation theory of $\mathfrak{L}(\mathring{\mathfrak{g}})$.

Proposition 2.2.1. *(Center has trivial action).*
Let $\rho : \hat{\mathring{\mathfrak{g}}} \longrightarrow End(V)$ be a finite-dimensional representation.
Then $\rho(c) = 0$.

Proof.
First note that $dim\,\hat{\mathring{\mathfrak{g}}} = \infty$ and $dim\,End(V) < \infty$ so $ker\rho \trianglelefteq \hat{\mathring{\mathfrak{g}}}$ is a non-zero ideal with $\hat{\mathring{\mathfrak{g}}}/ker\rho$ necessarily finite-dimensional. Recall from §2.1 how multiplication in $\hat{\mathring{\mathfrak{g}}}$ is defined:

$$\begin{aligned}[P \otimes x + \lambda c, Q \otimes y + \mu c]_{\hat{\mathring{\mathfrak{g}}}} &= [P \otimes x, Q \otimes y]_{\mathfrak{L}(\mathring{\mathfrak{g}})} + Res\left(\frac{dP}{dt} \otimes x\,,\, Q \otimes y\right) c \\ &= PQ \otimes [x, y]_{\mathring{\mathfrak{g}}} + Res\left(\frac{dP}{dt} Q\,(x, y)_\circ\right) c\end{aligned}$$

for all $P, Q \in \mathbb{C}[t, t^{-1}],\ x, y \in \mathring{\mathfrak{g}},\ \lambda, \mu \in \mathbb{C}$

Let $\displaystyle\sum_{\alpha \in \mathring{\Phi} \cup \{0\}} (Q_\alpha \otimes y_\alpha) + \mu c \in ker\rho \setminus \{0\},\ Q_\alpha \in \mathbb{C}[t, t^{-1}],\ y_\alpha \in \mathring{\mathfrak{g}}_\alpha,\ \mu \in \mathbb{C}.$

We argue by induction on the number of non-zero terms $Q_\alpha \otimes y_\alpha$ ($\alpha \in \mathring{\Phi}$). If all terms are zero then $\mu c \neq 0$ so $c \in ker\rho$ and we are done. Otherwise $\exists \beta \in \mathring{\Phi}$ such that $Q_\beta \otimes y_\beta \neq 0$. By standard theory of semisimple Lie algebras we have that $(\mathring{\mathfrak{g}}_\alpha, \mathring{\mathfrak{g}}_\beta)_\circ \neq 0 \Leftrightarrow \alpha = -\beta$ [Hum Prop 8.1]. Therefore $\exists x_\beta \in \mathring{\mathfrak{g}}_{-\beta}$ such that $(x_\beta, y_\beta)_\circ \neq 0$ and such that $(x_\beta, y_\alpha)_\circ = 0$ for $\alpha \neq \beta$. Let $P_\beta \in \mathbb{C}[t, t^{-1}]$ such that $Res(\frac{dP_\beta}{dt}Q_\beta) \neq 0$. Then

$$[P_\beta \otimes x_\beta, \sum_{\alpha\in\mathring{\Phi}\cup\{0\}} (Q_\alpha \otimes y_\alpha) + \mu c]_{\hat{\mathring{\mathfrak{g}}}} = \sum_{\alpha\in\mathring{\Phi}\cup\{0\}} (P_\beta Q_\alpha \otimes [x_\beta, y_\alpha]) + \underbrace{Res\left(\frac{dP_\beta}{dt}Q_\beta(x_\beta, y_\beta)_\circ\right)}_{\mu' \neq 0} c$$

$$\in ker\rho$$

If $[x_\beta, y_\beta] = 0$ then there are less non-zero terms and we are done by induction. Otherwise $[x_\beta, y_\beta] \in \mathring{\mathfrak{g}}_0 = \mathring{\mathfrak{h}}$ is non-zero. Therefore using that $(\ ,\ )_\circ|_{\mathring{\mathfrak{h}}}$ is non-degenerate $\exists h \in \mathring{\mathfrak{h}}$ such that $(h, [x_\beta, y_\beta])_\circ \neq 0$. Likewise $\exists R \in \mathbb{C}[t, t^{-1}]$ such that $Res(\frac{dR}{dt}P_\beta Q_\beta) \neq 0$. Then

$$\begin{aligned} &[R \otimes h, \sum_{\alpha\in\mathring{\Phi}\cup\{0\}} (P_\beta Q_\alpha \otimes [x_\beta, y_\alpha]) + \mu' c]_{\hat{\mathring{\mathfrak{g}}}} \\ = &\sum_{\alpha\in\mathring{\Phi}\cup\{0\}} (RP_\beta Q_\alpha \otimes [h, [x_\beta, y_\alpha]]) + \underbrace{Res\left(\frac{dR}{dt}P_\beta Q_\beta(h, [x_\beta, y_\beta])_\circ\right)}_{\mu'' \neq 0} c \\ &\in ker\rho \end{aligned}$$

Note that $[h, [x_\beta, y_\beta]] = 0$ since $[x_\beta, y_\beta] \in \mathring{\mathfrak{h}}$ so we have strictly less non-zero terms. Hence by induction $c \in ker\rho$. □

Remark 2.2.2.
The argument in Proposition 2.2.1 in fact shows something slightly stronger. Namely, c acts trivially on any unfaithful representation.

One of the usual questions about any representation theory is asking what the irreducibles are. Our next aim is to describe a tensor decomposition of finite-dimensional irreducible representations of $\mathfrak{L}(\mathring{\mathfrak{g}})$. The following key representation of $\mathfrak{L}(\mathring{\mathfrak{g}})$ is constructed by pulling back a representation of $\mathring{\mathfrak{g}}$ through an evaluation homomorphism.

Definition 2.2.3. (Evaluation representation).
Let V be a finite-dimensional representation of $\mathring{g}$.
For every $a \in \mathbb{C}^\times$ define the **evaluation homomorphism**

$$\begin{aligned} ev_a : \mathfrak{L}(\mathring{\mathfrak{g}}) &\longrightarrow \mathring{\mathfrak{g}} \\ P(t) \otimes x &\longmapsto P(a)x \end{aligned}$$

Then the **evaluation representation** $V(a)$ of $\mathfrak{L}(\mathring{\mathfrak{g}})$ is given by

$$(P(t) \otimes x).v = ev_a(P(t) \otimes x).v \quad (v \in V)$$

Tensor products of evaluation representations is defined the way tensor representations of Lie algebras are normally defined. Given representations $V_1, \ldots, V_n$ of $\mathring{\mathfrak{g}}$ and $a_1, \ldots, a_n \in \mathbb{C}^\times$ we define the representation $\bigotimes_{i=1}^{n} V_i(a_i)$ of $\mathfrak{L}(\mathring{\mathfrak{g}})$ via

$$(P(t) \otimes x).(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \ldots v_{i-1} \otimes ev_{a_i}(P(t) \otimes x).v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \quad (v_i \in V_i)$$

Every finite-dimensional irreducible representation of $\mathfrak{L}(\mathring{\mathfrak{g}})$ can be decomposed as a tensor product of evaluation representations. This fact was proved in [Rao]. We give a slightly alternative outline for the proof, still based on the main ideas in [Rao], but instead making use of facts from commutative algebra. To arrive at the decomposition we will make use of three mildly generic facts.

Lemma 2.2.4.
Let V be a faithful finite-dimensional completely reducible representation of a finite-dimensional Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is a direct sum of its center and a semisimple Lie algebra (i.e $\mathfrak{g}$ is reductive).

Proof.
(See [Kap Thm 23]). $\square$

Lemma 2.2.5.
Let $\mathfrak{i} \trianglelefteq \mathfrak{L}(\mathring{\mathfrak{g}})$. Then $\mathfrak{i} = I \otimes \mathring{\mathfrak{g}}$ for some ideal $I \trianglelefteq \mathbb{C}[t, t^{-1}]$.

Proof.
(See [Kac Lemma 8.6]). $\square$

Lemma 2.2.6.
Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. Then every irreducible finite-dimensional representation of $\bigoplus_{i=1}^{k} \mathfrak{g}$ is isomorphic to $\bigotimes_{i=1}^{k} V_i$ where V_i is an irreducible finite-dimensional representation of $\mathfrak{g}$ for $i = 1, \dots, k$.

Proof.
(See [Bou §7, no.7]). $\square$

Theorem 2.2.7. *(Tensor decomposition).*
Let V be a finite-dimensional irreducible representation of $\mathfrak{L}(\mathring{\mathfrak{g}})$.
Then

$$V \cong \bigotimes_{i=1}^{n} V_i(a_i)$$

where $V_1, \dots, V_n$ are irreducible representations of $\mathring{\mathfrak{g}}$ and $a_1, \dots, a_n \in \mathbb{C}^\times$ are pairwise distinct.

Proof.
Let $\rho : \mathfrak{L}(\mathring{\mathfrak{g}}) \longrightarrow End(V)$ be a finite-dimensional irreducible representation. Then $\mathfrak{L}(\mathring{\mathfrak{g}})/ker\rho$ is a finite-dimensional Lie algebra. Moreover ρ descends to an irreducible representation of $\mathfrak{L}(\mathring{\mathfrak{g}})/ker\rho$ which is necessarily faithful. Thus by Lemma 2.2.4, $\mathfrak{L}(\mathring{\mathfrak{g}})/ker\rho$ is a direct sum of its center and a semisimple Lie algebra. Hence $[\mathfrak{L}(\mathring{\mathfrak{g}})/ker\rho, \mathfrak{L}(\mathring{\mathfrak{g}})/ker\rho] = \mathfrak{L}(\mathring{\mathfrak{g}})/ker\rho$ is semisimple.
By Lemma 2.2.5, $ker\rho = I \otimes \mathring{\mathfrak{g}}$ for some ideal $I \trianglelefteq \mathbb{C}[t, t^{-1}]$. We claim I is a radical ideal. Let $f \in \sqrt{I}$. Then there exists $n \in \mathbb{N}$ such that $f^n \in I$. Let $x \in \mathring{\mathfrak{g}}$ such that $x \neq 0$ and consider the ideal $\mathfrak{i} = \langle f \otimes x \rangle$. Then $\mathcal{D}^{(n)}\mathfrak{i} \subseteq I \otimes \mathring{\mathfrak{g}} = ker\rho$. Therefore $(\mathfrak{i} + ker\rho)/ker\rho$ is soluble and so is contained in the (unique) maximal soluble ideal $Rad(\mathfrak{L}(\mathring{\mathfrak{g}})/ker\rho)$. But $\mathfrak{L}(\mathring{\mathfrak{g}})/ker\rho$ is semisimple, so by definition

$Rad(\mathfrak{L}(\mathring{\mathfrak{g}})/ker\rho) = 0$. Thus $\mathfrak{i} \subseteq ker\rho$. In particular $f \otimes x \in ker\rho = I \otimes \mathring{\mathfrak{g}}$ so $f \in I$. Therefore $\sqrt{I} \subseteq I$. Hence $\sqrt{I} = I$ and so I is a radical ideal.
Since I is radical and $\mathbb{C}[t, t^{-1}]$ is Noetherian, I is an intersection of finitely many prime ideals by a standard fact from commutative algebra. But prime ideals in $\mathbb{C}[t, t^{-1}]$ are generated by linear polynomials so

$$I = \left(\prod_{r=1}^{k}(t - a_r)\right) \quad \text{for some pairwise distinct } a_1, \ldots, a_k \in \mathbb{C}^{\times}.$$

By Chinese Remainder theorem we have

$$\mathbb{C}[t, t^{-1}] \Big/ \left(\prod_{r=1}^{k}(t - a_r)\right) \cong \bigoplus_{r=1}^{k} \mathbb{C}[t, t^{-1}]/(t - a_r).$$

Hence it follows that

$$\mathfrak{L}(\mathring{\mathfrak{g}})/ker\rho \cong \bigoplus_{r=1}^{k}(\mathbb{C}[t, t^{-1}]/(t - a_r)) \otimes \mathring{\mathfrak{g}}.$$

We now have a clear isomorphism

$$\begin{aligned} \Psi : \bigoplus_{r=1}^{k}(\mathbb{C}[t, t^{-1}]/(t - a_r)) \otimes \mathring{\mathfrak{g}} &\longrightarrow \bigoplus_{r=1}^{k} \mathring{\mathfrak{g}} \\ (f_1(a_1) \otimes x_1, \ldots, f_k(a_k) \otimes x_k) &\longmapsto (f_1(a_1)x_1, \ldots, f_k(a_k)x_k). \end{aligned}$$

By Lemma 2.2.7 it follows that the finite-dimensional irreducible representations of $\mathfrak{L}(\mathring{\mathfrak{g}})/ker\rho$ are given by

$$\mathfrak{L}(\mathring{\mathfrak{g}})/ker\rho \xrightarrow{\cong} \bigoplus_{r=1}^{k}(\mathbb{C}[t, t^{-1}]/(t - a_r)) \otimes \mathring{\mathfrak{g}} \xrightarrow{\Psi} \bigoplus_{r=1}^{k} \mathring{\mathfrak{g}} \xrightarrow{\bigotimes_{r=1}^{k} \rho_r} End\left(\bigotimes_{r=1}^{k} V_i\right)$$

where $\rho_i : \mathring{\mathfrak{g}} \longrightarrow End(V_i)$ is a finite-dimensional irreducible representation of $\mathring{\mathfrak{g}}$ for $i = 1, \ldots, k$. The finite-dimensional irreducible representations of $\mathfrak{L}(\mathring{\mathfrak{g}})/ker\rho$ are each lifted in correspondence with finite-dimensional irreducible representations of $\mathfrak{L}(\mathring{\mathfrak{g}})$ equipped with the evaluation action such that the restriction to $ker\rho = I \otimes \mathring{\mathfrak{g}}$ is zero (as given by evaluation in polynomials divisible by $\prod_{i=1}^{k}(t - a_i)$).

□

We will now look closer at more general finite-dimensional representations of $\mathfrak{L}.(\mathring{\mathfrak{g}})$.

Definition 2.2.8.
Let $\mathcal{F}$ denote the category of finite dimensional representations of $\mathfrak{L}(\mathring{\mathfrak{g}})$.

Since $\mathfrak{L}(\mathring{\mathfrak{g}})$ is neither finite-dimensional nor semisimple we cannot apply Weyl's theorem. In fact the category $\mathcal{F}$ is not semisimple [CP2]. This means one has the added complication of indecomposables which are not irreducible (see Weyl modules below). In particular it is not enough to know the irreducibles to understand the category $\mathcal{F}$. Since the loop algebra is closely dependent on $\mathring{\mathfrak{g}}$ one would still like to lift as much as possible of the representation theory of $\mathring{\mathfrak{g}}$ to the loop algebra. In particular one would like an analogue of highest weight module, which is a key concept in semisimple theory. The definition of a highest weight module in the semisimple case depended only on the triangular decomposition. The triangular decomposition

$$\mathring{\mathfrak{g}} = \mathring{\mathfrak{n}}_- \oplus \mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}}_+$$

induces a decomposition

$$\mathfrak{L}(\mathring{\mathfrak{g}}) = \mathfrak{L}(\mathring{\mathfrak{n}}_-) \oplus \mathfrak{L}(\mathring{\mathfrak{h}}) \oplus \mathfrak{L}(\mathring{\mathfrak{n}}_+)$$

which may be used to make an almost parallel definition ([Sen]):

Definition 2.2.9. (Highest weight module).
Let V be an $\mathfrak{L}(\mathring{\mathfrak{g}})$-module and $\lambda \in \mathring{\mathfrak{h}}^*$. We say V is a **highest weight module**, with highest weight λ, if there exists $v \in V$ with $U(\mathfrak{L}(\mathring{\mathfrak{g}})).v = V$ such that

$$\mathfrak{L}(\mathring{\mathfrak{n}}_+).v = 0, \qquad h.v = \lambda(h)v \text{ for all } h \in \mathfrak{L}(\mathring{\mathfrak{h}})$$

Here $U(\mathfrak{L}(\mathring{\mathfrak{g}}))$ is the *universal enveloping algebra* of $\mathfrak{L}(\mathring{\mathfrak{g}})$ (see Appendix B).

The category $\mathcal{F}$ has maximal indecomposable highest weight modules called *Weyl modules*. They are maximal in the sense that every object in $\mathcal{F}$ is projected uniquely onto by a Weyl module (up to isomorphism). Weyl modules therefore correspond to isomorphism classes of representations in $\mathcal{F}$. Hence these modules fulfill a similar role to the Verma module in semisimple theory, except they are always finite-dimensional and parametrized by n-tuples of polynomials with constant coefficient 1. Such n-tuples are also known as *Drinfeld polynomials*. We will spend the rest of this section introducing language to state these facts.

Definition 2.2.10. (The monoid $\mathcal{P}^+$ of Drinfeld polynomials).
Let $\mathcal{P}^+$ be the set of all n-tuples of polynomials $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n) \in (\mathbb{C}[u])^n$ such that $\pi_i(0) = 1$ for $i = 1, \ldots, n$. Here we take $n = rank(\mathring{\mathfrak{g}})$ i.e the dimension of $\mathring{\mathfrak{h}}$. The n-tuples $\boldsymbol{\pi}$ are called **Drinfeld polynomials**. The set $\mathcal{P}^+$ becomes a monoid under pointwise multiplication:

$$(\pi_1, \ldots, \pi_n)(\pi_1', \ldots, \pi_n') = (\pi_1\pi_1', \ldots, \pi_n\pi_n').$$

We introduce some further notation. For $a \in \mathbb{C}^\times$ and $i = 1, \ldots, n$ set

$$\boldsymbol{\pi}_{i,a} = ((1 - au)^{\delta_{ij}})_{1 \le j \le n} \in \mathcal{P}.$$

Recall that we may associate to $\mathring{\mathfrak{g}}$ a set of **fundamental weights**

$$w_1, \ldots, w_n \in \mathring{\mathfrak{h}}^* \text{ satisfying } w_i(\alpha_j^\vee) = \delta_{ij}.$$

We also define the **positive weight lattice**

$$\bigwedge\nolimits_W^+ = \bigoplus_{i=1}^n \mathbb{Z}_+ w_i.$$

Now for $\lambda \in \bigwedge_W^+$, $\lambda \neq 0$, set

$$\boldsymbol{\pi}_{\lambda,a} = \prod_{i=1}^n \boldsymbol{\pi}_{i,a}^{\lambda(\alpha_i^\vee)}.$$

Since every Drinfeld polynomial is an n-tuple of polynomials with constant coefficient 1 and every polynomial splits uniquely as a product of linear factors over $\mathbb{C}$, the constant term in each linear factor must be 1. We can therefore uniquely write every Drinfeld polynomial as a product of elements of the form $\boldsymbol{\pi}_{i,a}$ (up to reordering of terms). In terms of our compact notation $\boldsymbol{\pi}_{\lambda,a}$ (which simply picks out the multiplicity of the linear factor for each entry in the tuple), we get for every $\boldsymbol{\pi}^+ \in \mathcal{P}^+$ a unique decomposition

$$\boldsymbol{\pi}^+ = \prod_{k=1}^{l} \boldsymbol{\pi}_{\lambda_k,a_k}$$

for some $\lambda_1, \ldots, \lambda_l \in \bigwedge_W^+$ and pairwise distinct elements $a_1, \ldots, a_l \in \mathbb{C}^\times$.
We also associate to each $\boldsymbol{\pi}^+$ an element

$$\boldsymbol{\pi}^- = \prod_{k=1}^{l} \boldsymbol{\pi}_{\lambda_k,a_k^{-1}}.$$

Moreover we have a map

$$\begin{aligned} \lambda : \mathcal{P}^+ &\longrightarrow \wedge_W^+ \\ \boldsymbol{\pi}^+ &\longmapsto \lambda_{\boldsymbol{\pi}^+} := \sum_{i=1}^{n} deg(\pi_i) w_i. \end{aligned}$$

We can now define the Weyl module ([CFS]):

Definition 2.2.11. (Weyl module).
Let $\boldsymbol{\pi}^+ = (\pi_1^+, \ldots, \pi_n^+) \in \mathcal{P}^+$. Then the **Weyl module** $W(\boldsymbol{\pi}^+)$ is the $U(\mathfrak{L}(\mathring{\mathfrak{g}}))$-module generated by an element $w_{\boldsymbol{\pi}^+}$ with action:

$$\mathfrak{L}(\mathring{\mathfrak{n}}^+).w_{\boldsymbol{\pi}^+} = 0, \qquad h.w_{\boldsymbol{\pi}^+} = \lambda_{\boldsymbol{\pi}^+}(h) w_{\boldsymbol{\pi}^+}, \qquad (p_i^\pm - \pi_i^\pm(u)).w_{\boldsymbol{\pi}^+} = 0,$$

$$\left(\sum_{k=0}^{\infty} (f_i \otimes t^k) u^{k+1} \right)^{\lambda_{\boldsymbol{\pi}^+}(\alpha_i^\vee)+1} .w_{\boldsymbol{\pi}^+} = 0$$

where $h \in \mathring{\mathfrak{h}}$, $p_i^\pm = exp\left(-\sum_{k=1}^{\infty} \frac{\alpha_i^\vee \otimes t^{\pm k}}{k} u^k \right)$ and $i = 1, \ldots, n$.

The top three relations ensure $W(\boldsymbol{\pi})$ is a highest weight module. The bottom relation ensures $\mathfrak{L}(\mathring{\mathfrak{n}}^-)$ acts (locally) nilpotently on $W(\boldsymbol{\pi})$.

We end this section with a few important facts about Weyl modules.

Theorem 2.2.12. *($W(\boldsymbol{\pi}^+)$ is finite-dimensional).*
$dim\, W(\boldsymbol{\pi}^+) < \infty$ for all $\boldsymbol{\pi}^+ \in \mathcal{P}^+$.

Proof. (See [CP2]). □

Theorem 2.2.13. *($W(\boldsymbol{\pi}^+)$ is maximal with unique irreducible quotient).*
Let V be a highest weight module in $\mathcal{F}$ generated by v. Then there exists a unique $\boldsymbol{\pi}^+ \in \mathcal{P}^+$ such that V is a $\mathfrak{L}(\mathring{\mathfrak{g}})$-module quotient of $W(\boldsymbol{\pi}^+)$. Moreover $W(\boldsymbol{\pi}^+)$ has a unique irreducible $\mathfrak{L}(\mathring{\mathfrak{g}})$-module quotient $V(\boldsymbol{\pi}^+)$.

Proof. (See [CP2]) $\square$

Remark 2.2.14.
If $V \in \mathcal{F}$ is irreducible then by Theorem 2.2.7 it is a tensor product of irreducible evaluation representations. Irreducible evaluation representations of $\mathfrak{L}(\mathring{\mathfrak{g}})$ are irreducible as representations of the finite-dimensional simple Lie algebra $\mathring{\mathfrak{g}}$. Irreducible representations of finite-dimensional simple Lie algebras are known to be highest weight. Hence due to the induced action it follows that V is highest weight as a representation of $\mathfrak{L}(\mathring{\mathfrak{g}})$. Therefore by Theorem 2.2.13 there exists $\boldsymbol{\pi}^+ \in \mathcal{P}^+$ such that V is a quotient of $W(\boldsymbol{\pi}^+)$. Moreover V must be the unique irreducible quotient of $W(\boldsymbol{\pi}^+)$ by the second statement in Theorem 2.2.13. Therefore each irreducible V precisely corresponds to a unique $\boldsymbol{\pi}^+ \in \mathcal{P}^+$ giving a bijective parametrization of finite-dimensional irreducible representations by Drinfeld polynomials:

$$\begin{tikzcd} \mathcal{P}^+ \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & W(\boldsymbol{\pi}^+) \arrow[d, twoheadrightarrow] \\ & V(\boldsymbol{\pi}^+) \end{tikzcd}$$

Chapter 3

Quantum Affine Algebras

In this chapter we move up a step further in generalization from the object we started with, namely the finite-dimensional simple Lie algebra. In the first chapter we generalized simple Lie algebras by varying the Cartan matrix over a larger class of matrices. In the second chapter we described a procedure for affinizing the simple Lie algebra through a central extension of its loop algebra. We will here describe yet another way of generalizing (the universal enveloping algebra of) a simple Lie algebra through a so called q-deformation (or 'quantization') of its Chevalley-Serre presentation. We will then combine affinization with quantization to arrive at the quantum affine algebra. A q-deformation is roughly speaking the process of adding a formal parameter q from which a 'classical' object is recovered in the limit as $q \longrightarrow 1$. Many familiar mathematical objects admit a q-deformation, none the least the natural numbers! Simple finite-dimensional Lie algebras themselves appeared too rigid to admit a non-trivial deformation. However in 1985 it was astonishingly shown by Drinfeld[Dri] and Jimbo[Ji1] that the universal enveloping algebra $U(\mathfrak{g})$ of a simple Lie algebra (and more generally any symmetrizable Kac-Moody algebra) admits a q-deformation in the context of being Hopf-algebras. Generally speaking an object tends to lose its most characteristic features under deformation but the authors showed the axioms of a Hopf-algebra remain preserved and yields $U(\mathfrak{g})$ as $q \longrightarrow 1$. Hopf algebras (as opposed to general algebras) are better behaved because they have more manageable representation theory arising mainly from the fact that they allow for well-defined tensor and dual representations.
There are essentially two approaches one could take to define quantum affine algebras. We could either affinize and then quantize, which leads to Drinfeld's "new" realization or we could swap the process and be led to Drinfeld-Jimbo's standard q-analogue presentation. We will focus on the former in this chapter because it gives a better outlook for studying the finite-dimensional representation theory. In particular we will be keen to find q-analogues of notions studied in the previous chapter.

3.1 Quantized enveloping algebras

This section serves as a brief introduction to 'quantization', 'q-deformation' or 'q-analogue' (all here meaning the same thing) with emphasis on universal enveloping algebras (see Appendix B). In literature it is seemingly common practise to use the word quantization when q-deforming a Hopf algebra to distinguish it from other kinds of q-objects. The practise of finding q-analogues of classical functions/objects dates back to Euler and Gauss. Euler considered e.g q-analogues of the derivative as well as the partition function $p(n) = \#\text{partitions of } n$.

$$P(q) = \sum_{n=0}^{\infty} p(n)q^n = \prod_{k=0}^{\infty}(1 - q^{k+1})^{-1} \qquad \text{(q-partition function)}$$

$$D_q f(x) = \frac{f(qx) - f(x)}{x(q-1)} \qquad \text{(q-derivative)}$$

Since then mathematicians have tried to q-deform almost anything they could get their hands on. As described in the chapter introduction, constructing a q-analogue of a 'classical' object involves finding a family of objects that are similar (not meaning isomorphic) and depending on a formal parameter q which could be interpreted as a complex number, or more universally an indeterminate in the field of rational functions $\mathbb{C}(q)$. One requires that this family 'tend' to the classical object as $q \longrightarrow 1$ and importantly belong to the same underlying category as the classical object. Unless otherwise stated we assume over course of the chapter that

$$q \in \mathbb{C} \setminus \{0\} \textit{ is not a root of unity}$$

There is more subtlety involved in the root of unit case for reasons that will become apparent. General motivation for studying q-deformations is to understand how structures behave under perturbation. Clearly there can be geometrical motivations for doing so e.g in the study of manifolds, but they are also interesting to study in their own right because they provide new insight into the object under deformation. The q-analogues of objects such as universal enveloping algebras also have vast applications in physics (which is how the word 'quantum' enters the literature).

We begin by defining a q-analogue of the most fundamental of objects, namely the natural numbers. In fact a whole theory of q-arithmetic can be developed on top of it.

Definition 3.1.1. (q-numbers, q-factorials and q-binomial coefficients).
Let $q \in \mathbb{C} \setminus \{\pm 1\}$.
Define the **q-number** $[n]_q$ by

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

the **q-factorial** $[n]_q!$ by

$$[n]_q! = [n]_q[n-1]_q \dots [2]_q[1]_q,$$

and the **q-binomial coefficient** $\begin{bmatrix} n \\ r \end{bmatrix}_q$ by

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q![n-r]_q!}$$

Note that $[n]_q \longrightarrow n$ as $q \longrightarrow 1$ via e.g l'Hopital's rule.
Therefore $[n]_q! \longrightarrow n!$ and $\begin{bmatrix} n \\ r \end{bmatrix}_q \longrightarrow \binom{n}{r}$ as $q \longrightarrow 1$ as required

Before we consider the more esoteric *quantum affine* algebra we consider the q-analogue $U_q(\mathfrak{g})$ of the universal enveloping algebra of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$. Guided by what we know from semisimple theory the first object we should look to understand is $U_q(\mathfrak{sl}_2)$. Indeed in analogy with semisimple Lie algebras, it will turn out to be the prototype for the general construction also in the quantized setting.

Recall that $U(\mathfrak{sl}_2)$ is the algebra generated by E, H, F subject to

$$HE - EH = [H, E] = 2E, \quad HF - FH = [H, F] = -2F$$
$$EF - FE = [E, F] = H$$

(cf Appendix B).
Note that the formal multiplication symbol $\otimes$ on the left hand side has been omitted for brevity.

Recall that $U(\mathfrak{sl}_2)$ is a Hopf algebra (cf Appendix A).
The standard presentation of $U_q(\mathfrak{sl}_2)$ is given by:

Definition 3.1.2. (Presentation of $U_q(\mathfrak{sl}_2)$).
$U_q(\mathfrak{sl}_2)$ is the algebra generated by E, F, K, K^{-1} subject to

$$KK^{-1} = 1 = K^{-1}K$$
$$KE = q^2 EK$$
$$FK = q^2 KF$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

with co-multiplication, co-unit and antipode given by

$$\begin{aligned} &\Delta(K) = K \otimes K, && \epsilon(K) = 1, && \mathcal{S}(K) = K^{-1}, \\ &\Delta(E) = E \otimes 1 + K \otimes E, && \epsilon(E) = 0, && \mathcal{S}(E) = -K^{-1}E, \\ &\Delta(F) = F \otimes K^{-1} + 1 \otimes F, && \epsilon(F) = 0, && \mathcal{S}(F) = -FK \end{aligned}$$

It is not immediately clear what motivates this presentation. In particular the $q = 1$ specialization is not well-defined. Although the presentation can be modified to recover a structure isomorphic to $U(\mathfrak{sl}_2)$ at $q = 1$, we will offer a more intuitive explanation for the presentation at hand. The justification we give is strictly speaking informal but it does turn out to yield the right intuition. We remark that deforming $U(\mathfrak{sl}_2)$ in *some way* is easy since one can trivially devise a deformation with required classical limit that is merely a family of algebras. The challenge lies in finding a deformation which maintains the additional structure of a Hopf algebra (which is the main reason we can study its representation theory). Although it is straightforward to check the Hopf axioms given above definition, it is in general non-trivial that these assignments exist in the first place.

The idea behind this presentation is to deform the relation

$$EF - FE = H$$

by setting

$$EF - FE = [H]_q = \tfrac{q^H - q^{-H}}{q - q^{-1}}.$$

Although q^H does not make sense (and will be replaced by K) we could define it as a formal power series in terms of the exponential map

$$q^H = exp(log(q)H) = \sum_{n=0}^{\infty} \frac{log(q)^n}{n!} H^n.$$

Evaluating the limit of the summand using e.g l'Hopital rule we have

$$[H]_q = \tfrac{q^H - q^{-H}}{q - q^{-1}} = \sum_{n=0}^{\infty} \frac{log(q)^n - log(q)^{-n}}{(q - q^{-1})n!} H^n \longrightarrow H \text{ as } q \longrightarrow 1 \ .$$

By rewriting the relation $HE - EH = 2E$ as $HE = E(H + 2)$ we have by induction that

$$H^n E = E(H + 2)^n.$$

Therefore

$$\begin{aligned} q^H E &= \sum_{n=0}^{\infty} \frac{log(q)^n}{n!} H^n E = E \sum_{n=0}^{\infty} \frac{log(q)^n}{n!} (H + 2)^n \\ &= E\, exp(log(q)(H + 2)) = E\, exp(log(q)2)\, exp(log(q)H) = q^2 E q^H. \end{aligned}$$

Taking $K = q^H$ gives the relations in the presentation of $U_q(\mathfrak{sl}_2)$.

In the same way that copies of $\mathfrak{sl}_2$ were glued together to yield a presentation of an arbitrary semisimple Lie algebra, one can glue together copies of $U_q(\mathfrak{sl}_2)$ to obtain a presentation for $U_q(\mathfrak{g})$, which remarkably, is still consistent with the Hopf algebra structure.

Definition 3.1.3. (Presentation of $U_q(\mathfrak{g})$).
Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra with Cartan matrix (A_{ij}). Suppose $\mathfrak{g}$ has simple roots $\alpha_1, \ldots, \alpha_n$.
Then $U_q(\mathfrak{g})$ is the algebra generated by E_i, F_i, K_i, K_i^{-1} $(i = 1, \ldots, n)$ satisfying the following relations:

$$\begin{aligned} &K_i K_i^{-1} = K_i^{-1} K_i = 1, && K_i K_j = K_j K_i, \\ &K_i E_j K_i^{-1} = q_i^{A_{ij}} E_j, && K_i F_j K_i^{-1} = q_i^{-A_{ij}} F_j, \\ &E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i}, && \end{aligned}$$

$$\sum_{k=0}^{1-A_{ij}} (-1)^k \begin{bmatrix} 1-A_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-A_{ij}-k} = 0 \quad (i \neq j),$$

$$\sum_{k=0}^{1-A_{ij}} (-1)^k \begin{bmatrix} 1-A_{ij} \\ k \end{bmatrix}_{q_i} F_i^k F_j F_i^{1-A_{ij}-k} = 0 \quad (i \neq j),$$

where $q_i = q^{d_i}$, $d_i = (\alpha_i, \alpha_i)/2 \quad (i = 1, \dots, n)$
with co-multiplication, co-unit and antipode given by

$$\begin{aligned}
&\Delta(K_i) = K_i \otimes K_i, && \epsilon(K_i) = 1, && \mathcal{S}(K_i) = K_i^{-1} \\
&\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, && \epsilon(E_i) = 0, && \mathcal{S}(E_i) = -K_i^{-1}E_i, \\
&\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, && \epsilon(F_i) = 0, && \mathcal{S}(F_i) = -F_i K_i
\end{aligned}$$

Remark 3.1.4.
The above presentation is mainly dependent on the existence of a non-degenerate bilinear form so we don't have to restrict ourselves to finite-dimensional semisimple Lie algebras. The presentation works likewise for symmetrizable Kac-Moody algebras. However in such case one sometimes work with a slightly larger presentation containing generators K_h for all $h \in \mathfrak{h}$, in analogy to how we extended the original Chevalley-Serre presentation to Kac-Moody algebras. One then must impose the relation

$$K_0 = 1, \;\; K_a K_b = K_{a+b} \text{ for all } a, b \in \mathfrak{h}.$$

and replace the top relations by

$$K_h E_j K_{-h} = q^{\alpha_j(h)} E_j \text{ and } K_h E_j K_{-h} = q^{-\alpha_j(h)} E_j$$

while making corresponding changes to the Hopf algebra assignments.
Drinfeld and Jimbos original presentation was given in this generality.
We will therefore refer to this presentation as the **Drinfeld-Jimbo presentation**. In the quantum affine setting this generalized view of the presentation corresponds to the operations "quantization $\longrightarrow$ affinization" by first quantizing the relations of $U(\mathfrak{g})$ and then allow for generalized (affine) Cartan matrices. The sums are called **q-Serre** relations. Indeed their classical limit is given by

$$\begin{aligned}
\lim_{q \to 1} \sum_{k=0}^{1-A_{ij}} (-1)^k \begin{bmatrix} 1-A_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-A_{ij}-k} &= \sum_{k=0}^{1-A_{ij}} (-1)^k \binom{1-A_{ij}}{k} E_i^k E_j E_i^{1-A_{ij}-k} \\
&= (ad\, E_i)^{1-A_{ij}}(E_j)
\end{aligned}$$

The last identity follows easily by induction on the exponent. Finally notice that the presentation for $U_{q_i}(\mathfrak{sl}_2)$ is recovered if one restricts to the subalgebras generated by E_i, F_i, K_i, K_i^{-1} for $i = 1, \dots, n$.

We end this section by briefly mentioning a few results concerning the representation theory of $U_q(\mathfrak{g})$ for $\mathfrak{g}$ *finite-dimensional and semisimple.* As is well-known from the theory of semisimple Lie algebras, finite-dimensional irreducible representations are parametrized by highest weight. For $U_q(\mathfrak{g})$ this continues to hold, albeit with minor modification in that we must add a new parameter σ 'twisting' each weight component by a sign. The finite-dimensional representation theories of $U_q(\mathfrak{g})$ and $\mathfrak{g}$ are therefore very similar. In particular one has a complete reducibility result in analogy with Weyl's theorem. We begin by defining the quantum analogue of weight spaces.

Definition 3.1.5. (Weight).
A **weight** is an n-tuple $\mathbf{w} = (w_1, \ldots, w_n)$ where $w_i \in \mathbb{C} \setminus \{0\}$.

Definition 3.1.6. (Weight space).
Let V be a $U_q(\mathfrak{g})$-module and $\mathbf{w} = (w_1, \ldots, w_n)$ a weight.
Then the **weight space** $V_{\mathbf{w}}$ is given by

$$V_{\mathbf{w}} = \{v \in V : K_i.v = w_i v,\ i = 1, \ldots, n\}$$

We get a partial ordering on the weights via $\mathbf{w}' \leq \mathbf{w}$ iff $w_i'^{-1} w_i = q^{(\alpha_i, \beta)}$ for some $\beta \in \bigwedge_{\Pi}^{+}$. Notions such as highest weight and integrable module then carry over in immediate fashion.

Proposition 3.1.7.
Every finite-dimensional highest weight $U_q(\mathfrak{g})$-module is irreducible.

Proof. (See [CP3] Cor 10.1.6). □

Proposition 3.1.8.
Every finite-dimensional irreducible $U_q(\mathfrak{g})$-module is highest weight and integrable.

Proof. (See [CP3] Prop 10.1.2). □

Proposition 3.1.9.
The irreducible $U_q(\mathfrak{g})$-module $V_q(\mathbf{w})$ with highest weight $\mathbf{w}$ is integrable if and only if $\mathbf{w} = \mathbf{w}_{\sigma,\lambda}$ for some weight $\lambda \in \bigwedge_W^{+}$ and homomorphism $\sigma : \bigwedge_{\Pi} \longrightarrow \{\pm 1\}$ where $\mathbf{w}_{\sigma,\lambda} = (w_i)$, $w_i = \sigma(\alpha_i) q^{(\alpha_i, \lambda)}$ and $\alpha_1, \ldots, \alpha_n$ simple roots of $\mathfrak{g}$ for $i = 1, \ldots, n$.

Proof. (See [CP3] Prop 10.1.1). □

Combining the last two propositions gives us a complete description of the highest weight parametrization of finite-dimensional irreducible $U_q(\mathfrak{g})$-modules. In contrast to the semisimple case, the presence of the parameter σ implies there can be multiple irreducible highest weight modules associated with a given highest weight λ. In fact there are 2^n choices for σ as determined by where the fundamental weights go.

Example 3.1.10. ($U_q(\mathfrak{sl}_2)$-modules).
There are exactly two irreducible finite-dimensional $U_q(\mathfrak{sl}_2)$-modules up to isomorphism for each weight λ. They are parametrized by $\sigma = \pm 1$. For each $U_q(\mathfrak{sl}_2)$-module there exists a basis $\{v_1, \ldots, v_l\}$ on which the action of $U_q(\mathfrak{sl}_2)$ is given by

$$K_1.v_i = \sigma q^{l-2i} v_i \quad E_1.v_i = \sigma [l-i+1]_q v_{i-1} \quad F_1.v_i = [i+1]_q v_{i+1}$$

For fixed σ we also get an induced partial ordering $\mathbf{w}_{\sigma,\lambda'} \leq \mathbf{w}_{\sigma,\lambda}$ iff $\lambda' \leq \lambda$. Like mentioned previously there is also an important reducibility result for $U_q(\mathfrak{g})$ which implies it is enough to understand the finite-dimensional irreducible modules to understand *all* finite-dimensional modules.

Theorem 3.1.11.
Every finite-dimensional $U_q(\mathfrak{g})$-module is completely reducible.

Proof. (See [CP3] Thm 10.1.7). □

3.2 Drinfeld presentation

In the previous section we defined a presentation of $U_q(\mathfrak{g})$ which in the correct context corresponded to the procedure "quantization $\longrightarrow$ affinization". We now consider a different presentation due to Drinfeld corresponding to the opposite sequence "affinization $\longrightarrow$ quantization". By affinization of a simple finite-dimensional Lie algebra $\mathfrak{g}$ we take the following meaning

$$\hat{\mathfrak{g}} = \mathfrak{L}(\mathfrak{g}) \oplus \mathbb{C}c$$

where the construction is defined as in §2.1.

Definition 3.2.1. (Quantum affine algebra).
Let $\mathfrak{g}$ be a simple Lie algebra. Then $U_q(\hat{\mathfrak{g}})$ is called the **quantum affine algebra**.

In some sense the presentation we gave for $U_q(\mathfrak{g})$ in Definition 3.1.3 is too generic since we merely wish to quantize $U(\hat{\mathfrak{g}})$. We are also interested in finding q-analogues for the representation theory of $U(\hat{\mathfrak{g}})$. It is worth noting that $U(\hat{\mathfrak{g}})$ has representation theory equivalent to that of $\hat{\mathfrak{g}}$ (see Appendix A). In particular we look for a corresponding notion of highest weight representation. Since $\mathfrak{L}(\mathfrak{n}_+)$ and $\mathfrak{L}(\mathfrak{h})$ have no immediate description in terms of the standard generators for $\hat{\mathfrak{g}}$ it is not obvious how the highest weight concept quantizes to the entire family of $U_q(\hat{\mathfrak{g}})$ algebras. Nevertheless Drinfeld found a new realization in terms of q-analogues of the $\mathfrak{L}(\mathfrak{g})$-generators

$$\{e_i \otimes t^j, f_i \otimes t^j, h_i \otimes t^j \ : \ i = 1, \ldots, n, \ j \in \mathbb{Z}\}$$

in which one can assign a meaning to 'highest weight representation'.

Theorem 3.2.2. *(Drinfeld's presentation of $U_q(\hat{\mathfrak{g}})$).*
Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra with Cartan matrix (A_{ij}). Suppose $\mathfrak{g}$ has simple roots $\alpha_1, \ldots \alpha_n$.
Then $U_q(\hat{\mathfrak{g}})$ can be realized as the algebra generated by

$$x^{\pm}_{i,m},\ h_{i,r},\ K_i^{\pm 1},\ c^{\pm 1/2} \qquad (m \in \mathbb{Z},\ r \in \mathbb{Z} \setminus \{0\},\ i = 1, \ldots, n)$$

with relations:

$$
\begin{aligned}
c^{\pm 1/2} &\textit{ are central,} \\
c^{1/2}c^{-1/2} &= c^{-1/2}c^{1/2} = 1, \\
K_iK_i^{-1} &= K_i^{-1}K_i = 1, \\
K_iK_j &= K_jK_i, \\
K_ih_{j,r} &= h_{j,r}K_i, \\
K_ix_{j,m}^{\pm}K_i^{-1} &= q_i^{\pm A_{ij}}x_{j,m}^{\pm}, \\
[h_{i,r}, h_{j,s}] &= \delta_{r,-s}\frac{1}{r}[rA_{ij}]_{q_i}\frac{c^r - c^{-r}}{q_j - q_j^{-1}}, \\
[h_{i,r}, x_{j,m}^{\pm}] &= \pm\frac{1}{r}[rA_{ij}]_{q_i}c^{\mp|r|/2}x_{j,r+m}^{\pm}, \\
x_{i,m+1}^{\pm}x_{j,s}^{\pm} - q_i^{\pm A_{ij}}x_{j,s}^{\pm}x_{i,m+1}^{\pm} &= q_i^{\pm A_{ij}}x_{i,m}^{\pm}x_{j,s+1}^{\pm} - x_{j,s+1}^{\pm}x_{i,m}^{\pm}, \\
[x_{i,m}^{+}, x_{j,s}^{-}] &= \delta_{ij}\frac{c^{(r-s)/2}\phi_{i,m+s}^{+} - c^{-(r-s)/2}\phi_{i,m+s}^{-}}{q_i - q_i^{-1}},
\end{aligned}
$$

and for $i \neq j$

$$\sum_{\pi \in \sum_a}\sum_{k=0}^{a}(-1)^k\begin{bmatrix} a \\ k \end{bmatrix}_{q_i} x_{i,m_{\pi(1)}}^{\pm}\dots x_{i,m_{\pi(k)}}^{\pm}x_{j,s}^{\pm}x_{i,m_{\pi(k+1)}}^{\pm}\dots x_{i,m_{\pi(1-A_{ij})}}^{\pm} = 0,$$

for all sequences of integers $m_1, \dots, m_a$ where $a = 1 - A_{ij}$ and $\sum_a$ is the symmetric group on a letters, $q_i = q^{d_i}$, $d_i = (\alpha_i, \alpha_i)/2$ and $\phi_{i,m}^{\pm}$ determined by equating powers of z in the formal power series

$$\sum_{m=0}^{\infty}\phi_{i,\pm m}^{\pm}z^{\pm m} = K_i^{\pm 1}exp\left(\pm(q_i - q_i^{-1})\sum_{r=1}^{\infty}h_{i,\pm r}z^{\pm r}\right).$$

Let $\theta = \sum_{i=1}^{n} m_i\alpha_i$ be the highest root of $\mathfrak{g}$, set $q_\theta = q_i$ if θ is Weyl group conjugate to α_i and set $K_\theta = \prod_{i=1}^{n} K_i^{m_i}$. Suppose the root vector $\bar{E}_\theta$ of $\mathfrak{g}$ is expressed in terms of simple root vectors as

$$\bar{E}_\theta = \lambda[\bar{E}_{i_1}, [\bar{E}_{i_2}, [\dots, [\bar{E}_{i_k}, \bar{E}_j]\dots]]$$

for some $\lambda \in \mathbb{C}$.
Define maps

$$
\begin{aligned}
w_i^{\pm} : U_q(\hat{\mathfrak{g}}) &\longrightarrow U_q(\hat{\mathfrak{g}}) \\
a &\longmapsto x_{i,0}^{\pm}a - K_i^{\pm 1}aK_i^{\mp 1}x_{i,0}^{\pm}
\end{aligned}
$$

Then there exists an isomorphism Ψ between above presentation and the Drinfeld-Jimbo presentation of $U_q(\hat{\mathfrak{g}})$ in Definition 3.1.3 given by

$$
\begin{aligned}
K_0 &\longmapsto cK_\theta^{-1}, & K_i &\longmapsto K_i \\
E_i &\longmapsto x_{i,0}^{+}, & F_i &\longmapsto x_{i,0}^{-} \quad (i = 1, \dots, n) \\
E_0 &\longmapsto \mu w_{i_1}^{-}\dots w_{i_k}^{-}(x_{j,1}^{-})K_\theta^{-1}, & F_0 &\longmapsto \lambda K_\theta w_{i_1}^{+}\dots w_{i_k}^{+}(x_{j,-1}^{+}),
\end{aligned}
$$

where $\mu \in \mathbb{C}$ is determined by the condition

$$[E_0, F_0] = \frac{K_0 - K_0^{-1}}{q_\theta - q_\theta^{-1}}$$

Remark 3.2.3.
The correspondence between the q-analogue generators in the Drinfeld presentation and the generators for $\mathfrak{L}(\mathfrak{g})$ is given by

$$\begin{aligned} e_i \otimes t^j &\longleftrightarrow x^+_{i,j}, & f_i \otimes t^j &\longleftrightarrow x^-_{i,j}, \\ h_i \otimes 1 &\longleftrightarrow K_i, & h_i \otimes t^j &\longleftrightarrow h_{i,j} \quad (j \neq 0) \end{aligned}$$

Notice that in analogy with all previous presentations, this presentation is stringed together by copies of $U_{q_i}(\widehat{\mathfrak{sl}_2})$ $(i = 1, \dots, n)$ as given by the subalgebras

$$\langle x^\pm_{i,m}, h_{i,r}, K_i^{\pm 1}, c^{\pm 1/2} : m \in \mathbb{Z},\ r \in \mathbb{Z} \setminus \{0\} \rangle$$

The third takeaway from this presentation is that it is rather complicated, however its generators can be given a more clear interpretation. Using Drinfelds realization Beck [Bec] proved a q-analogue of the PBW-theorem for $U_q(\hat{\mathfrak{g}})$, in which taking $U_q^\pm, U_q^0$ to be the subalgebras generated by $x^\pm_{i,m}$ and $h_{i,r}, c^{\pm 1/2}, K_i^{\pm 1}$ respectively, yields a decomposition

$$U_q(\hat{\mathfrak{g}}) = U_q^+ U_q^0 U_q^- \quad \text{(as vector spaces)}$$

A q-PBW theorem means we get a family of bases for which the PBW-theorem holds for each $U_q(\hat{\mathfrak{g}})$ and such that the classical PBW-basis for $U(\hat{\mathfrak{g}})$ is recovered at $q = 1$. As we will see in the next section, these subalgebras will give us a way of interpreting the action of $U_q^\pm$ as raising/lowering operators on a vector space and U_q^0 as a kind of "Cartan subalgebra" in a highest weight representation theory of $U_q(\hat{\mathfrak{g}})$.

3.3 Quantum loop algebras

Starting from this section we study the finite-dimensional representation theory of $U_q(\hat{\mathfrak{g}})$. In the previous chapter we described a full parametrization of irreducible finite-dimensional representations of $\hat{\mathfrak{g}}$. In section §3.1 we stated results showing that the representation theory of $U_q(\mathfrak{g})$ is nearly congruent to $\mathfrak{g}$ up to a minor 'twisting'. In contrast, the representation theory of $U_q(\hat{\mathfrak{g}})$ is considerably more complex than either that of $\hat{\mathfrak{g}}$ and $U_q(\mathfrak{g})$ alone. This is to be anticipated since we lose both semisimplicity and finite-dimensionality by first affinizing and then get added twisting complexity by quantization. However several analogies can still be remedied from our previous discussions, although not in as full generality.

We begin by introducing some terminology.

Definition 3.3.1. (Type 1 representation).
A representation V of $U_q(\hat{\mathfrak{g}})$ is said to be of **Type 1** if $c^{1/2}$ acts by 1 and the generators K_i act diagonalizably on V with eigenvalues integer powers of q.

We proved in Proposition 2.2.1 that the finite-dimensional representation theory of $\hat{\mathfrak{g}}$ could be reduced to that of $\mathfrak{L}(\mathfrak{g})$. In the quantum case we do not have a full analogy to this but there is a partial one. The study of finite-dimensional irreducible representations of $U_q(\hat{\mathfrak{g}})$ can be reduced to the study of the **quantum loop algebra** $U_q(\mathfrak{L}(\mathfrak{g})) \cong U_q(\hat{\mathfrak{g}})/\langle c-1\rangle$. This is implied by the following proposition

Proposition 3.3.2. *(Reduction to quantum loop algebra).*
Every finite-dimensional irreducible representation V of $U_q(\hat{\mathfrak{g}})$ can be obtained from a Type 1 representation by twisting with a product of below automorphisms

$$\begin{array}{lllll} (1) & c^{1/2} \longmapsto -c^{1/2}, & x^{\pm}_{i,m} \longmapsto (-1)^m x^{\pm}_{i,m}, & K_i \longmapsto K_i, & h_{i,r} \longmapsto h_{i,r} \\ (2) & K_i \longmapsto \sigma_i K_i, & E_i \longmapsto \sigma_i E_i, & F_i \longmapsto F_i & \end{array}$$

where $\sigma_0, \dots, \sigma_n \in \{\pm 1\}$.

Proof.
Let V be a finite-dimensional irreducible representation of $U_q(\hat{\mathfrak{g}})$ and set

$$V^0 = \{v \in V : x^+_{i,m}.v = 0 \text{ for all } i, m\}$$

Suppose for a contradiction that $V^0 = 0$.
Since we are working over $\mathbb{C}$ it follows that V has at least one non-zero eigenspace with respect to K_i. Thus since $K_0, \dots, K_n$ act on V as a family of commuting invertible linear endomorphism (as implied by the relations) it follows by standard linear algebra that they must have a common eigenvector w with non-zero eigenvalues $\lambda_0, \dots, \lambda_n$. Since $V^0 = 0$ there must exist an infinite sequence of non-zero vectors

$$w, \; x^+_{i_1,m_1} w, \; x^+_{i_2,m_2} x^+_{i_1,m_1} w, \; x^+_{i_3,m_3} x^+_{i_2,m_2} x^+_{i_1,m_1} w, \dots \qquad (*)$$

By repeated application of the Drinfeld relation

$$K_i x^{\pm}_{j,m} = q_i^{\pm A_{ij}} x^{\pm}_{j,m} K_i$$

we calculate

$$\begin{aligned} K_r.(x^+_{i_k,m_k} x^+_{i_{k-1},m_{k-1}} \dots x^+_{i_1,m_1} w) &= q_r^{A_{ri_k}} x^+_{i_k,m_k} K_i.(x^+_{i_{k-1},m_{k-1}} x^+_{i_1,m_1} w) \\ &= q_r^{\sum_{j=1}^k A_{ri_j}} x^+_{i_k,m_k} \dots x^+_{i_1,m_1} K_r.w \\ &= \lambda_r q_r^{\sum_{j=1}^k A_{ri_j}} x^+_{i_k,m_k} \dots x^+_{i_1,m_1} w \end{aligned}$$

Note that A_{ri_k} cannot be zero for every r as that would give a zero column in the Cartan matrix. Therefore the non-zero elements in $(*)$ belong to distinct weight spaces and must therefore be linearly independent.
This contradicts finite-dimensionality of V.
Hence we may choose $v \in V^0 \setminus \{0\}$.
Recall from Remark 3.1.4 that $U_q(\hat{\mathfrak{g}})$ contains subalgebras isomorphic to $U_{q_i}(\mathfrak{sl}_2)$ in the Drinfeld-Jimbo generators

$$\langle E_i, F_i, K_i, K_i^{-1}\rangle \qquad i = 0, \dots, n.$$

By the isomorphism in Theorem 3.2.2 we have that E_i maps to $x_{i,0}^+$.
However by construction $x_{i,0}^+$ acts by 0 on v and therefore so does E_i.
Therefore v generates a highest weight $U_{q_i}(\mathfrak{sl}_2)$-module for each i.
By Proposition 3.1.7 every finite-dimensional highest weight module is irreducible. Therefore by Proposition 3.1.8 and Proposition 3.1.9 it follows that

$$K_i.v = \sigma_i q_i^{(\alpha,\lambda_i)} v, \quad i = 0, \dots, n$$

where $\sigma_i = \pm 1$, $(\alpha, \lambda_i) \in \mathbb{Z}$ and α is the simple root of $\mathfrak{sl}_2$.
By the isomorphism Ψ from Theorem 3.2.2 we have

$$K_0 \longmapsto cK_\theta^{-1}, \quad K_i \longmapsto K_i$$

Therefore $s_0 = d_0(\alpha, \lambda_0) \leq 0$ and $s_i = d_i(\alpha, \lambda_i) \geq 0 \quad$ for $i = 1, \dots, n$.
Moreover Ψ gives

$$\begin{aligned}
\Psi(K_0) = cK_\theta^{-1} &\Longrightarrow K_0 = \Psi^{-1}(c)\Psi^{-1}(K_\theta^{-1}) \\
&\Longrightarrow K_0 = \Psi^{-1}(c)\Psi^{-1}(K_n^{-m_n} \dots K_1^{-m_1}) \\
&\Longrightarrow K_0 = \Psi^{-1}(c)\Psi^{-1}(K_n)^{-m_n} \dots \Psi^{-1}(K_1)^{-m_1} \\
&\Longrightarrow K_0 = \Psi^{-1}(c)K_n^{-m_n} \dots K_1^{-m_1} \\
&\Longrightarrow K_0 \prod_{j=1}^{n} K_j^{m_j} = \Psi^{-1}(c)
\end{aligned}$$

Since c is central and V irreducible c acts on v by a scalar via Schur's lemma. Moreover since Ψ is an isomorphism $\Psi^{-1}(c)$ and c must act on v by the same scalar. Hence

$$c.v = \Psi^{-1}(c).v = \left(\sigma_0 \prod_{j=1}^{n} \sigma_j^{m_j} q^{s_0 + \sum_{k=0}^{n} m_k s_k} \right) v$$

There are more ways we can think of $\langle E_1, F_1, K_1, K_1^{-1}\rangle$ as embedding into $U_q(\hat{\mathfrak{g}})$ as a subalgebra isomorphic to $U_q(\mathfrak{sl}_2)$.
Consider below family of homomorphisms inducing actions on V

$$E_1 \longmapsto x_{i,r}, \qquad F_1 \longmapsto x_{i,-r}, \qquad K_1 \longmapsto c^r K_i \qquad (i = 1, \dots, n,\ r \in \mathbb{Z}).$$

Computing the induced action of K_1 under each homomorphism we get

$$c^r K_i.v = \left(\sigma_i \sigma_0^r \prod_{j=1}^{n} \sigma_j^{r m_j} q^{s_i + r\left(s_0 + \sum_{k=1}^{n} m_k s_k\right)} \right) v$$

Since K_1 acts on v with a non-negative power of q it follows that $c^r K_i$ will likewise and therefore

$$s_i + r\left(s_0 + \sum_{k=1}^{n} m_k s_k\right) \geq 0$$

for every $r \in \mathbb{Z}$.

Thus

$$s_0 + \sum_{k=1}^{n} m_k s_k = 0$$

so

$$c.v = \sigma_0 \prod_{j=1}^{n} \sigma_j^{m_j} v$$

Given that $\sigma_i^2 = 1$ we may, by invoking automorphisms in (2) where necessary, assume $\sigma_i = 1$ for all $i = 0, \dots, n$
Hence $c.v = v$.
Since V is irreducible it is generated as a $U_q(\hat{\mathfrak{g}})$-module by v.
Therefore since c is central in $U_q(\hat{\mathfrak{g}})$ it acts by 1 on all of V.
By Schur's lemma $c^{1/2}.v = \lambda v$ for some $\lambda \in \mathbb{C}$.
Thus

$$v = c.v = c^{1/2} c^{1/2}.v = \lambda^2 v$$

Therefore $c^{1/2}$ acts on V by ± 1.
If the action is by -1 then we can twist by the automorphism in (1) to negate the action of $c^{1/2}$. We may therefore assume $c^{1/2}$ acts by 1 and this shows V is a Type 1 representation. □

Corollary 3.3.3. *(Highest weight analogue).*
Let U_q^+ and U_q^0 be the subalgebras generated by $\{x_{i,m} : m \in \mathbb{Z},\ i = 1, \dots, n\}$ and $\{h_{i,r}, c^{\pm 1/2}, K_i^{\pm 1} : r \in \mathbb{Z} \setminus \{0\},\ i = 1, \dots, n\}$ respectively.
Then every non-zero finite-dimensional irreducible Type 1 representation V of $U_q(\hat{\mathfrak{g}})$ contains a non-zero vector v which is annihilated by U_q^+ and is a simultaneous eigenvector for the elements of U_q^0.

Proof.
The Drinfeld relations imply V^0 is a U_q^0-module. They also imply that the generators of U_q^0 all commute. Since V^0 is non-zero (by proof of Proposition 3.3.2) the elements of U_q^0 will simultaneously diagonalize a non-zero vector $v \in V^0$. This vector is also annihilated by U_q^+ by construction. Hence v satisfies the requirements.
□

Remark 3.3.4.
The corollary motivates a quantum affine analogue of highest weight representation. However it will not be a proper q-analogue since it does not give back the classical highest weight notion for $U(\hat{\mathfrak{g}})$. The notion we are about to define is therefore more commonly referred to as 'pseudo-highest weight' in literature (see e.g [CP3]). Since every irreducible representation can be twisted into a Type 1 representation we can without loss of generality restrict ourselves to the study of the quantum loop algebra $U_q(\mathfrak{L}(\mathfrak{g}))$.

By comparing coefficients in the formal power series given in the Drinfeld presentation one obtains

$$\phi_{i,r}^{\pm} = \pm K_i^{\pm 1}((q_i - q_i^{-1})h_{i,r} + g_{i,r}^{\pm}(h_{i,\pm 1}, \dots, h_{i,\pm(r-1)}))$$

where $g_{i,r}^{\pm}$ are homogeneous polynomials of degree r.
Unwinding this recursive definition with respect to $h_{i,r}$ we may rearrange

$$h_{i,r} = \frac{\mp K_i^{\mp 1}\left(-\phi_{i,r}^{\pm} \pm K_i^{\pm 1} g_{i,r}^{\pm}\left(h_{i,\pm 1},\dots,h_{i,\pm(r-1)}\right)\right)}{q_i - q_i^{-1}}$$

and apply induction on $h_{i,\pm 1}, \dots, h_{i,\pm(r-1)}$ to write each $h_{i,r}$ in terms of $\phi_{i,m}^{\pm}$ and $K_i^{\pm}$. Hence U_q^0 is equally defined by generators $\phi_{i,m}^{\pm}$ in place of $h_{i,r}$.

Definition 3.3.5. (Pseudo-highest weight representation).
A Type 1 representation V of $U_q(\mathfrak{L}(\mathfrak{g}))$ is **pseudo-highest weight** if it is generated by a vector v_0 which is annihilated by U_q^+ and is a simultaneous eigenvector for the elements in U_q^0. If $\phi_{i,m}.v_0 = \varphi_{i,m}^{\pm} v_0$, then the collection of complex numbers $\varphi_{i,m}^{\pm}$, denoted $\boldsymbol{\varphi}$ is called the pseudo-highest weight of V.

We may thus summarize our efforts into the following corollary

Corollary 3.3.6. *(Type 1 irreducibles are pseudo-highest weight).*
Every finite-dimensional irreducible Type 1 representation of $U_q(\mathfrak{L}(\mathfrak{g}))$ is pseudo-highest weight.

3.4 Chari-Pressley theorem

In the previous section we established a partial quantum affine analogy to the reduction made in chapter 2 from the finite-dimensional representation theory of affine Lie algebras to loop algebras. In chapter 2 this was followed up with discussions regarding parametrization of finite-dimensional irreducible representations by Drinfeld polynomials. It is therefore natural to ask the same question for finite-dimensional irreducible Type 1 representations of $U_q(\mathfrak{L}(\mathfrak{g}))$. It turns out the analogy carries over again with such representations also admitting a bijective parametrization by Drinfeld polynomials. The result was proved for $U_q(\widehat{\mathfrak{sl}_2})$ in [CP4] but the general case is very similar. After quoting some technical identities we shall prove it in the general setting following [CP3][CP4] expanding on the proof outline and be very explicit about calculations therein.

Theorem 3.4.1. *(Chari-Pressley Theorem).*
Let V be a finite-dimensional irreducible (pseudo-highest weight) Type 1 $U_q(\mathfrak{L}(\mathfrak{g}))$-module and let v_0 be a pseudo-highest weight vector of V. Then there exists unique monic polynomials $\pi_i \in \mathbb{C}[u]$, $i = 1, \dots, n$ with non-zero constant term, such that

$$\sum_{m=0}^{\infty} \varphi_{i,m}^{+} u^m = q_i^{deg(\pi_i)} \frac{\pi_i(q_i^{-2}u)}{\pi_i(u)} = \sum_{m=0}^{\infty} \varphi_{i,-m}^{-} u^{-m} \tag{3.1}$$

where left and right-hand sides are Laurent expansions of the middle term about 0 and ∞ respectively.

Remark 3.4.2.
The monic polynomials in the theorem may be normalized to Drinfeld polynomials with constant coefficient 1.

Define elements $\pi^{\pm}_{i,\pm m} \in U^0_q$ recursively for $i = 1, \dots n,\ m \in \mathbb{Z}$ by setting $\pi^{\pm}_{i,0} = 1$, and for $m > 0$

$$\pi^{\pm}_{i,\pm m} = \frac{\mp q_i^{\pm m}}{q_i^m - q_i^{-m}} \sum_{s=0}^{m-1} \phi^{\pm}_{i,\pm(s+1)} \pi^{\pm}_{i,\pm(m-s-1)} K_i^{\mp 1} \tag{3.2}$$

To prove the theorem we shall make use of a few congruence identities involving $\pi^{\pm}_{i,\pm m}$. These are proved via somewhat lengthy inductions and we refer to [CP4] for their details.

Lemma 3.4.3. *(Congruence identities).*
Let $N_q^+ = \sum_{i=1}^{n} \sum_{m\in\mathbb{Z}} U_q(\mathfrak{L}(\mathfrak{g})).x^+_{i,m}$.
Then for $m \in \mathbb{Z}_+$

$$\begin{aligned} \pi^{+}_{i,m} &\equiv (-1)^m q_i^{r^2} \frac{(x^+_{i,0})^m (x^-_{i,1})^m}{([m]_{q_i})^2} \\ \pi^{-}_{i,-m} &\equiv (-1)^m q_i^{-r^2} \frac{(x^+_{i,-1})^m (x^-_{i,0})^m}{([m]_{q_i})^2} \end{aligned} \tag{3.3}$$

$$\begin{aligned} (-1)^m q_i^{m(m-1)} \frac{(x^+_{i,0})^{m-1} (x^-_{i,1})^m}{[m-1]_{q_i}[m]_{q_i}} &\equiv -\sum_{s=0}^{m-1} x^-_{i,s+1} \pi^+_{i,m-s-1} K_i^{m-1} \\ (-1)^m q_i^{-m(m-1)} \frac{(x^+_{i,-1})^{m-1} (x^-_{i,0})^m}{[m-1]_{q_i}[m]_{q_i}} &\equiv -\sum_{s=0}^{m-1} x^-_{i,-s} \pi^-_{i,-m+s+1} K_i^{-m+1}) \end{aligned} \tag{3.4}$$

all congruences are $(mod\ \ N_q^+)$

Proof. (See [CP4]). □

Lemma 3.4.4. *(Formal power series definition of* $\pi^{\pm}_{i,\pm m}$*).*
Let

$$\boldsymbol{\pi}^{\pm}_i(u) = \sum_{m=0}^{\infty} \pi^{\pm}_{i,\pm m} u^{\pm m}, \qquad \boldsymbol{\phi}^{\pm}_i(u) = \sum_{m=0}^{\infty} \phi^{\pm}_{i,\pm m} u^{\pm m}$$

Then

$$\boldsymbol{\phi}^{\pm}_i(u) = K_i^{\pm 1} \frac{\boldsymbol{\pi}^{\pm}_i(q_i^{\mp 2} u)}{\boldsymbol{\pi}^{\pm}_i(u)} \tag{3.5}$$

Proof.
We have

$$\frac{\mp q_i^{\pm m}}{q_i^m - q_i^{-m}} = \frac{1}{q^{\mp 2m} - 1}$$

after multiplying by $q_i^{\mp m}$ at top and bottom.
So multiplying both sides of (3.2) by $q^{\mp 2m}-1$ and $K_i^{\pm 1}$ we get

$$\pi^{\pm}_{i,\pm m}K_i^{\pm 1}q_i^{\mp 2m}-\pi^{\pm}_{i,\pm m}K_i^{\pm 1}=\sum_{s=0}^{m-1}\phi^{\pm}_{i,\pm(s+1)}\pi^{\pm}_{i,\pm(m-s-1)}$$

Noticing that $\phi^{\pm}_{i,0}=K_i^{\pm}$ we have

$$\pi^{\pm}_{i,\pm m}K_i^{\pm 1}q_i^{\mp 2m}=\sum_{s=0}^{m}\phi^{\pm}_{i,\pm s}\pi^{\pm}_{i,\pm(m-s)}$$

The right hand side is precisely the m^{th} coefficient in the power series expansion of $\boldsymbol{\phi}_i^{\pm}(u)\boldsymbol{\pi}_i^{\pm}(u)$ and the left hand side that of $K_i^{\pm}\boldsymbol{\pi}_i^{\pm}(q_i^{\mp 2}u)$.
Hence the identity follows. $\square$

Proof. (Chari-Pressley Theorem).
Let $V(\boldsymbol{\varphi})$ be a finite-dimensional irreducible Type 1 $U_q(\mathfrak{L}(\mathfrak{g}))$-module with pseudo-highest weight $\boldsymbol{\varphi}$, and let v_0 be a pseudo-highest weight vector in $V(\boldsymbol{\varphi})$. Since $V(\boldsymbol{\varphi})$ is Type 1 we have

$$K_i.v_0=q_i^{s_i}\quad i=1,\dots,n$$

with $s_i\in\mathbb{Z}_{\geq 0}$ as observed in Proposition 3.3.2.
The Drinfeld presentation makes it clear that $U_q(\mathfrak{L}(\mathfrak{g}))$ contains an isomorphic copy of $U_q(\mathfrak{g})$ generated by $\langle x^{\pm}_{i,0},K_i^{\pm}:i=1,\dots,n\rangle$. Indeed one observes that the Drinfeld presentation degenerates to the Drinfeld-Jimbo presentation for $U_q(\mathfrak{g})$ restricting to this subalgebra. We may therefore regard $V(\boldsymbol{\varphi})$ as a $U_q(\mathfrak{g})$-module by restricting the action. In particular the $U_q(\mathfrak{g})$-module generated by v_0 becomes a highest weight module, since $x^+_{i,0}$ acts by 0, and K_i acts diagonally on v_0 with the induced action. By Proposition 3.1.7 it follows that the module is irreducible and by Proposition 3.1.9 that irreducible modules are highest weight of the form $V_q(\boldsymbol{w}_{\sigma,\lambda})$ where $\lambda=\sum_{i=1}^{n}s_i\lambda_i\in\bigwedge_W^+$ and $\sigma=id$ (since $K_i.v_0=q_i^{s_i}$ $\forall i$). In particular $(x^-_{i,0})^{s_i+1}v_0=0$. Similarly $(x^-_{i,1})^{s_i+1}v_0=0$. Thus by the first congruence in (3.3) from Lemma 3.4.3 it follows that the action of $\pi^+_{i,m}$ is zero for $m>s_i$. Moreover by definition of pseudo-highest weight module $\pi^+_{i,m}\in U_q^0$ has diagonal action on v_0, and hence

$$\boldsymbol{\pi}_i^+(u)v_0=P_i(u)v_0$$

for some polynomial $P_i(u)=\sum_{m=0}^{s_i}p_{i,m}u^m$ with $deg(P_i)=s_i$ and $p_{i,m}\in\mathbb{C}$.
By the identity in Lemma 3.4.4 we now have

$$\begin{aligned}\sum_{m=0}^{\infty}\varphi^+_{i,m}u^m v_0=\boldsymbol{\phi}_i^+(u)v_0=K_i\frac{\boldsymbol{\pi}_i^+(q_i^{-2}u)}{\boldsymbol{\pi}_i^+(u)}v_0&=q_i^{s_i}\frac{P_i^+(q_i^{-2}u)}{P_i^+(u)}v_0\\&=q_i^{deg(P_i)}\frac{P_i^+(q_i^{-2}u)}{P_i^+(u)}v_0\end{aligned}$$

Hence the first equality in the theorem follows.
To prove the second equality we invoke the congruences in (3.4) from Lemma 3.4.3, letting both sides act on v_0, setting $m = s_i + 1$.
Since $(x^-_{i,0})^{s_r+1} v_0 = 0$ and $(x^-_{i,1})^{s_r+1} v_0 = 0$ we get

$$\sum_{m=0}^{s_i} x^-_{i,m+1} \pi^+_{i,s_i-m} K_i^{s_i} v_0 = 0$$

Recall the Drinfeld relation

$$[x^+_{i,m}, x^-_{i,s}] = \frac{\phi^+_{i,m+s} - \phi^-_{i,m+s}}{q_i - q_i^{-1}}$$

and note in particular that we have regarded c as 1 since we are working over the quantum loop algebra.
Acting by $x^+_{i,-r-1}$ for $r \geq 0$ and applying the above relation we get

$$\sum_{m=0}^{s_i} \left(x^-_{i,m+1} x^+_{i,-r-1} + \frac{\phi^+_{i,m-r} - \phi^-_{i,m-r}}{q_i - q_i^{-1}} \right) \pi^+_{i,s_i-m} K_i^{s_i} v_0 = 0$$

since π^+_{i,s_i-m} and $K_i^{s_i}$ act diagonally and $x^+_{i,-r-1}$ kills v_0.
Reindexing and carrying through the remaining action we get

$$\sum_{t=0}^{r} \varphi^-_{i,-t}\, p_{i,s_i-r+t} = \sum_{t=0}^{s_i-r} \varphi^+_{i,t}\, p_{i,s_i-r-t} \tag{3.6}$$

for $0 \leq r \leq s_i$, and for exponent reasons

$$\sum_{t=r-s_i}^{r} \varphi^+_{i,-t}\, p_{i,s_i-r+t} = 0$$

whenever $r > s_i$. Letting both sides of (3.2) act on v_0, taking $m = s_i - r$, and doing some minor rearrangement we have

$$\frac{q_i^{s_i}(q_i^{s_i-r} - q_i^{-(s_i-r)})}{-q_i^{s_i-r}} p^+_{i,s_i-r} = \sum_{t=0}^{s_i-r-1} \varphi^+_{i,s_i-r+1}\, p^+_{i,s_i-r-t-1}$$

We thus compute the right hand side of (3.6) as

$$\begin{aligned}
\sum_{t=0}^{s_i-r} \varphi^+_{i,t}\, p_{i,s_i-r-t} &= \varphi^+_{i,0}\, p_{i,s_i-r} + \sum_{t=1}^{s_i-r-1} \varphi^+_{i,t+1}\, p_{i,s_i-r-t-1} \\
&= q_i^{s_i} p_{i,s_i-r} + \frac{q_i^{s_i}(q_i^{s_i-r} - q_i^{-(s_i-r)})}{-q_i^{s_i-r}} p_{i,s_i-r} \\
&= q_i^{s_i} q_i^{-2(s_i-r)} p_{i,s_i-r}
\end{aligned}$$

Finally notice that the left hand side of (3.6) is the r^{th} coefficient in the power series expansion of

$$\left(\sum_{r=0}^{\infty} \varphi^-_{i,-r} u^{-r} \right) P_i(u) u^{-(s_i-r)}$$

so multiplying both sides of (3.6) by u^{s_i-r} and summing $r = 0$ to ∞ gives

$$\left(\sum_{r=0}^{\infty} \varphi_{i,-r}^{-} u^{-r}\right) P_i(u) = q_i^{s_i} P_i(q_i^{-2} u)$$

as required.

□

Remark 3.4.5.
Theorem 3.4.1 shows that every finite-dimensional irreducible Type 1 representation can be uniquely associated with a Drinfeld polynomial (after normalization). Chari and Pressley proved in [CP5] that the converse also holds, namely the polynomials associated with the finite-dimensional irreducible Type 1 modules exhaust all Drinfeld polynomials. We hence have the desired bijection

$$\begin{aligned} \mathcal{P}^+ &\longleftrightarrow \{\text{finite-dimensional irreducible Type 1 } U_q(\mathcal{L}(\mathfrak{g}))\text{-modules}\} \\ \boldsymbol{\pi} &\longleftrightarrow V_q(\boldsymbol{\pi}) \end{aligned}$$

One of the keys to understanding the converse to Theorem 3.4.1 is the existence (see [CP5]) of a surjective $U_q(\mathfrak{L}(\mathfrak{g}))$-module homomorphism

$$f : M \longrightarrow V_q(\boldsymbol{\pi}_1 \boldsymbol{\pi}_2)$$

where M is the $U_q(\mathfrak{L}(\mathfrak{g}))$-submodule of $V_q(\boldsymbol{\pi}_1) \otimes V_q(\boldsymbol{\pi}_2)$ generated by tensor products of all pseudo-highest weight vectors in $V_q(\boldsymbol{\pi}_1)$ and $V_q(\boldsymbol{\pi}_2)$ respectively. Thus if $V_q(\boldsymbol{\pi}_1 \boldsymbol{\pi}_2)$ exists it must be a quotient of M and hence finite-dimensional whenever $V_q(\boldsymbol{\pi}_1)$ and $V_q(\boldsymbol{\pi}_2)$ are finite-dimensional. Over $\mathbb{C}$ every polynomial splits as a product of linear factors. This gives a possible approach for the proof of the converse. One could identify finite-dimensional irreducible representations for linear polynomials, and then realize irreducible representations as quotients under tensor products of such representations.

This motivates the definition of a **fundamental representation** $V_q(\boldsymbol{\pi}_{i,a})$ where $\boldsymbol{\pi}_{i,a} = ((1 - au)^{\delta_{ij}})_{1 \leq j \leq n} \in \mathcal{P}$ as defined in chapter 2. Indeed it is proved in [CP5] that the finite-dimensional irreducible representations fall out as submodule quotients of tensor products of fundamental representations. Although this gives the required bijection it does not serve as an explicit description of the finite-dimensional irreducible representations.

3.5 Evaluation representations of $U_q(\widehat{\mathfrak{sl}_2})$

In this section we look at how the notion of evaluation representation from chapter 2 generalizes to $U_q(\widehat{\mathfrak{sl}_2})$. The analogue was originally defined by Jimbo in [Ji2].

Recall that $\mathfrak{sl}_2$ has Cartan matrix with single entry (2). There are only two ways (up to basis reordering) that this matrix admits an extension to a 2×2 affine GCM since it must have determinant 0, namely

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

By the Dynkin diagram classification of affine Lie algebras given in e.g [Kac] there are no quadruply laced Dynkin diagrams in the class of untwisted affine Lie algebras. We thus conclude that A must be the GCM of $\widehat{\mathfrak{sl}_2}$. Plugging this matrix into the Drinfeld-Jimbo presentation we obtain the following explicit presentation for $U_q(\widehat{\mathfrak{sl}_2})$ in terms of generators

$$E_1, E_2, F_1, F_2, K_1^{\pm 1}, K_2^{\pm 1}$$

and relations:

$$\begin{aligned}
&K_iK_i^{-1} = K_i^{-1}K_i = 1, && K_1K_2 = K_2K_1,\\
&K_iE_iK_i^{-1} = q^2E_i, && K_iF_iK_i^{-1} = q^{-2}F_i,\\
&K_iE_jK_i^{-1} = q^{-2}E_j, && K_iF_jK_i^{-1} = q^2F_j,\\
&[E_1, F_2] = 0, && [F_1, E_2] = 0,\\
&[E_i, F_i] = \frac{K_i - K_i^{-1}}{q - q^{-1}}, &&
\end{aligned}$$

$$\begin{aligned}
&E_i^3E_j - [3]_qE_i^2E_jE_i + [3]_qE_iE_jE_i^2 - E_jE_i^3 = 0 && (i \neq j)\\
&F_i^3F_j - [3]_qF_i^2F_jF_i + [3]_qF_iF_jF_i^2 - F_jF_i^3 = 0 && (i \neq j)
\end{aligned}$$

Developing a q-analogue of the evaluation representation requires the existence of a homomorphism

$$ev_a : U_q(\widehat{\mathfrak{sl}_2}) \longrightarrow U_q(\mathfrak{sl}_2) \quad (a \in \mathbb{C}^\times)$$

with classical limit reducing to the ordinary evaluation homomorphism. In addition the map needs to restrict to the identity on the subalgebra $\langle E_2, F_2, K_2^{\pm 1}\rangle \cong U_q(\mathfrak{sl}_2)$ in order for the $U_q(\mathfrak{sl}_2)$-representation V_q to be isomorphic (as $U_q(\mathfrak{sl}_2)$-representations) to its pullback by ev_a. Below proposition provides a working definition, for which we shall provide the necessary checks.

Proposition 3.5.1. *(Evaluation homomorphism for $U_q(\widehat{\mathfrak{sl}_2})$).*
For every $a \in \mathbb{C}^\times$ there is an algebra homomorphism

$$ev_a : U_q(\widehat{\mathfrak{sl}_2}) \longrightarrow U_q(\mathfrak{sl}_2)$$

$$E_1 \mapsto q^{-1}aF,\ F_1 \mapsto qa^{-1}E, \quad E_2 \mapsto E,\ F_2 \mapsto F, \quad K_1 \mapsto K^{-1},\ K_2 \mapsto K$$

Proof.
We have to show that the relations in the above stated presentation for $U_q(\widehat{\mathfrak{sl}_2})$ are preserved under ev_a using the relations for $U_q(\mathfrak{sl}_2)$ (see Definition 3.1.2).
We check:
$ev_a(K_1)ev_a(K_1)^{-1} = K^{-1}(K^{-1})^{-1} = K^{-1}K = 1$
$ev_a(K_1)ev_a(K_2) = K^{-1}K = 1 = KK^{-1} = ev_a(K_2)ev_a(K_1)$
$ev_a(K_1)ev_a(E_1)ev_a(K_1)^{-1} = K^{-1}(q^{-1}aF)K = qaF = q^2ev_a(E_1)$

$ev_a(K_1)ev_a(E_2)ev_a(K_1)^{-1} = K^{-1}EK = K^{-1}Kq^{-2}E = q^{-2}ev_a(E_2)$
$[ev_a(E_1), ev_a(F_2)] = [q^{-1}aF, F] = q^{-1}a[F, F] = 0$
$[ev_a(E_1), ev_a(F_1)] = [q^{-1}aF, qa^{-1}E] = [F, E] = \frac{K^{-1}-K}{q-q^{-1}} = \frac{ev_a(K_1)-ev_a(K_1)^{-1}}{q-q^{-1}}$

Note that the first q-Serre relation may be written more compactly as the equation

$$[E_1^3, E_2] = [3]_q E_1[E_1, E_2]E_1$$

By Leibniz rule we have

$$\begin{aligned}[E_1^3, E_2] &= -(ad\, E_2)(E_1^2 E_1)\\ &= -E_1^2(ad\, E_2)(E_1) - (ad\, E_2)(E_1^2)E_1\\ &= E_1^2[E_1, E_2] - (E_1(ad\, E_2)(E_1) + (ad\, E_2)(E_1)E_1)E_1\\ &= E_1^2[E_1, E_2] + E_1[E_1, E_2]E_1 + [E_1, E_2]E_1^2\end{aligned}$$

Thus

$$ev_a(E_1)^2[ev_a(E_1), ev_a(E_2)] + ev_a(E_1)[ev_a(E_1), ev_a(E_2)]ev_a(E_1) + [ev_a(E_1), ev_a(E_2)]ev_a(E_1)^2$$
$$= ev_a(E_1)(q^{-1}a)^2F\frac{K^{-1}-K}{q-q^{-1}} + ev_a(E_1)(q^{-1}a)\frac{K^{-1}-K}{q-q^{-1}}ev_a(E_1) + \frac{K^{-1}-K}{q-q^{-1}}(q^{-1}a)^2Fev_a(E_1)$$

$$= ev_a(E_1)(q^{-1}a)\frac{q^{-2}K^{-1}-q^2K}{q-q^{-1}}(q^{-1}aF) + ev_a(E_1)(q^{-1}a)\frac{K^{-1}-K}{q-q^{-1}}ev_a(E_1)$$
$$+ (q^{-1}aF)(q^{-1}a)\frac{q^2K^{-1}-q^{-2}K}{q-q^{-1}}ev_a(E_1)$$

$$= (q^2+1+q^{-2})ev_a(E_1)(q^{-1}a)\frac{K^{-1}-K}{q-q^{-1}}ev_a(E_1) = \frac{q^3-q^{-3}}{q-q^{-1}}ev_a(E_1)[ev_a(E_1), ev_a(E_2)]ev_a(E_1)$$
$$= [3]_q\, ev_a(E_1)[ev_a(E_1), ev_a(E_2)]ev_a(E_1)$$

Remaining checks follow by symmetrical calculations. □

Remark 3.5.2.
The fact that ev_a reduces to the classical evaluation homomorphism is easiest seen by restating the definition of ev_a via the isomorphism provided by Drinfeld's presentation in Theorem 3.2.2:

$$\begin{aligned} ev_a(c) &= 1, & ev_a(K) &= K\\ ev_a(x_{1,0}^+) &= E, & ev_a(x_{1,0}^-) &= F\\ ev_a(x_{1,-1}^+) &= qa^{-1}K^{-1}E, & ev_a(x_{1,1}^-) &= q^{-1}aFK\end{aligned}$$

Using the Drinfeld relation

$$[h_{1,1}, x_{1,k}^{\pm}] = \pm(q+q^{-1})x_{1,k+1}^{\pm}$$

it follows by induction that

$$ev_a(x_{1,k}^+) = q^{-k}a^kK^kE$$
$$ev_a(x_{1,k}^-) = q^{-k}a^kFK^k$$

The Drinfeld generators have clear correspondence with the generators of $\mathfrak{L}(\mathfrak{g})$ from which it follows that ev_a is given by polynomial evaluation at a when $q = 1$.

Evaluation representations are defined the same way as in §2.2. Given $a \in \mathbb{C}^\times$ and a representation V_q of $U_q(\mathfrak{sl}_2)$ we pull back a representation $V_q(a)$ of $U_q(\widehat{\mathfrak{sl}}_2)$ with action

$$x.v = ev_a(x).v \quad (x \in U_q(\widehat{\mathfrak{sl}}_2))$$

on the same vector space.

Via the action of $U_q(\mathfrak{sl}_2)$-modules from Example 3.1.10 we see that the action of $U_q(\widehat{\mathfrak{sl}_2})$ on $V_q(a)$ is given by

$$x_{1,k}^+.v_i = q^{-k}a^kK^kE.v_i = a^kq^{k(l-2i+1)}[l-i+1]_q\, v_{i-1} \tag{3.7}$$
$$x_{1,k}^-.v_i = q^{-k}a^kFK^k.v_i = a^kq^{k(l-2i-1)}[i+1]_q\, v_{i+1} \tag{3.8}$$

Unfortunately analogues of evaluation representations only exist when $\mathfrak{g} = \mathfrak{sl}_n$, in contrast to the classical case where evaluation representations exist for all affine Lie algebras. In the case $\mathfrak{g} = \mathfrak{sl}_2$ it was proved in [CP5] that, in analogy with the classical case, one gets an explicit decomposition of finite-dimensional irreducible representations as tensor products of evaluation representations. Although it may not seem as much, it will go a long way into defining quantum affine analogues of characters for settings far more general than this.

3.6 q-characters

In classical theory characters are powerful invariants for understanding isomorphism classes of representations. Importantly they flag isomorphic representations and tell apart non-isomorphic representations. In the quantum affine setting we have seen that one has to work harder to establish basic facts about finite-dimensional representations. It has therefore become of interest to develop a quantum affine analogue of character theory. The q-characters were first introduced by Frenkel and Reshetikhin in [FR]. We end this chapter by examining its basic development.

In this section we denote by $U_q(\hat{\mathfrak{g}})$-*mod* the category of finite-dimensional Type 1 representations of $U_q(\hat{\mathfrak{g}})$. Since $U_q(\hat{\mathfrak{g}})$ is a Hopf algebra $U_q(\hat{\mathfrak{g}})$-*mod* has precisely the structure of a monoidal tensor category i.e it has well-defined associative tensor product and well-defined unit (the trivial representation, $\mathbf{1} = \mathbb{C}$). Module categories are moreover canonical examples of Abelian categories with Hom-addition, kernels and cokernels etc. Finally since the objects of $U_q(\hat{\mathfrak{g}})$-*mod* are finite-dimensional modules, we have Jordan-Hölder theorem so the composition factors and their multiplicities are the same up to reordering. We may hence regard $U_q(\hat{\mathfrak{g}})$-*mod* as a monoidal abelian tensor category with a commutative ring structure - the so called Grothendieck ring $Rep(U_q(\hat{\mathfrak{g}}))$ of $U_q(\hat{\mathfrak{g}})$.

Definition 3.6.1. (Grothendieck ring of $U_q(\hat{\mathfrak{g}})$).
The **Grothendieck ring** $Rep(U_q(\hat{\mathfrak{g}}))$ is the ring generated as a free abelian group by isomorphism classes of simple modules $\{X_i\}_{i\in I}$ in $U_q(\hat{\mathfrak{g}})$-*mod* such that for each object $X \in U_q(\hat{\mathfrak{g}})$-*mod* we can associate its class $[X] \in Rep(U_q(\hat{\mathfrak{g}}))$ via

$$[X] = \sum_{k\in I} |X : X_i|[X_i]$$

where $|X : X_i|$ is the multiplicity of X_i in the composition series of X. Addition and multiplication in $Rep(U_q(\hat{\mathfrak{g}}))$ is defined by

$$\begin{aligned}[X_i] + [X_j] &= [X_i \oplus X_j] \\ [X_i][X_j] &= [X_i \otimes X_j]\end{aligned}$$

Remark 3.6.2.
Although in general $X \otimes Y \ncong Y \otimes X$ the composition factors of $X \otimes Y$ and $Y \otimes X$ remain the same so

$$[X_i][X_j] = [X_i \otimes X_j] = [X_j \otimes X_i] = [X_j][X_i]$$

Hence $Rep(U_q(\hat{\mathfrak{g}}))$ is a commutative ring.
We shall omit the square bracket when there is no risk for confusion.

Before we move on to q-characters for quantum affine algebras let us first recall how characters are defined for $U_q(\mathfrak{g})$. The definition parallels the classical definition, and is completely congruent for representations with trivial twisting i.e when $\sigma = id$.

Definition 3.6.3. ($U_q(\mathfrak{g})$-characters).
Let $\mathfrak{g}$ be a simple Lie algebra with fundamental weights $w_1, \dots, w_n$ and define (the injective homomorphism)

$$\begin{aligned}\chi : Rep(U_q(\mathfrak{g})) &\longrightarrow \mathbb{Z}[y_1^{\pm}, \dots, y_1^{\pm}] \\ V &\longmapsto \sum_{\lambda=\sum_{i=1}^n m_i w_i \in \bigwedge_W} dim(V_\lambda) \prod_{i=1}^n y_i^{m_i}\end{aligned}$$

where $V_\lambda = \{v \in V : h.v = \lambda(h)v \ \ \forall h \in \mathfrak{h}\}$.
Given a representation V of $U_q(\mathfrak{g})$ we say that $\chi(V)$ is the **character** of V.

Let us initially see if we can straightforwardly extend the definition again, this time from $U_q(\mathfrak{g})$ to $U_q(\hat{\mathfrak{g}})$. We have seen that $U_q(\hat{\mathfrak{g}})$ possesses subalgebras isomorphic to $U_q(\mathfrak{g})$. We could therefore restrict the action of the $U_q(\hat{\mathfrak{g}})$-modules to $U_q(\mathfrak{g})$ and then recycle the character homomorphism for $U_q(\mathfrak{g})$. The definition can be tested on $U_q(\widehat{\mathfrak{sl}_2})$. In this case we know by §3.5 that there exists evaluation representations $V_q(a)$ which are pullbacks of $U_q(\mathfrak{sl}_2)$-representations V_q such that $V_q(a) \cong V_q$ as $U_q(\mathfrak{sl}_2)$-representations. But then since χ is injective we have

$$\chi(V_q(a)) = \chi(V_q) = \chi(V_q(b))$$

for any $a, b \in \mathbb{C}^\times$.
Thus the character χ cannot tell apart even the most basic of non-isomorphic representations. This is not a desired feature, so something different (or larger) is required. We get a better idea by taking into account the information provided by the Chari-Pressley theorem which uniquely identifies irreducible representations in terms of eigenvalues of $\phi^{\pm}_{i,m}$ with respect to a pseudo-highest weight vector. To define a character homomorphism χ_q over $Rep(U_q(\hat{\mathfrak{g}}))$ we need something more general than the Chari-Pressley theorem that applies to all weight spaces. A theorem by Frenkel and Reshetikhin [FR] generalizes the Chari-Pressley theorem into something that can be used to build a character homomorphism able to discriminate between objects in $Rep(U_q(\hat{\mathfrak{g}}))$.

Let $V \in Rep(U_q(\hat{\mathfrak{g}}))$.
The family $\{\phi^{\pm}_{i,\pm m} : i = 1, \dots, n,\ m \in \mathbb{N}\}$ commutes since they are defined in terms of the Drinfeld generators $\{K_i^{\pm 1}, h_{i,r} : i = 1, \dots, n,\ r \in \mathbb{Z} \setminus \{0\}$ which in turn commute because $c = 1$ since we are working over a category of Type 1 representations. We thus get a pseudo-weight space decomposition

$$V = \bigoplus_{\boldsymbol{\varphi} = (\boldsymbol{\varphi}^+, \boldsymbol{\varphi}^-)} V_{\boldsymbol{\varphi}}$$

where $\boldsymbol{\varphi}^{\pm} = (\varphi^{\pm}_{i,\pm m})_{i \in \{1,\dots,n\},\ m \in \mathbb{N}}$ and
$V_{\boldsymbol{\varphi}} = \{v \in V : \exists p \in \mathbb{N} \text{ s.t } (\phi^{\pm}_{i,m} - \varphi^{\pm}_{i,m})^p.v = 0\ \ \forall m \in \mathbb{Z},\ i = 1, \dots, n\}$

Theorem 3.6.4. *(Frenkel-Reshetikhin).*
Let $V \in Rep(U_q(\hat{\mathfrak{g}}))$ and $\boldsymbol{\varphi}^{\pm} = (\varphi^{\pm}_{i,\pm m})_{i \in \{1,\dots,n\},\ m \in \mathbb{N}}$ such that $V_{\boldsymbol{\varphi}} \neq \{0\}$. Then there exists polynomials $R_i(z), Q_i(z) \in \mathbb{C}[z],\ i = 1, \dots, n$ such that $R_i(0) = Q_i(0) = 1$ and for any $i \in \{1, \dots, n\}$

$$\sum_{m \in \mathbb{N}} \varphi^{\pm}_{i,\pm m} u^m = q_i^{deg\, R_i - deg\, Q_i} \frac{R_i(q_i^{-1}u) Q_i(q_i u)}{R_i(q_i u) Q_i(q_i^{-1}u)} \tag{3.9}$$

Proof.
Recall from Remark 3.2.3 that the collection of Drinfeld generators

$$\{x^{\pm}_{i,m}, h_{i,r}, K_i^{\pm 1}, c^{\pm 1/2} : m \in \mathbb{Z},\ r \in \mathbb{Z} \setminus \{0\}\} \quad (i = 1, \dots, n)$$

generate subalgebras of $U_q(\hat{\mathfrak{g}})$ isomorphic to $U_{q_i}(\widehat{\mathfrak{sl}_2})$. By the Drinfeld presentation, the families $(\phi^{\pm}_{i,\pm m})_{m \in \mathbb{N}}$ are defined only in terms of the generators contained in each $U_{q_i}(\widehat{\mathfrak{sl}_2})$-copy for $i = 1, \dots, n$. Therefore the pseudo-eigenvalues $(\varphi^{\pm}_{i,\pm m})_{m \in \mathbb{N}}$ of $(\phi^{\pm}_{i,\pm m})_{m \in \mathbb{N}}$ coincide with the pseudo-eigenvalues of the restriction of V to $U_{q_i}(\widehat{\mathfrak{sl}_2})$ for each $i = 1, \dots, n$.
Hence we may assume

$$\mathfrak{g} = \mathfrak{sl}_2$$

Thus from now on $n = 1$.
The plan is to prove the statement in stages of increasing generality:

(i) Evaluation representations
(ii) Irreducible representations
(iii) $Rep(U_q(\hat{\mathfrak{g}}))$

(i) :

Let $V_q(b)$ be an evaluation representation of $U_q(\widehat{\mathfrak{sl}_2})$. By factoring out the complex number q (with foresight) we may set $b = aq$ for some $a \in \mathbb{C}^\times$. Recall from Remark 3.5.2 that there is a basis $v_0, \dots, v_l$ with respect to which the action on $V_q(aq)$ is given by

$$\begin{aligned} K_1.v_i &= q^{l-2i} v_i \\ x^+_{1,m}.v_i &= (aq)^m q^{m(l-2i+1)}[l-i+1]_q v_{i-1} \\ x^-_{1,m}.v_i &= (aq)^m q^{m(l-2i-1)}[i+1]_q v_{i+1} \end{aligned}$$

Recall the Drinfeld relation

$$[x^+_{1,m}, x^-_{1,0}] = \frac{\phi^+_{1,m} - \phi^-_{1,m}}{q - q^{-1}}$$

where we have set $c = 1$ without loss of generality by underlying category assumption.
We also remark that by definition $\phi^+_{i,m} = 0$ if $m < 0$ and $\phi^-_{i,m} = 0$ if $m > 0$. Calculating for $m > 0$ we have

$$\begin{aligned} \phi^+_{1,m}.v_i &= (\phi^+_{1,m} - \phi^-_{1,m}).v_i \\ &= (q-q^{-1})[x^+_{1,m}, x^-_{1,0}]_q.v_i \\ &= (q-q^{-1})(x^+_{1,m}x^-_{1,0}.v_i - x^-_{1,0}x^+_{1,m}.v_i) \\ &= (q-q^{-1})(x^+_{1,m}[i+1]_q.v_{i+1} - (aq)^m q^{m(l-2i+1)}[l-i+1]_q.v_{i-1}) \\ &= (q-q^{-1})([i+1]_q(aq)^m q^{m(l-2i-1)}[l-i]_q v_i - (aq)^m q^{m(l-2i+1)}[l-i+1]_q[i]_q v_i) \\ &= (q-q^{-1})(aq^{l-2i})^m([i+1]_q[l-i]_q - q^{2m}[l-i+1]_q[i]_q)v_i \\ &= (\varphi^{(i)}_{1,m})^+ v_i \end{aligned}$$

The similar hold for $m < 0$ and for $m = 0$ we have

$$\phi^+_{1,0}.v_i = K_1.v_i = q^{l-2i} v_i$$

Thus computing the power series on the left hand side of (3.9) we get

$$\begin{aligned} &\left(\sum_{m=0}^{\infty} (\varphi^{(i)}_{1,m})^+ u^m\right) \\ &= \left(q^{l-2i} + \sum_{m=1}^{\infty}(q-q^{-1})(aq^{l-2i})^m([i+1]_q[l-i]_q - q^{2m}[l-i+1]_q[i]_q)u^m\right) \\ &= q^{l-2i} \\ &+ (q-q^{-1})\left([i+1]_q[l-i]_q aq^{l-2i}u\sum_{k=0}^{\infty}(aq^{l-2i})^k u^k - [l-i+1]_q[i]_q aq^{l-2i+2}u\sum_{k=0}^{\infty}(aq^{l-2i+2})^k u^k\right) \\ &= q^{n-2i} + (q-q^{-1})\left([i+1]_q[l-i]_q\frac{aq^{l-2i}u}{1-aq^{l-2i}u} - [l-i+1]_q[i]_q\frac{aq^{l-2i+2}u}{1-aq^{l-2i+2}u}\right) \end{aligned}$$

After putting under common denominator and simplifying one obtains

$$\left(\sum_{m=0}^{\infty} (\varphi^{(i)}_{1,m})^+ u^m\right) = q^{l-2i}\frac{(1-aq^{l+2}u)(1-aq^{-l}u)}{(1-aq^{l-2i}u)(1-aq^{l-2i+2}u)}$$

Setting

$$R_1^{(i)}(u) = \prod_{k=1}^{l}(1 - aq^{l-2k+1}u)$$
$$Q_1^{(i)}(u) = \prod_{k=1}^{i}(1 - aq^{l-2k+3}u)\prod_{k=1}^{i}(1 - aq^{l-2k+1}u)$$

we have

$$\frac{R_1^{(i)}(q^{-1}u)}{R_1^{(i)}(qu)} = \frac{(1-aq^{l-2}u)(1-aq^{l-4}u)\ldots(1-aq^{-l+2}u)(1-aq^{-l}u)}{(1-aq^{l}u)(1-aq^{l-2}u)(1-aq^{l-4}u)\ldots(1-aq^{-l+2}u)} = \frac{1-aq^{-l}u}{1-aq^{l}u}$$

and similar cancellation yields

$$\frac{Q_1^{(i)}(qu)}{Q_1^{(i)}(q^{-1}u)} = \frac{(1-aq^{l+2}u)(1-aq^{l}u)}{(1-aq^{l-2i}u)(1-aq^{l-2i+2}u)}$$

Hence

$$\left(\sum_{m=0}^{\infty}(\varphi_{1,m}^{(i)})^{+}u^{m}\right) = q^{l-2i}\frac{R_1^{(i)}(q^{-1}u)\,Q_1^{(i)}(qu)}{R^{(i)}(qu)\,Q_1^{(i)}(q^{-1}u)} = q^{deg\,R_1^{(i)}-deg\,Q_1^{(i)}}\frac{R_1^{(i)}(q^{-1}u)\,Q_1^{(i)}(qu)}{R_1^{(i)}(qu)\,Q_1^{(i)}(q^{-1}u)}$$

The negative series is computed similarly.

(ii) :

Now let V be an irreducible representation in $Rep(U_q(\widehat{\mathfrak{sl}_2}))$.
Then as quoted at the end of Remark 3.5.2 we have

$$V = \bigotimes_{i=1}^{r} W_i(a_i)$$

for some evaluation representations $W_i(a_i)$ of $U_q(\widehat{\mathfrak{sl}_2})$ $(i = 1, \ldots, r)$.
Let $\boldsymbol{\varphi}^{\pm} = (\varphi_{1,\pm m}^{\pm})_{m\in\mathbb{N}}$ be such that $V_{\boldsymbol{\varphi}} \neq \{0\}$ and set

$$d(u) = \sum_{m\in\mathbb{N}}\varphi_{1,\pm m}^{\pm}u^{m}.$$

We claim there exists power series $d^{(1)}(u), \ldots, d^{(r)}(u)$ of pseudo-eigenvalues of $(\phi_{1,m}^{\pm})_{m\in\mathbb{N}}$ corresponding to $W_1(a_1), \ldots, W_1(a_r)$ respectively such that

$$d(u) = d^{(1)}(u)\ldots d^{(r)}(u) \tag{3.10}$$

Then by previous part there exists for each $i = 1, \ldots, n$ polynomials $R_1^{(1)}, \ldots, R_1^{(r)}$ and $Q_1^{(1)}, \ldots, Q_1^{(r)}$ such that

$$d^{(k)}(u) = q^{deg\,R_1^{(k)}-deg\,Q_1^{(k)}}\frac{R_1^{(k)}(q^{-1}u)\,Q_1^{(k)}(qu)}{R_1^{(k)}(qu)\,Q_1^{(k)}(q^{-1}u)}$$

So if the claim is true then taking $R_1(u) = R_1^{(1)}(u)\ldots R_1^{(r)}(u)$ and $Q_1(u) = Q_1^{(1)}(u)\ldots Q_1^{(r)}(u)$ will give us the required polynomials for $d(u)$.
It suffices to prove (3.10) for $r = 2$ and have the claim follow by induction for $r > 2$. We prove the claim more generally for any two representations V and W

of $U_q(\widehat{\mathfrak{sl}_2})$. The argument depends on below co-multiplication formula, proved (in more precise form) in [CP5] for $\mathfrak{g} = \mathfrak{sl}_2$

$$\Delta(h_{1,\pm m}) = h_{1,\pm m} \otimes 1 + 1 \otimes h_{1,\pm m} \ \ (mod\, U_q^- \otimes U_q^+) \tag{3.11}$$

We treat the positive case only, the negative case being similar. By the Drinfeld relations, the generators $\{K_1, h_{1,m} : m \in \mathbb{N}\}$ commute. By standard linear algebra it follows that V and W admit bases simultaneously upper-triangularizing the family of endomorphisms $\{K_1, h_{1,m} : m \in \mathbb{N}\}$. By introducing a lexicographical ordering on the corresponding basis for $V \otimes W$, we see from (3.11) that the family $\{\Delta(K_1), \Delta(h_{1,m}) : m \in \mathbb{N}\}$ (where $\Delta(K_1) = K_1 \otimes K_1$) is again simultaneously upper triangular. The eigenvalues of this family on $V \otimes W$ is therefore given by sums of eigenvalues of $\{K_1, h_{1,m} : m \in \mathbb{N}\}$ on V and W respectively. The claim now follows by applying the co-unit axiom $(H4)$, using the definition of $\sum_{m=0}^{\infty} \phi_{1,m}^{+} u^m$

$$\begin{aligned}\sum_{m=0}^{\infty} \phi_{1,m}^{+} u^m &= (\epsilon \otimes id) \circ \Delta \left(\sum_{m=0}^{\infty} \phi_{1,m}^{+} u^m \right) \\ &= (\epsilon \otimes id) \left(\Delta(K_1) exp \left(\pm (q - q^{-1}) \sum_{r=1}^{\infty} \Delta(h_{1,r}) u^r \right) \right)\end{aligned}$$

and splitting the sums of eigenvalues coming from each co-multiplication using the multiplicative property of exp, for any given eigenvector.

(iii) :

For the final part let $V \in Rep(U_q(\widehat{\mathfrak{sl}_2}))$.
Since V is finite-dimensional there exists a composition series

$$0 = V_0 < V_1 < V_2 < \cdots < V_m = V$$

such that V_i/V_{i-1} is simple for $i = 1, \dots, m$.
If $v \in V_{\boldsymbol{\varphi}} \setminus \{0\}$ then $\exists\, i \in \{1, \dots, m\}$ such that $v \in V_i$ and $v \notin V_{i-1}$.
Thus $v + V_{i-1}$ corresponds to the same eigenvalues in the irreducible quotient V_i/V_{i-1}, where we can apply previous part to get required polynomials.

□

Since we are working over $\mathbb{C}$ we may write

$$R_i(u) = \prod_{r=1}^{l_i} (1 - a_{i,r} u) \qquad Q_i(u) = \prod_{s=1}^{k_i} (1 - b_{i,s} u)$$

for some $a_{i,r}, b_{i,s} \in \mathbb{C}^{\times}$.
These complex numbers encode essential information about the space. They can be used to define a map into a ring of Laurent polynomials in infinitely many indeterminates, parametrized by tuples in the form above.

Definition 3.6.5. (q-character).
For $V \in Rep(U_q(\hat{\mathfrak{g}}))$ with $\boldsymbol{\varphi} = (\varphi_{i,\pm m}^{\pm})_{i \in \{1,\dots,n\},\, m \in \mathbb{N}}$ such that $V_{\boldsymbol{\varphi}} \neq 0$ let

$$m_{\boldsymbol{\varphi}} = \prod_{i=1}^{n} \prod_{r=1}^{l_i} Y_{i,a_{i,r}} \prod_{s=1}^{k_i} Y_{i,b_{i,s}}^{-1}.$$

Define the map

$$\begin{aligned}\chi_q : Rep(U_q(\hat{\mathfrak{g}})) &\longrightarrow \mathbb{Z}[Y_{i,a_i}^{\pm 1}]_{i\in\{1,\dots,n\},\ a_i\in\mathbb{C}^\times}\\ V &\longmapsto \sum_{\boldsymbol{\varphi}} dim(V_{\boldsymbol{\varphi}}) m_{\boldsymbol{\varphi}}.\end{aligned}$$

We call $\chi_q(V)$ the **q-character** of V.

Recall the naive attempt from the beginning of this section at constructing χ_q by composition of the maps

$$\begin{aligned}res : Rep(U_q(\hat{\mathfrak{g}})) &\longrightarrow Rep(U_q(\mathfrak{g}))\\ \chi : Rep(U_q(\mathfrak{g})) &\longrightarrow \mathbb{Z}[y_i^{\pm 1}]_{i\in\{1,\dots,n\}}\end{aligned}$$

By associating $Y_{i,a}^{\pm 1}$ with the fundamental weight $\pm w_i$ through

$$\begin{aligned}\beta : \mathbb{Z}[Y_{i,a_i}^{\pm 1}]_{i\in\{1,\dots,n\},\ a_i\in\mathbb{C}^\times} &\longrightarrow \mathbb{Z}[y_i^{\pm 1}]_{i\in\{1,\dots,n\}}\\ Y_{i,a_i}^{\pm 1} &\longmapsto y_i^{\pm 1}\end{aligned}$$

we get a commutative diagram

$$\begin{tikzcd} Rep(U_q(\hat{\mathfrak{g}})) \arrow[r, "\chi_q"] \arrow[d, "\text{res}"'] & \mathbb{Z}[Y_{i,a_i}^{\pm 1}]_{i\in\{1,\dots,n\},\ a_i\in\mathbb{C}^\times} \arrow[d, "\beta"] \\ Rep(U_q(\mathfrak{g})) \arrow[r, "\chi"'] & \mathbb{Z}[y_i^{\pm 1}]_{i\in\{1,\dots,n\}} \end{tikzcd}$$

We have left to check that χ_q does what it is intended to do, namely distinguish between objects of $Rep(U_q(\hat{\mathfrak{g}}))$. In other words we want to show that χ_q is an injective homomorphism. The implicit claim here is that the eigenvalue information of $\phi_{i,m}^{\pm}$ (or equivalently that of $K_i^{\pm 1},\ h_{i,r}$) is enough to determine whether two representations have the same irreducible subrepresentations (including multiplicity). In particular this means that irreducible representations are uniquely identified up to isomorphism by χ_q. Note that the same is not true for arbitrary representations since in general objects of $Rep(U_q(\hat{\mathfrak{g}}))$ need not be semisimple.

Proposition 3.6.6. *χ_q is an injective homomorphism.*

Proof.
We first check that χ_q is a homomorphism.
Let $V, W \in Rep(U_q(\hat{\mathfrak{g}}))$ with pseudo-weight space decompositions

$$V = \bigoplus_{i=1}^{n} V_{\boldsymbol{\varphi}'_i} \qquad W = \bigoplus_{j=1}^{m} W_{\boldsymbol{\varphi}''_j}$$

Then

$$\chi_q(V \oplus W) = \sum_{\boldsymbol{\varphi}} dim(V_{\boldsymbol{\varphi}}) m_{\boldsymbol{\varphi}} = \sum_{i=1}^{n} dim(V_{\boldsymbol{\varphi}'_i}) m_{\boldsymbol{\varphi}'_i} + \sum_{j=1}^{m} dim(W_{\boldsymbol{\varphi}''_j}) m_{\boldsymbol{\varphi}''_j}$$
$$= \chi_q(V) + \chi_q(W)$$

In the proof of Theorem 3.6.4 it was shown, in the case of $\mathfrak{g} = \mathfrak{sl}_2$, that $m_{\boldsymbol{\varphi}}$ has a multiplicative property. The same argument works for general $\mathfrak{g}$ since the proof is only dependent on the co-multiplication formula, which continues to exist provided c acts as 1 [Dam]. We denote by $m_{\boldsymbol{\varphi}'\boldsymbol{\varphi}''}$ the monomial corresponding to the pseudo-weight space $(V \otimes W)_{\boldsymbol{\varphi}'\boldsymbol{\varphi}''}$ where $\boldsymbol{\varphi}'$ is a pseduo-weight of V and $\boldsymbol{\varphi}''$ a pseudo-weight of W such that $m_{\boldsymbol{\varphi}'\boldsymbol{\varphi}''} = m_{\boldsymbol{\varphi}'} m_{\boldsymbol{\varphi}''}$. Then since

$$\bigoplus_{\boldsymbol{\varphi}} (V \otimes W)_{\boldsymbol{\varphi}} = V \otimes W = \bigoplus_{i=1}^{n} V_{\boldsymbol{\varphi}'_i} \otimes \bigoplus_{j=1}^{m} W_{\boldsymbol{\varphi}''_j} = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} (V_{\boldsymbol{\varphi}'_i} \otimes W_{\boldsymbol{\varphi}''_j})$$

we have

$$\begin{aligned}
\chi_q(V)\chi_q(W) &= \left(\sum_{i=1}^{n} dim(V_{\boldsymbol{\varphi}'_i}) m_{\boldsymbol{\varphi}'_i} \right) \left(\sum_{j=1}^{m} dim(W_{\boldsymbol{\varphi}''_j}) m_{\boldsymbol{\varphi}''_j} \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} dim(V_{\boldsymbol{\varphi}'_i}) dim(W_{\boldsymbol{\varphi}''_j}) m_{\boldsymbol{\varphi}'_i} m_{\boldsymbol{\varphi}''_j} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} dim(V_{\boldsymbol{\varphi}'_i} \otimes W_{\boldsymbol{\varphi}''_j}) m_{\boldsymbol{\varphi}'_i \boldsymbol{\varphi}''_j} \\
&= \sum_{\boldsymbol{\varphi}} dim(V \otimes W)_{\boldsymbol{\varphi}} m_{\boldsymbol{\varphi}} \\
&= \chi_q(V \otimes W)
\end{aligned}$$

Hence χ_q is a homomorphism.

To prove injectivity consider first an irreducible representation V in $Rep(U_q(\hat{\mathfrak{g}}))$. By Chari-Pressley theorem V is associated to a unique Drinfeld polynomial $\boldsymbol{\pi} = \boldsymbol{\pi}_{i_1,a_{i_1}} \dots \boldsymbol{\pi}_{i_n,a_{i_n}}$ corresponding to a pseudo-highest weight space $V_{\boldsymbol{\varphi}}$. Thus $m_{\boldsymbol{\varphi}} = Y_{i_1,a_{i_1}} \dots Y_{i_n,a_{i_n}}$ with the monomial corresponding to the highest weight $w_{i_1} + \dots + w_{i_n}$. Therefore $\chi_q(V) \neq 0$ since remaining monomials in $\chi_q(V)$ are of lower weight. An arbitrary non-zero class in $Rep(U_q(\hat{\mathfrak{g}}))$ is a linear combination of irreducible classes, each with a unique highest weight monomial term from the correspondence with a unique Drinfeld polynomial. At least one of these unique monomials will be of absolute highest weight. In case of a tie the highest weight monomials are algebraically independent since they come from distinct Drinfeld polynomials. Thus the value of the character is non-zero.

Hence χ_q is injective.

□

Appendix A

Hopf Algebras

Definition A.0.1. (Hopf Algebra).
A **Hopf algebra** A over a field k is an algebra over k with operations

$$\begin{aligned} \mathcal{M} &: A \otimes A \longrightarrow A && \text{(multiplication)} \\ \eta &: k \longrightarrow A && \text{(unit map)} \end{aligned}$$

equipped with algebra homomorphisms:

$$\begin{aligned} \Delta &: A \longrightarrow A \otimes A && \text{(co-multiplication)} \\ \epsilon &: A \longrightarrow k && \text{(co-unit map)} \\ \mathcal{S} &: A \longrightarrow A && \text{(antipode)} \end{aligned}$$

satisfying:

(H1) $\mathcal{M} \circ (id \otimes \mathcal{M}) = \mathcal{M} \circ (\mathcal{M} \otimes id)$ (associativity)
(H2) $\mathcal{M} \circ (id \otimes \eta) = id = \mathcal{M} \circ (\eta \otimes id)$ (existence of unit)
(H3) $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$ (co-associativity)
(H4) $(\epsilon \otimes id) \circ \Delta = id = (id \otimes \epsilon) \circ \Delta$ (existence of co-unit)
(H5) $\mathcal{M} \circ (id \otimes \mathcal{S}) \circ \Delta = \eta \circ \epsilon = \mathcal{M} \circ (\mathcal{S} \otimes id) \circ \Delta$ (antipode property)
(H6) $\Delta \circ \mathcal{M} = (\mathcal{M} \otimes \mathcal{M}) \circ (\Delta \otimes \Delta)$ (connecting axiom)

The Hopf algebra is a natural generalization of several frequently occurring objects such as group algebras, universal enveloping algebras and as described in Chapter 3, quantized enveloping algebras. Apart from being a usual algebra with formalized multiplication and unit, it also has a dual notion of "unmultiplication" along with a dual notion of unit, having properties making it a so called *bialgebra*. In addition the antipode is generalizing the notion of an inverse. There is much more to be said about Hopf algebras than there is room for here.

Appendix B

Universal Enveloping Algebras

Let $\mathfrak{g}$ be a Lie algebra over a field k.

The universal enveloping algebra $U(\mathfrak{g})$ can be thought of as the smallest associative algebra containing all the information of $\mathfrak{g}$. It is an example of a Hopf algebra (see appendix A). We will however begin by giving its more standard definition.

Definition B.0.1. (Tensor Algebra).
The **tensor algebra** $T(\mathfrak{g})$ is given by

$$T(\mathfrak{g}) = k \oplus \bigoplus_{n=1}^{\infty} \bigotimes_{i=1}^{n} \mathfrak{g} \quad \text{(as a vector space)}$$

with formal multiplication

$$\begin{aligned} \mathcal{M}:\ T(\mathfrak{g}) \otimes T(\mathfrak{g}) &\longrightarrow T(\mathfrak{g}) \\ (x_1 \otimes \cdots \otimes x_m) \otimes (y_1 \otimes \cdots \otimes y_n) &\longmapsto (x_1 \otimes \cdots \otimes x_m \otimes y_1 \cdots \otimes y_n) \end{aligned}$$

extended linearly to $T(\mathfrak{g})$.

Definition B.0.2. (Universal Enveloping Algebra).
The **universal enveloping algebra** $U(\mathfrak{g})$ is obtained from $T(\mathfrak{g})$ by adding the relations

$$[x, y] = x \otimes y - y \otimes x \quad (x, y \in \mathfrak{g})$$

where $[\,,\,]$ is the Lie bracket on $\mathfrak{g}$.

$U(\mathfrak{g})$ is a universal object in the sense that any homomorphism from $\mathfrak{g}$ to another associative algebra factors uniquely through the canonical homomorphism $i : \mathfrak{g} \longrightarrow U(\mathfrak{g})$ satisfying $i([x, y]) = i(x) \otimes i(y) - i(y) \otimes i(x)$.

The important Poincaré-Birkhoff-Witt (PBW) theorem provides an explicit basis for $U(\mathfrak{g})$:

Theorem B.0.3. *(Poincaré-Birkhoff-Witt Theorem).*
Let $(x_1, x_2, x_3, \dots)$ be an ordered basis of $\mathfrak{g}$ (possibly infinite).
Then $U(\mathfrak{g})$ has a basis

$$\{x_{i_1} \otimes \cdots \otimes x_{i_n} \,:\, i_1 \leq \cdots \leq i_n\} \cup \{1\}$$

of ordered monomials.

Proof. (See [Hum 17.4]). □

A few immediate facts that follow from this theorem are:
- $U(\mathfrak{g})$ is always infinite-dimensional (unless $\mathfrak{g}$ is trivial).
- $\mathfrak{g}$ embeds into $U(\mathfrak{g})$
- Given a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ we can reorder basis elements of $\mathfrak{g}$ such that $U(\mathfrak{g}) = U(\mathfrak{n}_+) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_-)$

$U(\mathfrak{g})$ is mainly useful because it has representation theory equivalent to that of $\mathfrak{g}$ and is a Hopf algebra (so its module category has well-defined tensor products, trivial module and duals). Since $U(\mathfrak{g})$ contains a copy of $\mathfrak{g}$ by the PBW-theorem, every representation of $U(\mathfrak{g})$ induces a representation of $\mathfrak{g}$ by restricting the action of $U(\mathfrak{g})$ to $\mathfrak{g}$. Conversely if $\rho : \mathfrak{g} \longrightarrow End(V)$ is a representation of $\mathfrak{g}$ then the linear map given by

$$\begin{aligned} \tilde{\rho} : U(\mathfrak{g}) &\longrightarrow End(V) \\ x_1 \otimes \cdots \otimes x_n &\longmapsto \rho(x_1) \dots \rho(x_n) \end{aligned}$$

is a representation of $U(\mathfrak{g})$ because

$$\begin{aligned} \tilde{\rho}(x \otimes y - y \otimes x) &= \rho(x)\rho(y) - \rho(y)\rho(x) \\ &= [\rho(x), \rho(y)]_{End(V)} = \rho([x, y]_{\mathfrak{g}}) = \tilde{\rho}([x, y]_{\mathfrak{g}}) \end{aligned}$$

so the linear map $\tilde{\rho}$ preserves the defining relations of $U(\mathfrak{g})$ and is therefore a homomorphism. In some sense this allows for the transfer of questions about Lie algebras to questions about associative algebras (albeit necessarily infinite-dimensional ones).

We finally remark that $U(\mathfrak{g})$ is a Hopf algebra by setting for all $x \in \mathfrak{g}$

$$\begin{aligned} \Delta(x) &= x \otimes 1 + 1 \otimes x, & \Delta(1) &= 1 \otimes 1, \\ \epsilon(x) &= 0, & \epsilon(1) &= 1, \\ \mathcal{S}(x) &= -x, & \mathcal{S}(1) &= 1, \end{aligned}$$

and extending to $U(\mathfrak{g})$.

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