THE CONE OF CYCLIC SIEVING PHENOMENA

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Abstract. We study cyclic sieving phenomena (CSP) on combinatorial objects from an abstract point of view by considering a rational polyhedral cone determined by the linear equations that define such phenomena. Each lattice point in the cone corresponds to a non-negative integer matrix which jointly records the statistic and cyclic order distribution associated with the set of objects realizing the CSP. In particular we consider a *universal* subcone onto which every CSP matrix linearly projects such that the projection realizes a CSP with the same cyclic orbit structure, but via a *universal* statistic that has even distribution on the orbits.

Reiner et.al. showed that every cyclic action give rise to a unique polynomial (mod $q^n - 1$) complementing the action to a CSP. We give a necessary and sufficient criterion for the converse to hold. This characterization allows one to determine if a combinatorial set with a statistic give rise (in principle) to a CSP without having a combinatorial realization of the cyclic action. We apply the criterion to conjecture a new CSP involving stretched Schur polynomials and prove our conjecture for certain rectangular tableaux. Finally we study some geometric properties of the CSP cone. We explicitly determine its half-space description and in the prime order case we determine its extreme rays.

Contents

Key words and phrases. Cyclic sieving, universal, polytope, roots of unity.

1. Introduction

1.1. **Background on cyclic sieving phenomena.** The cyclic sieving phenomenon was introduced by Reiner, Stanton and White in [RSW04]. For a survey, see [Sag].

Definition 1.1. Let C_n be a cyclic group of order n generated by σ_n, X a finite set on which C_n acts and $f(q) \in \mathbb{N}[q]$. Let $X^g \coloneqq \{x \in X : g \cdot x = x\}$ denote the fixed point set of X under $g \in C_n$. We say that the triple $(X, C_n, f(q))$ exhibits the *cyclic sieving phenomenon (CSP)* if

$$f(\omega_n^k) = |X^{\sigma_n^k}|, \text{ for all } k \in \mathbb{Z}, \tag{1.1}$$

where ω_n is any fixed primitive n^{th} root of unity.

Since $f(1)$ is always the cardinality of X, it is common that $f(q)$ is given as $f_\tau(q) \coloneqq \sum_{x \in X} q^{\tau(x)}$ for some statistic on X. With this in mind, we say that the triple (X, C_n, τ) exhibits CSP if $(X, C_n, f_\tau(q))$ does.

Here is a short list of cyclic sieving phenomena found in the literature (see [RSW04, Sag] for a more comprehensive list):

- Words $X = W_{n,k}$ of length n over an alphabet of size k, C_n acting via cyclic shift,
$$f(q) \coloneqq \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q = \sum_{w \in W_{n,k}} q^{\operatorname{maj} w}.$$
- Standard Young tableaux $X = \operatorname{SYT}(\lambda)$ of rectangular shape $\lambda = (n^m)$, C_n acting via jeu-de-taquin promotion [Rho10],
$$f(q) \coloneqq \frac{[n]_q!}{\prod_{(i,j)\in\lambda}[h_{i,j}]_q} = q^{-n\binom{m}{2}} \sum_{T \in \operatorname{SYT}(\lambda)} q^{\operatorname{maj}(T)},$$
this expression being the q-hook-length formula [Sta71].
- Triangulations X of a regular $(n+2)$-gon, C_{n+2} acting via rotation of the triangulation, $f(q) \coloneqq \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}_q$, MacMahon's q-analogue of the Catalan numbers [Mac16]. Note that through well-known bijections (see [Sta15]) we get induced CSPs with the sets $X = \operatorname{Dyck}(n)$, the set of Dyck paths of semi-length n, and $X = \mathfrak{S}_n(231)$, the set of permutations in $\mathfrak{S}_n$ avoiding the classical pattern 231. Moreover one has
$$f(q) \coloneqq \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}_q = \sum_{P \in \operatorname{Dyck}(n)} q^{\operatorname{maj}(P)} = \sum_{\pi \in \mathfrak{S}_n(231)} q^{\operatorname{maj}(\pi)+\operatorname{maj}(\pi^{-1})},$$
where the last equality is due to Stump [Stu09].

1.2. **Outline of the paper.** The examples presented in the previous subsection have one or more of the following pair of common features:

- The action of C_n on X has a *natural* definition.
- The polynomial $f(q)$ is generated by a *natural* statistic on X.

What is *natural* largely lies in the eyes of the beholder, but broadly it could be taken to mean a definition with combinatorial substance.

The following equivalent condition for a triple $(X, C_n, f(q))$ to exhibit the cyclic sieving phenomenon was given by Reiner–Stanton–White in [RSW04]:

$$f(q) \equiv \sum_{\mathcal{O} \in \mathrm{Orb}_{C_n}(X)} \frac{q^n - 1}{q^{n/|\mathcal{O}|} - 1} \pmod{q^n - 1}, \tag{1.2}$$

where $\mathrm{Orb}_{C_n}(X)$ denotes the set of orbits of X under the action of C_n.

Therefore the coefficient of q^i in $f(q) \pmod{q^n - 1}$ is generically interpreted as the number of orbits whose stabilizer-order divides i. This alternative condition also means that every cyclic action of C_n on a finite set X give rise to a (not necessarily natural) polynomial $f(q)$, unique modulo $q^n - 1$, such that $(X, C_n, f(q))$ exhibits the cyclic sieving phenomenon.

In this paper we consider when the converse of the above property holds. *Given a combinatorial set X with a natural statistic $\tau : X \to \mathbb{N}$, when does it give rise to a (not necessarily natural) action of C_n on X such that (X, C_n, τ) exhibits the cyclic sieving phenomenon?*

Having a necessary and sufficient criteria for the existence of such a CSP adds a couple of benefits:

- Given a polynomial $f(q) = \sum_{x \in X} q^{\tau(x)}$ generated by a natural statistic $\tau : X \to \mathbb{N}$, we can determine if a CSP exists in principle without knowing a combinatorial realization of the cyclic action. The criteria thus serves as a tool for confirming or refuting the existence of cyclic sieving phenomena involving a candidate polynomial.
- Generic evidence that a CSP exists provides motivation to search for a combinatorially meaningful cyclic action on the set X.

The main result in Section 2 is the following: Theorem 2.7 provides the necessary and sufficient conditions for $(X, C_n, f(q))$ to exhibit CSP. The natural (necessary) condition is that $f(q) \in \mathbb{Z}[q]$ take non-negative integer values at all n^{th} roots of unity, which is evident from the definition of a cyclic sieving phenomena.

We prove the following: Define

$$S_k := \sum_{j|k} \mu(k/j) f(\omega_n^j), \qquad \text{where } k|n.$$

Then $(X, C_n, f(q))$ exhibits CSP for some C_n acting on X if and only if $S_k \geq 0$ for all $k|n$. We warn that merely having a polynomial $f(q) \in \mathbb{N}[q]$ that takes non-negative integer values at all n^{th} roots of unity is no guarantee for the existence of a cyclic action complementing $f(q)$ to a CSP. A polynomial demonstrating this is given in Example 2.9.

In Section 3, we conjecture a new cyclic sieving phenomena involving stretched Schur polynomials. In a special case, we prove this conjecture by applying Theorem 2.7, see Theorem 3.7 below. That is, we prove existence of CSP without having to provide a natural cyclic group action.

Section 4 and onwards treat the cyclic sieving phenomenon from a more geometric perspective. We record the joint cyclic order and statistic distribution of the elements of X in a matrix and reformulate the CSP condition in terms of linear equations in the matrix entries. The set of matrices that satisfy these linear equations we call *CSP matrices* and we prove via Theorem 7.1 that they form a convex rational polyhedral cone whose integer lattice points correspond to realizable instances of CSP. Inspired by [AS17], we further proceed to identify a certain subcone which we call the *universal CSP cone* containing all matrices corresponding to realizable instances of CSP with evenly distributed statistic on all its orbits. We prove that all integer CSP matrices can be obtained from a universal CSP matrix through a sequence of *swaps* without going outside of the CSP cone (Proposition 6.4). The swaps can be interpreted as a sequence of statistic interchanges between pairs of elements in the corresponding CSP-instance.

Finally we explicitly determine all extreme rays of the universal CSP cone (Corollary 7.4) and in Section 5 we prove some general properties for all CSP cones.

1.3. **Notation.** The following notation will be used throughout the paper.

- $[n] \coloneqq \{1, \dots, n\}$.
- $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers.
- $K^{n\times n}$ denotes the set of $n \times n$ matrices over the set K.
- $\mu(n) \coloneqq \begin{cases} 0, & \text{if } n \text{ is not square-free,} \\ (-1)^r, & \text{if } n \text{ is a product of } r \text{ distinct primes,} \end{cases}$
  denotes the classical Möbius function.
- ω_n denotes a primitive n^{th} root of unity.
- $\Phi_n(q) \coloneqq \prod_{\substack{1\leq k\leq n \\ \gcd(n,k)=1}} (q - \omega_n^k)$ denotes the n^{th} cyclotomic polynomial.
- $[n]_q \coloneqq \dfrac{q^n - 1}{q - 1}$, $[n]_q! \coloneqq [n]_q[n-1]_q \cdots [1]_q$, $\begin{bmatrix} n \\ k \end{bmatrix}_q \coloneqq \dfrac{[n]_q!}{[k]_q![n-k]_q!}$,
  denotes the q-integer, q-factorial and q-binomial coefficients respectively.

2. Integer-valued polynomials at roots of unity

In the context of discovering cyclic sieving phenomena, one may sometimes have a candidate polynomial (*e.g.* a natural q-analogue of the enumeration formula for the underlying set) that takes integer values at all roots of unity, but the cyclic action complementing it to a CSP is unknown. In such situations one may like to know if a CSP could exist even in principle. In this section we characterize the set polynomials $f(q) \in \mathbb{Z}[x]$ of degree less than n such that $f(\omega_n^j) \in \mathbb{Z}$ for all $j = 1, \dots, n$ and show that they are indeed $\mathbb{Z}$-linear combinations of polynomials of the form

$$\frac{q^n - 1}{q^{n/d} - 1} = \sum_{i=0}^{n/d-1} q^{di} \qquad \text{for } d|n.$$

Using the characterization one can quickly determine if a CSP is present and get the count of the number of elements of each order in terms of evaluations of the polynomial at roots of unity. Often it is much simpler to determine the evaluations at roots of unity than it is to write the polynomial in terms of the above basis.

Finally note that not all polynomials $f(q) \in \mathbb{N}[q]$ such that $f(\omega_n^j) \in \mathbb{N}$ for all $j = 1, \ldots, n$ may necessarily be paired with a cyclic action to produce a CSP, see Example 2.9.

The set

$$M(n) := \{f(q) \in \mathbb{Z}[q] : \deg(f) < n,\ f(\omega_n^j) \in \mathbb{Z} \text{ for } j = 1, \ldots, n\}$$

forms a $\mathbb{Z}$-module. First we identify two useful bases for $M(n)$ using the following proposition and Lemma 2.2.

Proposition 2.1 (Désarménien [Dé89])**.** *Let $f(q) \in \mathbb{Z}[q]$ be a polynomial of degree less than n. Then the following two properties are equivalent:*

(i) For every $d|n$,

$$f(q) \equiv r_d \pmod{\Phi_d(q)} \text{ for some } r_d \in \mathbb{Z},$$

where $\Phi_d(q)$ denotes the d^{th} cyclotomic polynomial.

(ii) The polynomial $f(q)$ has the form

$$f(q) = \sum_{j=0}^{n-1} a_j q^j, \quad \text{where } a_j = a_{gcd(n,j)}. \tag{2.1}$$

Lemma 2.2. *For each $n \in \mathbb{N}$, the following sets form $\mathbb{Z}$-bases for $M(n)$:*

(i) $\mathcal{B}_1(n) = \{g_d(q) : d|n\}$ where

$$g_d(q) = \sum_{\substack{0 \le j < n \\ \gcd(j,n) = d}} q^j,$$

(ii) $\mathcal{B}_2(n) = \{h_d(q) : d|n\}$ where

$$h_d(q) = \sum_{j=0}^{n/d-1} q^{dj}.$$

Proof. Let $f(q) \in M(n)$ and suppose $d|n$. Then $\omega_n^{n/d}$ is a d^{th} root of unity. Note that $f(q) - f(\omega_n^{n/d})$ vanishes at $q = \omega_n^{n/d}$ so it is divisible by the minimal polynomial of $\omega_n^{n/d}$ over $\mathbb{Z}$, that is, $\Phi_d(q)$. Hence $f(q) \equiv r_d \pmod{\Phi_d(q)}$ where $r_d = f(\omega_n^{n/d}) \in \mathbb{Z}$. By Proposition 2.1 it follows that $f(q)$ has the form (2.1). Such polynomials are clearly spanned by $\mathcal{B}_1(n)$.

Now, the elements in $\mathcal{B}_2(n)$ are linearly independent, since the lowest-degree terms of $h_d(q) - 1$ are all different. By inclusion-exclusion we see that for each $d|n$,

$$g_d(q) = \sum_{d|r} \mu(r/d) h_r(q)$$

and hence $\mathcal{B}_1(n)$ and $\mathcal{B}_2(n)$ both form bases of $M(n)$. □

We may in fact extend the characterization in Lemma 2.2 to multivariate polynomials $f \in \mathbb{Z}[q_1, \dots, q_m]$ of degree less than n_i in variable q_i for $i = 1, \dots, m$ taking integer values at all points $(\omega_{n_1}^{j_1}, \dots, \omega_{n_m}^{j_m}) \in \mathbb{C}^m$ for $j_i = 1, \dots, n_i$, $i = 1, \dots, m$.

Theorem 2.3. *Let $M(n_1, \dots, n_m) = \{f \in \mathbb{Z}[q_1, \dots, q_m] : deg_i f < n_i,\ f(\omega_{n_1}^{j_1}, \dots, \omega_{n_m}^{j_m}) \in \mathbb{Z}$ for $j_i = 1, \dots, n_i,\ i = 1, \dots, m\}$ where $n_1, \dots, n_m \in \mathbb{N}$ and $deg_i f$ denotes the degree of x_i in f. Then the following sets form $\mathbb{Z}$-bases for $M(n_1, \dots, n_m)$:*

(i) $\mathcal{B}_1(n_1, \dots, n_m) = \left\{ \prod_{i=1}^m g_{d_i}^{(i)}(q_i) : d_i | n_i,\ i = 1, \dots, m \right\}$ *where*

$$g_{d_i}^{(i)}(q_i) = \sum_{\substack{0 \le j < n_i \\ gcd(j, n_i) = d_i}} q_i^j,$$

(ii) $\mathcal{B}_2(n_1, \dots, n_m) = \left\{ \prod_{i=1}^m h_{d_i}^{(i)}(q_i) : d_i | n_i,\ i = 1, \dots, m \right\}$ *where*

$$h_{d_i}^{(i)}(q_i) = \sum_{j=0}^{n_i/d_i - 1} q_i^{d_i j}.$$

Proof. We prove that $\mathcal{B}_1(n_1, \dots, n_m)$ is a $\mathbb{Z}$-basis of $M(n_1, \dots, n_m)$ by induction on m. The proof for $\mathcal{B}_2$ is similar and therefore omitted. The base case $m = 1$ follows from Lemma 2.2. Let $f \in M(n_1, \dots, n_{m+1})$. Write

$$f = f_{n_{m+1}-1}(q_1, \dots, q_m) q_{m+1}^{n_{m+1}-1} + \dots + f_1(q_1, \dots, q_m) q_{m+1} + f_0(q_1, \dots, q_m),$$

where $f_0, f_1, \dots, f_{n_{m+1}-1} \in \mathbb{Z}[q_1, \dots, q_m]$ with $f_k(\omega_{n_1}^{j_1}, \dots, \omega_{n_m}^{j_m}) \in \mathbb{Z}$ for all $k = 0, \dots, n_{m+1} - 1$, $j_i = 1, \dots, n_i$ and $i = 1, \dots, m$. The univariate polynomials

$$F_{\omega_{n_1}^{j_1}, \dots, \omega_{n_m}^{j_m}}(q_{m+1}) = f(\omega_{n_1}^{j_1}, \dots, \omega_{n_m}^{j_m}, q_{m+1}) \in \mathbb{Z}[q_{m+1}],$$

take integer values at $q_{m+1} = \omega_{n_{m+1}}^j$ for all $j = 1, \dots, n_{m+1}$. By Proposition 2.1 we therefore have that

$$f_k(\omega_{n_1}^{j_1}, \dots, \omega_{n_m}^{j_m}) = f_{\gcd(n_{m+1}, k)}(\omega_{n_1}^{j_1}, \dots, \omega_{n_m}^{j_m}),$$

for all $(\omega_{n_1}^{j_1}, \dots, \omega_{n_m}^{j_m}) \in \mathbb{C}^m$. Since the $\prod_{i=1}^m n_i$ points $(\omega_{n_1}^{j_1}, \dots, \omega_{n_m}^{j_m}) \in \mathbb{C}^m$ lie in general position the polynomials must coincide on all points in $\mathbb{C}^m$. Hence

$$f_k(q_1, \dots, q_m) = f_{\gcd(n_{m+1}, k)}(q_1, \dots, q_m)$$

for all $k = 0, \dots, n_{m+1} - 1$. It follows that f is uniquely spanned by $\mathcal{B}_1(n_{m+1})$ over $\mathbb{Z}[q_1, \dots, q_m]$. By induction $f_k(q_1, \dots, q_m)$ is uniquely spanned by $\mathcal{B}_1(n_1, \dots, n_m)$ over $\mathbb{Z}$ for all $k = 0, \dots, n_{m+1} - 1$. Hence f is uniquely spanned by $\mathcal{B}_1(n_1, \dots, n_{m+1})$ over $\mathbb{Z}$ completing the induction. □

Lemma 2.4. *Let $f(q) \in \mathbb{Z}[q]$ such that $f(\omega_n^j) \in \mathbb{Z}$ for all $j = 1, \dots, n$. Then for each $m, p, e \in \mathbb{N}$ where p is prime we have*

$$f(\omega_n^{mp^e}) \equiv f(\omega_n^{mp^{e-1}}) \pmod{p^e}.$$

In particular if $p \not| n$, then $f(\omega_n^{mp^{e-1}}) = f(\omega_n^{mp^e})$.

Proof. Since we are only concerned with evaluations of $f(q)$ at n^{th} roots of unity, we may assume $f(q) \in M(n)$. Furthermore by Lemma 2.2 and linearity it suffices to show the statement for the basis elements $\mathcal{B}_2$ of $M(n)$. For each $d|n$ and $k \in \mathbb{Z}$ we have

$$h_d(\omega_n^k) = \sum_{j=0}^{n/d-1} (\omega_{n/d}^k)^j = \begin{cases} n/d, & \text{if } k \equiv 0 \pmod{n/d}, \\ 0, & \text{otherwise.} \end{cases}$$

Now suppose $k = mp^e$ for some $m, p, e \in \mathbb{N}$ with p prime, and consider the different cases: Suppose first $mp^{e-1} \equiv 0 \pmod{n/d}$. This implies that $mp^e \equiv 0 \pmod{n/d}$, so $h_d(\omega_n^{mp^e}) = n/d = h_d(\omega_n^{mp^{e-1}})$. Secondly, suppose $mp^{e-1} \not\equiv 0 \pmod{n/d}$. If $mp^e \not\equiv 0 \pmod{n/d}$, then $h_d(\omega_n^{mp^e}) = 0 = h_d(\omega_n^{mp^{e-1}})$. On the other hand if $mp^e \equiv 0 \pmod{n/d}$, then $n/d = p^f a$ for some $f \geq e$ and $a \in \mathbb{N}$. Therefore $h_d(\omega_n^{mp^e}) - h_d(\omega_n^{mp^{e-1}}) = p^f a - 0 \equiv 0 \pmod{p^e}$. Hence the lemma follows. □

Lemma 2.5. *Let $f(q) \in \mathbb{Z}[q]$ such that $f(\omega_n^j) \in \mathbb{Z}$ for all $j = 1, \ldots, n$. Then for each $k = 1, \ldots, n$ we have that*

$$\sum_{j|k} \mu(k/j) f(\omega_n^j) \equiv 0 \pmod{k}.$$

Moreover if $k \not| n$, then $\sum_{j|k} \mu(k/j) f(\omega_n^j) = 0$.

Proof. Let $1 \leq k \leq n$ and write $k = mp^e$ where $p, m \in \mathbb{N}$, p prime and $p \not| m$. By Lemma 2.4 we have

$$\begin{aligned} \sum_{j|k} \mu(k/j) f(\omega_n^j) &= \sum_{j|m} \mu(k/(jp^{e-1})) f(\omega_n^{jp^{e-1}}) + \sum_{j|m} \mu(k/(jp^e)) f(\omega_n^{jp^e}) \\ &\equiv \sum_{j|m} \mu(k/(jp^{e-1})) f(\omega_n^{jp^{e-1}}) + \sum_{j|m} \mu(k/(jp^e)) f(\omega_n^{jp^{e-1}}) \pmod{p^e} \\ &\equiv 0 \pmod{p^e}. \end{aligned}$$

If $k \not| n$, then we may write $k = mp^e$ for some $m, p \in \mathbb{N}$ with p prime such that $p \not| n$. Then by the second assertion in Lemma 2.4 the congruences above hold with equality and we are done. □

Construction 2.6. Let $X = \mathcal{O}_1 \sqcup \mathcal{O}_2 \sqcup \cdots \sqcup \mathcal{O}_m$ be a partition of a finite set X into m parts such that $|\mathcal{O}_i|$ divides n for $i = 1, \ldots, m$. Fix a total ordering on the elements of $\mathcal{O}_i$ for $i = 1, \ldots, m$. Let C_n act on X by permuting each element $x \in \mathcal{O}_i$ cyclically with respect to the total ordering on $\mathcal{O}_i$ for $i = 1, \ldots, m$.

This *ad-hoc* cyclic action in Construction 2.6 lacks combinatorial context and depends only on the choice of partition and total order.

Theorem 2.7. *Let $f(q) \in \mathbb{N}[q]$ and suppose $f(\omega_n^j) \in \mathbb{N}$ for each $j = 1, \ldots, n$. Let X be any set of size $f(1)$. Then there exists an action of C_n on X such that $(X, C_n, f(q))$ exhibits CSP if and only if for each $k|n$,*

$$\sum_{j|k} \mu(k/j) f(\omega_n^j) \geq 0. \tag{2.2}$$

Proof. The forward direction follows from [RSW04, Prop. 4.1]. Conversely if we put

$$S_k = \sum_{j|k} \mu(k/j) f(\omega_n^j) \tag{2.3}$$

for each $k = 1, \ldots, n$ and consider X of size $f(1)$, then by Möbius inversion

$$|X| = f(\omega_n^n) = \sum_{j|n} S_j.$$

Thus by hypothesis and Lemma 2.5, we may partition X into orbits, such that for each $k|n$, there are $\frac{1}{k}S_k$ orbits of size k. We then let C_n act on X as in Construction 2.6. The fixed points of X under $\sigma_n^k \in C_n$ are given by the elements of order dividing k. This gives (by Möbius inversion)

$$|X^{\sigma_n^k}| = \sum_{j|k} S_j = f(\omega_n^k).$$

Hence $(X, C_n, f(q))$ exhibits CSP. □

Remark 2.8. The sums S_k in (2.3) represent the number of elements with order k under the action of C_n.

Example 2.9. The following example demonstrates that even if $f(q) \in \mathbb{N}[q]$ satisfies $f(\omega_n^j) \in \mathbb{N}$ for all $j = 1, \ldots, n$, there might not be an associated cyclic action complementing $f(q)$ to a CSP.

Let $f(q) = q^5 + 3q^3 + q + 10$. Then $f(\omega_6^j)$ takes values $8, 12, 5, 12, 8, 15$ for $j = 1, \ldots, 6$. On the other hand $S_k = \sum_{j|k} \mu(k/j) f(\omega_6^j)$ takes values $8, 4, -3, 0, 0, 6$ for $k = 1, \ldots, 6$. Since we cannot have a negative number of elements of order 3, there is no action of C_6 on a set X of size $f(1) = 15$ such that $(X, C_6, f(q))$ is a CSP-triple.

Rao and Suk [RS17] generalized the notion of cyclic sieving to arbitrary groups with finitely generated representation ring, so called *G-sieving*. In particular, Berget, Eu and Reiner [BER11] considered the case where G is an Abelian group, whence $G \cong C_{n_1} \times \cdots \times C_{n_m}$, acting pointwise on a set $X_1 \times \cdots \times X_m$. Unfortunately G-sieving depends in general on the particular choices of representations ρ_i of G over $\mathbb{C}$ generating the representation ring. However, given the characterization in Theorem 2.3 it would be interesting to understand what conditions are necessary and sufficient for a polynomial $f \in M(n_1, \ldots, n_m)$ to be complemented to a G-sieving phenomenon for an Abelian group $G \cong C_{n_1} \times \cdots \times C_{n_m}$ with respect to the canonical representations sending the generator σ_{n_i} of C_{n_i} to ω_{n_i}.

3. Applications

In this section we demonstrate how one can use Theorem 2.7 to find new cyclic sieving phenomena arising from natural polynomials.

By Theorem 2.7 any polynomial $f(q) \in \mathbb{N}[q]$ such that $f(\omega_n^j) \in \mathbb{N}$ for $j = 1, \ldots, n$ satisfying the positivity condition (2.2), can be completed to a CSP with an ad-hoc cyclic action. Although this action lacks combinatorial context, it often helps to know that a CSP can exist even in principle, particularly if one is considering a combinatorial set where the cyclic action is not immediately apparent. The following example illustrates this point for the polynomial $C_n(q) := \frac{1}{[n+1]_q}\begin{bmatrix}2n\\n\end{bmatrix}_q$ which is generated by statistics on multiple combinatorial (Catalan) objects, but where the naturalness of the action varies depending on the object under consideration.

Example 3.1. Stump [Stu09] showed that $C_n(q) = \sum_{\sigma\in\mathfrak{S}_n(231)} q^{\mathrm{maj}(\sigma)+\mathrm{maj}(\sigma^{-1})}$. There is no obvious natural cyclic action on $\mathfrak{S}_n(231)$ that is compatible with $C_n(q)$. However we can check the positivity condition (2.2) in Theorem 2.7 to reveal that a CSP is nevertheless present for $C_n(q)$ with an ad-hoc cyclic action on $\mathfrak{S}_n(231)$. Indeed rewriting $C_n(q) = \frac{1}{[2n+1]_q}\begin{bmatrix}2n+1\\n+1\end{bmatrix}_q$ and using [RSW04, Prop. 4.2 (iii)] we have for $j|n$,

$$C_n(\omega_n^j) = \begin{cases}\binom{2j}{j}, & \text{if } j < n,\\ \frac{1}{n+1}\binom{2n}{n}, & \text{if } j = n.\end{cases}$$

By Wallis formula, $\prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right) = \frac{2}{\pi}$, the sequences

$$2n\left(\binom{2n}{n}\frac{1}{4^n}\right)^2 = \frac{1}{2}\prod_{j=2}^{n}\left(1+\frac{1}{4j(j-1)}\right),$$

$$(2n+1)\left(\binom{2n}{n}\frac{1}{4^n}\right)^2 = \prod_{j=1}^{n}\left(1-\frac{1}{4j^2}\right)$$

monotonically increase and decrease respectively towards $\frac{2}{\pi}$ as $n\to\infty$. Thus

$$\frac{4^n}{\sqrt{\pi(n+1/2)}} \le \binom{2n}{n} \le \frac{4^n}{\sqrt{\pi n}}.$$

A trivial bound for the number of divisors of n, excluding n, is given by $2\sqrt{n}-1$. Hence for each divisor $k<n$ we have

$$\begin{aligned}\sum_{j|k}\mu(k/j)C_n(\omega_n^j) &= \sum_{j|k}\mu(k/j)\binom{2j}{j}\\ &\ge \binom{2k}{k} - \sum_{\substack{j|k\\ j<k}}\binom{2j}{j}\\ &\ge \frac{4^k}{\sqrt{\pi(k+1/2)}} - (2\sqrt{k}-1)\frac{4^{k/2}}{\sqrt{\pi(k/2)}} \ge 0.\end{aligned}$$

Moreover for $k=n$ we have by a similar calculation that

$$\sum_{j|n}\mu(n/j)C_n(\omega_n^j) \ge \frac{4^n}{(n+1)\sqrt{\pi(n+1/2)}} - (2\sqrt{n}-1)\frac{4^{n/2}}{\sqrt{\pi(n/2)}} \ge 0,$$

for $n\ge 5$. The required inequality can be verified explicitly by hand for $n<5$. Hence $C_n(q)$ exhibits CSP with an ad-hoc cyclic action on $\mathfrak{S}_n(231)$.

With this evidence one could now either proceed to search for a natural cyclic action on $\mathfrak{S}_n(231)$ matching the orbit structure of the ad-hoc cyclic action, or find a natural cyclic action on an object in bijection with $\mathfrak{S}_n(231)$. In this case there happens to exist known candidates *e.g.* the set of Dyck paths $\mathrm{Dyck}(n)$ of semi-length n where C_n acts by changing peaks to valleys (and vice versa) from left to right whenever possible, or the set of triangulation of a regular $(n+2)$-gon where C_{n+2} acts by rotating the triangulation. In the latter case we instead lack a simple natural statistic (as opposed to a natural action) on the set of triangulations that generates $C_n(q)$.

3.1. A new CSP with stretched Schur polynomials.

In this section we conjecture a new cyclic sieving phenomenon involving stretched Schur polynomials. We prove our conjecture in the case of certain rectangular shapes for which it is straightforward to explicitly compute the data needed to verify the positivity condition (2.2) in Theorem 2.7. We begin by recalling the basic definitions required to state the conjecture.

A *partition* $\lambda = (\lambda_1, \dots, \lambda_r)$ is a finite weakly decreasing sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$. The *parts* of λ are the positive entries and the number of positive parts is the *length* of λ, denoted $l(\lambda)$. The quantity $|\lambda| \coloneqq \lambda_1 + \cdots + \lambda_r$ is called the *size* of λ. The *empty partition* $\emptyset$ is the partition with no parts. We use exponents to denote multiplicities *e.g.* $\lambda = (5,3,3,2,1,1,1) = (5,3^2,2,1^3)$. Scalar multiplication on partitions is performed elementwise *e.g.* with $n \in \mathbb{N}$ and λ as above we have $n\lambda = (5n,(3n)^2,2n,n^3)$. If $\mu = (\mu_1,\dots,\mu_r)$ is a partition such that $\lambda_i \geq \mu_i$ for all $i = 1,\dots,r$ then we say that $\mu \subseteq \lambda$. This is called the *inclusion order* on partitions.

Partitions are commonly visualized in at least two different ways. The first and most common way to represent a partition is via its Young diagram. A *skew Young diagram* of *shape* λ/μ is an arrangement of boxes in the plane with coordinates given by $\{(i,j) \in \mathbb{Z}^2 : \mu_i \leq j \leq \lambda_i\}$. The first coordinate represents the row and the second coordinate the column. If $\mu = \emptyset$, then we simply write λ instead of λ/μ and refer to the corresponding skew Young diagram as the *(regular) Young diagram* of λ. A *border strip* (or *rim hook*) of size d is a connected skew Young diagram consisting of d boxes and containing no 2×2 square. The *height* of a border strip is one less than its number of rows. A *border strip tableau* of shape λ/μ and type $\alpha = (\alpha_1,\dots,\alpha_d)$ is a sequence $\mu = \lambda^1 \subset \lambda^2 \subset \cdots \subset \lambda^r = \lambda$ such that λ^i/λ^{i-1} is a border strip of size α_i.

A second way to visually represent a partition λ is via an *abacus* with $m \geq r$ beads: Let $d \in \mathbb{N}$. For $i = 1,\dots,m$, write $\lambda_i + m - i = s + dt$, with $0 \leq s \leq d-1$, and place a bead on the s^{th} runner in the t^{th} row. The operation of sliding a bead one row upwards on its runner into a vacant position corresponds to removing a border strip of size d from λ. Sliding all beads up as far as possible produces an abacus representation of the *d-core* partition of λ, a partition from which no further border strip tableaux of size d can be removed. It is worth mentioning that the d-core of λ is independent of the way in which border strip tableaux are removed. For $i = 0,1,\dots,d-1$, let $\lambda_j^{(i)}$ be the number of unoccupied positions on the i^{th} runner above the j^{th} bead from the bottom. Then $\lambda^i = (\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_d^{(i)})$ is a partition and the d-tuple $[\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(d-1)}]$ is called the *d-quotient* of λ.

A *semi-standard Young tableau (SSYT)* is a Young diagram whose boxes are filled with non-negative integers, such that each row is weakly increasing and each column is strictly increasing. Denote the set of SSYT of shape λ with entries in $\{0,\dots,m-1\}$ by $\mathrm{SSYT}(\lambda,m)$. Given $T \in \mathrm{SSYT}(\lambda,m)$, the *type* of T is the vector

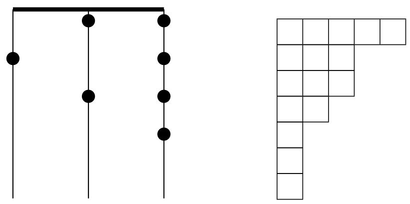

FIGURE 1. The abacus representation of $\lambda = (5, 3^2, 2, 1^3)$ with $m = 7$ beads and $d = 3$ runners, next to the Young diagram representation of λ.

$\alpha(T) = (\alpha_0(T), \alpha_1(T), \ldots, \alpha_{m-1}(T))$ where $\alpha_k(T)$ counts the number of boxes of T containing the number k.

The *Schur polynomial* is defined as

$$s_\lambda(x_0, \ldots, x_{m-1}) = \sum_{T \in \mathrm{SSYT}(\lambda,m)} x_0^{\alpha_0(T)} x_1^{\alpha_1(T)} \cdots x_{m-1}^{\alpha_{m-1}(T)}.$$

The polynomial $s_\lambda(x_0, \ldots, x_{m-1})$ is symmetric and has several alternative definitions, see [Sta99]. The *principal specialization* of $s_\lambda(x_0, \ldots, x_{m-1})$ is given by

$$s_\lambda(1, q, q^2, \ldots, q^{m-1}) = \sum_{T \in \mathrm{SSYT}(\lambda,m)} q^{|T|},$$

where $|T|$ denotes the sum of all entries in T. The following explicit formula is referred to as the *q-hook-content formula* and is due to Stanley (see [Sta99, Thm 7.21.2]),

$$s_\lambda(1, q, q^2, \ldots, q^{m-1}) = q^{b(\lambda)} \prod_{(i,j)\in\lambda} \frac{[m + c_{i,j}]_q}{[h_{i,j}]_q}, \tag{3.1}$$

where $b(\lambda) = \sum_{i=1}^{r}(i-1)\lambda_i$, $c_{i,j} = j - i$ (the *content*) and $h_{i,j}$ is defined as the number of boxes in λ to the right of (i, j) in row i plus the number of boxes below (i, j) in column j plus 1 (the *hook length*). In particular

$$|\mathrm{SSYT}(\lambda, m)| = s_\lambda(1^m) = \prod_{(i,j)\in\lambda} \frac{m + c_{i,j}}{h_{i,j}}. \tag{3.2}$$

If G is a group and V a (finite-dimensional) vector space over $\mathbb{C}$, then a *representation* of G is a group homomorphism $\rho : G \to \mathrm{GL}(V)$ where $\mathrm{GL}(V)$ is the group of invertible linear transformations of V. A representation $\rho : G \to \mathrm{GL}(V)$ is *irreducible* if it has no proper subrepresentation $\rho|_W : G \to \mathrm{GL}(W)$, $0 < W < V$ closed under the action of $\{\rho(g) : g \in G\}$. The *character* of G on V is a function $\chi : G \to \mathbb{C}$ defined by $\chi(g) = \mathrm{tr}(\rho(g))$. Note that characters are invariant under conjugation by G. A character χ is said to be *irreducible* if the underlying representation is irreducible. If $G = \mathfrak{S}_m$, then the irreducible characters χ^λ of $\mathfrak{S}_m$ are indexed by partitions λ of weight m and may be computed combinatorially (on each conjugacy class of type α in $\mathfrak{S}_m$) using the *Murnaghan–Nakayama rule* [Sta99, Thm 7.17.3]

$$\chi^\lambda_\alpha = \sum_{T \in \mathrm{BST}(\lambda,\alpha)} (-1)^{\mathrm{ht}(T)}, \tag{3.3}$$

where the sum runs over all border strip tableaux $\mathrm{BST}(\lambda, \alpha)$ of shape λ and type α and $\mathrm{ht}(T)$ is the sum of all heights of the border strips in T. In particular this implies χ^λ takes integer values.

The following theorem provides an expression for the root of unity evaluation of the principal specialization $s_\lambda(1, q, \ldots, q^{m-1})$.

Theorem 3.2 (Reiner–Stanton–White [RSW04]). *Let $d|m$ and ω_d be a primitive d^{th} root of unity. Then $s_\lambda(1, \omega_d, \ldots, \omega_d^{m-1})$ is zero unless the d-core of λ is empty, in which case*

$$\mathrm{s}_\lambda(1, \omega_d, \omega_d^2, \ldots, \omega_d^{m-1}) = sgn(\chi^\lambda_{d^{|\lambda|/d}}) \prod_{i=0}^{d-1} \mathrm{s}_{\lambda^{(i)}}(1^{m/d}),$$

where χ^λ is the irreducible character of the symmetric group $\mathfrak{S}_{|\lambda|}$ indexed by λ.

Lemma 3.3. *Suppose ω_d is a primitive d^{th} root of unity with $d|m$, then*

$$\mathrm{s}_{n\lambda}(1, \omega_d, \omega_d^2, \ldots, \omega_d^{m-1}) = \prod_{i=0}^{d-1} \mathrm{s}_{(n\lambda)^{(i)}}(1^{m/d}) \in \mathbb{N}.$$

Proof. If d does not divide $n|\lambda|$, then $\mathrm{s}_{n\lambda}(1, \omega_d, \omega_d^2, \ldots, \omega_d^{m-1}) = 0$ by Theorem 3.2, so there is nothing to prove. Thus we may assume d divides $n|\lambda|$. By Theorem 3.2 we only need to verify that $\chi^{n\lambda}_{d^{n|\lambda|/d}} \geq 0$. A result by White [Whi83, Cor. 10] (see also [Pak00, Thm. 3.3]), implies that the Murnaghan–Nakayama rule (3.3) is cancellation-free in this instance. Furthermore, it is clear that there is a border-strip tableau of shape $n\lambda$ with border-strips of size d with positive sign. For example, take all strips to be horizontal — this is possible since $d|n$. □

We are now ready to state our conjecture.

Conjecture 3.4. *Let $n, m \in \mathbb{N}$ and let λ be a partition. Then the triple*

$$(\mathrm{SSYT}(n\lambda, m), C_n, \mathrm{s}_{n\lambda}(1, q, q^2, \ldots, q^{m-1}))$$

exhibits a CSP for some C_n acting on $\mathrm{SSYT}(n\lambda, m)$.

We believe that a natural action is realized by some type of promotion on semi-standard Young tableaux similar to [Rho10]. In the case $\lambda = (1)$ we have

$$\mathrm{s}_{n\lambda}(1, q, q^2, \ldots, q^m) = \begin{bmatrix} n+m-1 \\ n \end{bmatrix}_q$$

and this polynomial exhibits a cyclic sieving phenomenon under C_n, see [RSW04].

We have verified Conjecture 3.4 using Theorem 2.7 for all partitions λ such that $|\lambda| \leq 6$, all $m \leq 6$ and all $n \leq 12$.

Below we prove the conjecture for certain rectangular shapes λ.

Lemma 3.5. *The n-quotient of the rectangular shape $(na)^{nb+r}$ with $0 \leq r < n$ is given by*

$$[\underbrace{a^b, a^b, \ldots, a^b}_{n-r \text{ times}}, \underbrace{a^{b+1}, a^{b+1}, \ldots, a^{b+1}}_{r \text{ times}}].$$

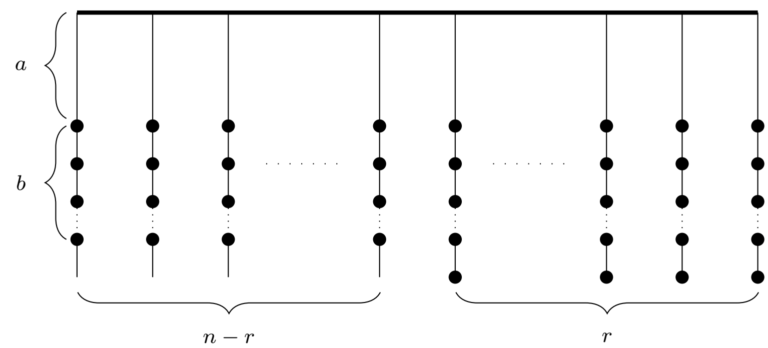

FIGURE 2. The abacus representation of $\lambda = (na)^{nb+r}$ with $m = nb + r$ beads and $d = n$ runners.

Proof. The abacus representation of $\lambda = (na)^{nb+r}$ with $m = nb+r$ beads and $d = n$ runners is given via

$$na + (nb + r) - i = s + nt,$$

for $i = 1, \dots, nb + r$ where $0 \le s \le n - 1$, see Figure 2. Thus we see that each of the n runners have no bead in the first a rows. Since all parts of λ are the same, we also note that the $nb + r$ beads are distributed evenly from right to left on the n runners with no vacant positions in between the beads on each runner. Thus there are b beads on the first $n - r$ runners and $b + 1$ beads on the last r runners. Moreover each bead have exactly a vacant positions above it on its runner, so the n-quotient is given as in the lemma. □

Lemma 3.6. *We have*

$$\mathrm{s}_{(a^b)}(1^m) = \prod_{j=0}^{a-1} \binom{m+j}{b} \binom{b+j}{b}^{-1}$$

Proof. By the hook-content formula (3.2) we have

$$s_{(a^b)}(1^m) = \prod_{(i,j)\in(a^b)} \frac{m+j-i}{(a-j)+(b-i)+1},$$

which after rearrangement equals

$$\prod_{j=0}^{a-1}\prod_{i=0}^{b-1} \frac{m+j-i}{b+j-i} = \prod_{j=0}^{a-1} \frac{(m+j)!}{(m-b+j)!} \frac{j!}{(b+j)!} = \prod_{j=0}^{a-1} \binom{m+j}{b} \binom{b+j}{b}^{-1}.$$

□

Theorem 3.7. *Let $n, m, a, b \in \mathbb{N}$ with $b < m$ and $n|b, m$. If $\lambda = (a^b)$, then the triple*

$$(\mathrm{SSYT}(n\lambda, m), C_n, s_{n\lambda}(1, q, q^2, \dots, q^{m-1}))$$

exhibits a CSP for some C_n acting on $\mathrm{SSYT}(\lambda, m)$*.*

Proof. By Lemma 3.3 it follows that $s_{n\lambda}(1, \omega_n^j, \omega_n^{2j}, \dots, \omega_n^{(m-1)j}) \in \mathbb{N}$ for all $j = 1, \dots, n$. By Theorem 2.7 it therefore remains to show that for all $k|n$,

$$\sum_{j|k} \mu(k/j) s_{n\lambda}(1, \omega_n^j, \omega_n^{2j}, \dots, \omega_n^{(m-1)j}) \geq 0. \tag{3.4}$$

Note that ω_n^j is a $(n/j)^{th}$ root of unity. By Lemma 3.3 and Lemma 3.5 the left hand side of (3.4) rewrites as

$$\sum_{j|k} \mu(k/j) \prod_{i=0}^{n/j-1} \underbrace{s_{(ja)^{bj/n}}(1^{mj/n})}_{\text{independent of } i} = \sum_{j|k} \mu(k/j) \left(s_{(ja)^{bj/n}}(1^{mj/n}) \right)^{n/j}. \tag{3.5}$$

Using Lemma 3.6, this equals

$$\sum_{j|k} \mu(k/j) \left(\prod_{i=0}^{ja-1} \binom{mj/n + i}{bj/n} \binom{bj/n + i}{bj/n}^{-1} \right)^{n/j}, \tag{3.6}$$

which is greater or equal to

$$\left(\prod_{i=0}^{ka-1} \binom{mk/n + i}{bk/n} \binom{bk/n + i}{bk/n}^{-1} \right)^{\frac{n}{k}} - \sum_{\substack{j|k \\ j<k}} \left(\prod_{i=0}^{ja-1} \binom{mj/n + i}{bj/n} \binom{bj/n + i}{bj/n}^{-1} \right)^{\frac{n}{j}}. \tag{3.7}$$

By Lemma 8.4 and the fact that the number of divisors of k, excluding k, is bounded above by $2\sqrt{k} - 1$ we get that (3.7) is greater than or equal to

$$\left(\prod_{i=k'a}^{ka-1} \frac{\binom{mk/n+i}{bk/n}^{n/k}}{\binom{bk/n+i}{bk/n}^{n/k}} - (2\sqrt{k} - 1) \right) \left(\prod_{i=0}^{k'a-1} \frac{\binom{mk/n+i}{bk/n}^{n/k}}{\binom{bk/n+i}{bk/n}^{n/k}} - \prod_{i=0}^{k'a-1} \frac{\binom{mk'/n+i}{bk'/n}^{n/k'}}{\binom{bk'/n+i}{bk'/n}^{n/k'}} \right), \tag{3.8}$$

where $k' = \lfloor k/2 \rfloor$. The remaining steps needed are given in the appendix Section 8, where it is shown that the left factor in (3.8) is non-negative by Lemma 8.6 and the right factor is non-negative by Lemma 8.4 for all $k|n$. This concludes the proof of the theorem. □

4. The CSP cone

In the following sections we offer a geometric perspective on the cyclic sieving phenomenon by associating a polyhedral cone that captures joint information about the cyclic action and statistics on the object X. The cone has the property that all cyclic sieving phenomena with a polynomial generated by a choice of statistic (modulo n) on the set X corresponds to a lattice point in the cone.

As presented in the introduction, the polynomial $f(q)$ is often given by some natural statistic $\tau : X \to \mathbb{N}$ on X. Define

$$f_\tau(q) \coloneqq \sum_{x \in X} q^{\tau(x)}.$$

Moreover for each $n \in \mathbb{N}$, define $\tau_n : X \to \mathbb{Z}_n$ by

$$\tau_n(x) \coloneqq \tau(x) \pmod{n}.$$

More than understanding the individual components of the CSP triple $(X, C_n, f_\tau(q))$, one is also interested in the behaviour and distribution of the statistic τ with respect to the cyclic action. Given an action of C_n on X and a statistic $\tau : X \to \mathbb{N}$, we can associate a $n \times n$ matrix $A_{(X,C_n,\tau)} = (a_{ij})$ which keeps track of the coefficients of the generating function

$$\sum_{x\in X} q^{\tau_n(x)} t^{o(x)} := \sum_{i=0}^{n-1} \sum_{j=1}^{n} a_{ij} q^i t^j,$$

where $o(x) := \min\{j \in [n] : \sigma_n^j \cdot x = x\}$ denotes the order of $x \in X$ under C_n. We remark that the rows of $A_{(X,C_n,\tau)}$ are indexed from 0 to $n-1$.

We can now restate CSP as follows:

Proposition 4.1. *Suppose X is a finite set on which C_n acts and let $\tau : X \to \mathbb{N}$ be a statistic. Then the triple $(X, C_n, f_\tau(q))$ exhibits CSP if and only if $A_{(X,C_n,\tau)} = (a_{ij})$ satisfies the condition that for each $1 \le k \le n$,*

$$\sum_{\substack{0\le i<n\\1\le j\le n}} a_{ij}\omega_n^{ki} = \sum_{0\le i<n} \sum_{j|k} a_{ij}. \tag{4.1}$$

where ω_n is a primitive nth root of unity.

Proof. For each $1 \le k \le n$ we have that

$$\begin{aligned} X^{\sigma_n^k} &= \bigcup_{i=0}^{n-1} \{x \in X : \tau_n(x) = i,\ \sigma_n^k \cdot x = x\} \\ &= \bigcup_{i=0}^{n-1} \bigcup_{j|k} \{x \in X : \tau_n(x) = i,\ o(x) = j\}. \end{aligned}$$

Hence $(X, C_n, f_\tau(q))$ exhibits CSP if and only if for each $1 \le k \le n$,

$$\sum_{\substack{0\le i<n\\1\le j\le n}} a_{ij}\omega_n^{ki} = f_{\tau_n}(\omega_n^k) = |X^{\sigma_n^k}| = \sum_{0\le i<n} \sum_{j|k} a_{ij}. \tag{4.2}$$

□

This motivates the following definition.

Definition 4.2. A $n \times n$-matrix $A = (a_{ij}) \in \mathbb{R}_{\ge 0}^{n\times n}$ is called a *CSP-matrix* if it fulfills the conditions in Equation (4.1). Let $\mathrm{CSP}(n)$ denote the set of all $n \times n$ CSP-matrices and $\mathrm{CSP}_{\mathbb{Z}}(n) := \mathrm{CSP}(n) \cap \mathbb{Z}^{n\times n}$ the set of integer CSP-matrices.

Example 4.3. Consider all binary words of length 6, with group action being shift by 1 and τ being the the major index statistic. Then

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 11 \\ 0 & 0 & 2 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 11 \\ 0 & 1 & 2 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 11 \\ 0 & 0 & 2 & 0 & 0 & 7 \end{pmatrix}$$

is the corresponding CSP matrix. The entry in the upper left hand corner correspond to the two binary words 000000 and 111111. These have major index 0 and are fixed under a single shift. The words corresponding to the second column are 010101 and 101010. These have major index $6 \equiv 0 \pmod 6$ and $9 \equiv 3 \pmod 6$ respectively and are fixed under two consecutive shifts etc.

By linearity of the CSP-condition (4.1), it follows that for all $A, B \in \mathrm{CSP}(n)$ we have $sA + tB \in \mathrm{CSP}(n)$ for any $s, t \geq 0$. Hence $\mathrm{CSP}(n)$ forms a real convex cone. In fact by Theorem 7.1 in Section 7 we have the following corollary.

Corollary 4.4. *The set* $\mathrm{CSP}(n)$ *forms a real convex rational polyhedral cone.*

5. General properties of the CSP cone

Since $\mathrm{CSP}(n)$ is a rational cone by Corollary 4.4, its extreme rays are spanned by integer matrices. Every element in $\mathrm{CSP}(n)$ is therefore a conic combination of elements in $\mathrm{CSP}_{\mathbb{Z}}(n)$. In particular, properties of $\mathrm{CSP}_{\mathbb{Z}}(n)$ closed under conic combinations can be lifted to $\mathrm{CSP}(n)$.

A priori an integer lattice point $A \in \mathrm{CSP}_{\mathbb{Z}}(n)$ need not be realizable by a cyclic sieving phenomenon with CSP-matrix A. However thanks to Lemma 5.1 we shall see that this property does indeed hold.

Lemma 5.1. *Let* $A = (a_{ij}) \in \mathrm{CSP}_{\mathbb{Z}}(n)$*. Then there exists a CSP-triple* (X, C_n, τ) *with* $A_{(X,C_n,\tau)} = A$*.*

Proof. According to (4.1), the polynomial $f(q) = \sum_{i=0}^{n-1} r_i q^i$ where $r_i = \sum_{j=1}^{n} a_{ij}$ for $i = 0, \dots, n-1$ defines a polynomial such that $f(\omega_n^k) = \sum_{j|k} S_j \in \mathbb{N}$ for $k = 1, \dots, n$, where $S_j = \sum_{i=0}^{n-1} a_{ij}$ for $j = 1, \dots, n$. By Möbius inversion as in Theorem 2.7 we have $S_k = \sum_{j|k} \mu(k/j) f(\omega_n^j)$. Hence by Lemma 2.5, $k|S_k$. Therefore a CSP-instance having CSP-matrix A can be realized through any triple (X, C_n, τ) with C_n acting in an ad-hoc manner on a set X with $\sum_{i,j} a_{ij}$ elements divided into S_k/k orbits of size k for each $k|n$ where $\tau : X \to \mathbb{N}$ is any statistic distributed according to A. □

Let $\{E_{ij} : 0 \leq i < n, 1 \leq j \leq n\}$ denote the standard basis of $\mathbb{R}^{n \times n}$.

Definition 5.2. Call a matrix $\delta_a(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{n \times n}$ a *swap* if

$$\delta_a(\mathbf{u}, \mathbf{v}) \coloneqq a(E_{u_1 u_2} + E_{v_1 v_2} - E_{v_1 u_2} - E_{u_1 v_2}),$$

where $a \in \mathbb{R}$.

Lemma 5.3. *Let* $A \in \mathrm{CSP}(n)$ *and suppose* $\delta_a(\mathbf{u}, \mathbf{v}) + A \in \mathbb{R}_{\geq 0}^{n \times n}$*. Then* $\delta_a(\mathbf{u}, \mathbf{v}) + A \in \mathrm{CSP}(n)$*.*

Proof. Since adding $\delta_a(\mathbf{u}, \mathbf{v})$ does not alter column nor row-sums we have that the CSP-condition (4.1) remains intact. Hence $\delta_a(\mathbf{u}, \mathbf{v}) + A \in \mathrm{CSP}(n)$. □

The next lemma follows by repeated applications of Lemma 5.3.

Lemma 5.4. *Let $A = (a_{ij}) \in \mathrm{CSP}(n)$. Suppose i and i' are two row indices such that $\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{i'j}$. If A' is the matrix obtained from A by interchanging rows i and i', then $A' \in \mathrm{CSP}(n)$.*

Remark 5.5. The corresponding statement of Lemma 5.4 also holds for the column indices instead of row indices.

Proposition 5.6. *Let $n \in \mathbb{N}$ and suppose i and i' are row indices such that $\gcd(n,i) = \gcd(n,i')$. If $A \in \mathrm{CSP}(n)$, then $A' \in \mathrm{CSP}(n)$ where A' is obtained from A by interchanging rows i and i'.*

Proof. Let $A \in \mathrm{CSP}_{\mathbb{Z}}(n)$. Then the polynomial $f(q) = \sum_{i=0}^{n-1} c_i q^i \in \mathbb{N}[q]$, where $c_i = \sum_{j=1}^{n} a_{ij}$, satisfies $f(\omega_n^j) \in \mathbb{N}$ for all $j = 1, \ldots, n$. By Lemma 2.2 it follows that $c_{i \pmod n} = c_{\gcd(n,i)}$ for all $i = 1, \ldots, n$. Hence $A' \in \mathrm{CSP}_{\mathbb{Z}}(n)$ by Lemma 5.4. Moreover from above, row i and i' clearly have the same row sum in $sA + tB$ for any $A, B \in \mathrm{CSP}_{\mathbb{Z}}(n)$ and $s, t \geq 0$. Hence the property can be lifted to all matrices in $\mathrm{CSP}(n)$. □

6. The universal CSP cone

Let W_α be the set of words with content α, that is, α_i is the number of occurrences of the letter i in the words, and let n be the length of the words. Then C_n acts on such words by cyclic shift. In [AS17], the authors construct a statistic, $\mathrm{flex}(\cdot)$, which is equidistributed modulo n with major index on W_α. Furthermore, flex has the property that for every orbit $\mathcal{O}$, the triple $(\mathcal{O}, C_n, \mathrm{flex})$ exhibits the cyclic sieving phenomenon. They show that flex is universal in the following sense:

Definition 6.1. A cyclic sieving phenomena (X, C_n, τ) is called *universal* if $(\mathcal{O}, C_n, \tau)$ exhibits the cyclic sieving phenomenon for every orbit C_n-orbit $\mathcal{O}$ of X. This is shown in [AS17] to be equivalent with the property that for every C_n-orbit $\mathcal{O} \subseteq X$ with length k, the sets

$$\{\tau_n(x) : x \in \mathcal{O}\} \text{ and } \left\{0, \frac{n}{k}, \frac{2n}{k}, \ldots, \frac{(k-1)n}{k}\right\}$$

coincide. In other words, the statistic τ is "evenly distributed" on each C_n-orbit modulo n. We also refer to τ as being a universal statistic (with respect to X and C_n).

Clearly a universal statistic is uniquely determined modulo n by the orbit structure of X under C_n (up to a choice of total order on the orbits). We remark that most cyclic sieving phenomena in the literature are *not* universal. We shall see below how a non-universal statistic can be turned into a universal one without changing the generating polynomial.

Definition 6.2. A matrix $A = (a_{ij}) \in \mathrm{CSP}(n)$ is called *universal* if there are constants $K_1, \ldots, K_n \in \mathbb{R}_{\geq 0}$ such that

$$a_{ij} = \begin{cases} K_j, & \text{if } i \equiv 0 \pmod{\frac{n}{j}}, \\ 0, & \text{otherwise.} \end{cases}$$

for all $1 \leq i, j \leq n$. Let $\widetilde{\mathrm{CSP}}(n)$ denote the subset of all universal CSP-matrices. Moreover if $\mathbf{s} = (S_1, \dots, S_n) \in \mathbb{N}^n$ is a sequence such that $j | S_j$ for $j = 1, \dots, n$ and $S_j = 0$ for $j \not| n$, then we let $U(\mathbf{s}) \in \widetilde{\mathrm{CSP}}(n)$ denote the unique universal CSP-matrix with column sums given by $S_1, \dots, S_n$.

Remark 6.3. Note that $\widetilde{\mathrm{CSP}}(n)$ forms a subcone of $\mathrm{CSP}(n)$ and that the lattice points $\widetilde{\mathrm{CSP}}_{\mathbb{Z}}(n)$ are realized by universal cyclic sieving phenomena.

Every CSP-matrix can be linearly projected onto a universal CSP-matrix. Indeed the map

$$P : \mathrm{CSP}(n) \to \widetilde{\mathrm{CSP}}(n)$$
$$a_{ij} \mapsto \begin{cases} \frac{1}{j} \sum_{i=1}^{n} a_{ij}, & \text{if } i \equiv 0 \pmod{\frac{n}{j}}, \\ 0, & \text{otherwise,} \end{cases}$$

is clearly linear in each entry with $P^2 = P$. By Proposition 5.1 the projection P restricts to a map $P : \mathrm{CSP}_{\mathbb{Z}}(n) \to \widetilde{\mathrm{CSP}}_{\mathbb{Z}}(n)$.

If $A \in \mathrm{CSP}_{\mathbb{Z}}(n)$, then $\delta_1(\mathbf{u}, \mathbf{v}) + A$ corresponds to swapping statistic between two elements belonging to orbits of different size.

Next we show that every CSP matrix $A \in \mathrm{CSP}_{\mathbb{Z}}(n)$ can be obtained from a universal CSP-matrix with the same column sums via a sequence of such swaps while keeping inside $\mathrm{CSP}_{\mathbb{Z}}(n)$. We prove this fact by showing a slightly more general result over the class of non-negative integer matrices with matching row and column sums.

Proposition 6.4. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be integer $n \times n$ matrices with non-negative entries having matching row and column sums i.e. $\sum_{i=1}^{n} a_{ij_0} = \sum_{i=1}^{n} b_{ij_0}$ and $\sum_{j=1}^{n} a_{i_0 j} = \sum_{j=1}^{n} b_{i_0 j}$ for $1 \leq i_0, j_0 \leq n$. Then there exists swaps $\delta_1(\mathbf{u}_r, \mathbf{v}_r)$ for $r = 1, \dots, t$ such that*

$$A = B + \sum_{r=1}^{t} \delta_1(\mathbf{u}_r, \mathbf{v}_r). \tag{6.1}$$

Moreover the swaps $\delta_1(\mathbf{u}_r, \mathbf{v}_r)$ can be chosen such that $B + \sum_{r=1}^{t_0} \delta_1(\mathbf{u}_r, \mathbf{v}_r)$ has non-negative entries for all $1 \leq t_0 \leq t$.

Proof. Define $\Delta(A)$ to be the quantity

$$\Delta(A) := ||A - B||$$

where $||A|| = \sum_{i,j} |a_{ij}|$. We say that an entry a_{ij} is in *deficit* if $a_{ij} < b_{ij}$ and in *surplus* if $a_{ij} > b_{ij}$. We argue by induction on $\Delta(A)$. If $\Delta(A) = 0$, then clearly $A = B$ since A and B both have non-negative entries. Suppose $\Delta(A) > 0$. Then there exists indices i and j such that $a_{ij} - b_{ij} \neq 0$. If a_{ij} is in surplus, then there must exists some row index i' such that $a_{i'j}$ is in deficit, otherwise the sum of column j in A is strictly greater than sum of column j in B which leads to a contradiction. Therefore we may assume a_{ij} is in deficit. Since a_{ij} is in deficit there exists $j' \neq j$ such that $a_{ij'}$ is in surplus, otherwise the sum of row i in B is strictly greater than the sum of row i in A. Similarly, there exists a row index $i' \neq i$ such that $a_{i'j}$ is in surplus. It follows that

$$A' := A - \delta_1((i, j'), (i', j))$$

has non-negative entries by construction with row and column sums matching that of A (and hence that of B). Moreover

$$\Delta(A') = \begin{cases} \Delta(A) - 4, & \text{if } a_{i'j'} \text{ is in deficit,} \\ \Delta(A) - 2, & \text{otherwise.} \end{cases}.$$

Hence by induction

$$\begin{aligned} A &= A' + \delta_1((i,j'),(i',j)) \\ &= B + \sum_{r=1}^{t} \delta(\mathbf{u}_r, \mathbf{v}_r) + \delta_1((i,j'),(i',j)). \end{aligned}$$

□

Corollary 6.5. *Let $A = (a_{ij}) \in \mathrm{CSP}_{\mathbb{Z}}(n)$. Write $S_j = \sum_{i=0}^{n-1} a_{ij}$ for the column sums of A for $j = 1, \dots, n$ and set $\mathbf{s} = (S_1, \dots, S_n)$. Then there exists swaps $\delta_1(\mathbf{u}_r, \mathbf{v}_r)$ for $r = 1, \dots, t$ such that*

$$A = U(\mathbf{s}) + \sum_{r=1}^{t} \delta_1(\mathbf{u}_r, \mathbf{v}_r). \tag{6.2}$$

Moreover $U(\mathbf{s}) + \sum_{r=1}^{t_0} \delta_1(\mathbf{u}_r, \mathbf{v}_r) \in \mathrm{CSP}_{\mathbb{Z}}(n)$ for all $1 \le t_0 \le t$.

Proof. Let $R_i = \sum_{j=1}^{n} a_{ij}$ denote the row sums of A for $i = 0, 1, \dots, n-1$. Note that the row sums of A are determined uniquely by the column sums of A via

$$R_i = \sum_{j:\frac{n}{j}|i} \frac{1}{j} S_j, \quad \text{for } i = 0, \dots, n-1,$$

since both sides count the number of orbits whose stabilizer-order divides i in the corresponding CSP-instance, according to (1.2) and Remark 2.8. Since A and $U(\mathbf{s})$ have the same column sums they must therefore have the same row sums. The corollary now follows from Proposition 6.4 and Lemma 5.3. □

Remark 6.6. Proposition 6.4 shows that every $A \in \mathrm{CSP}_{\mathbb{Z}}(n)$ can be uniquely expressed as $U(\mathbf{s}) + B$ where $B = (b_{ij}) \in \mathbb{Z}^{n \times n}$ is a matrix with zero row and column-sums and non-negative values in all entries $b_{k\ell}$ unless $(k, \ell) = (\frac{ni}{j}, j)$ where $0 \le i < j$ and $j|n$.

Construction 6.7. If $(C_m, X, f(q))$ and $(C_n, Y, g(q))$ are two CSP-triples, then we can construct a new CSP-triple of the form $(C_{mn}, X \times Y, h(q))$ where $h(q)$ is a polynomial of degree less than mn which may be expressed as certain convolution of f and g.

Let $(x, y) \in X \times Y$ and suppose $o(x) = i$, $o(y) = j$ with respect to the actions of C_m on X and C_n on Y respectively. Let C_{mn} act on (x, y) via

$$\sigma_{mn}^{is+t} \cdot (x, y) \coloneqq (\sigma_m^t \cdot x, \sigma_n^s \cdot y)$$

where $0 \le t < i$ and $s \in \mathbb{Z}$. Note that (x, y) has order ij under the above action. By Remark 2.8, the number of elements of order i and j with respect to the actions of C_m on X and C_n on Y are given respectively by

$$S_i = \sum_{\ell | i} \mu(\ell/i) f(\omega_m^\ell), \quad T_j = \sum_{\ell | j} \mu(\ell/j) g(\omega_n^\ell).$$

Therefore the action of C_{mn} on $X \times Y$ has

$$\sum_{ij=k} S_i T_j,$$

elements of order k. By (1.2) the coefficients c_r of the unique polynomial $h(q) = \sum_{r=0}^{mn-1} c_r q^r \pmod{q^{mn}-1}$ complementing the action of C_{mn} on $X \times Y$ to a CSP is given by the number of orbits whose stabilizer-order divides r, that is,

$$c_r = \sum_{k:\frac{mn}{k}|r} \sum_{ij=k} \frac{1}{k} S_i T_j.$$

The above construction gives rise to a natural product on universal CSP-matrices. Given a vector $\mathbf{s} = (S_d)$, we define its *number-theoretical series* as the formal power-series

$$NS(\mathbf{s}) \coloneqq \sum_{1 \leq d} S_d x_{p_1}^{e_1} \dots x_{p_\ell}^{e_\ell} \tag{6.3}$$

where $d = p_1^{e_1} \dots p_\ell^{e_\ell}$ is the prime factorization of d.

Given two vectors $\mathbf{s}$ and $\mathbf{t}$ of length m and n, respectively, define the vector $\mathbf{s} \boxtimes \mathbf{t}$ of length mn via the identity

$$NS(\mathbf{s} \boxtimes \mathbf{t}) = NS(\mathbf{s}) \cdot NS(\mathbf{t}).$$

In other words, coordinate k in $\mathbf{s} \boxtimes \mathbf{t}$ is given by $\sum S_i T_j$, where the sum ranges over all natural numbers i, j such that $ij = k$. Note that $\boxtimes$ is symmetric and transitive, and $|\mathbf{s} \boxtimes \mathbf{t}| = |\mathbf{s}| \cdot |\mathbf{t}|$ where $|\cdot|$ denotes the sum of the entries.

Proposition 6.8. *Let $U(\mathbf{s}) \in \widetilde{\mathrm{CSP}}(m)$ and $U(\mathbf{t}) \in \widetilde{\mathrm{CSP}}(n)$. Then*

$$U(\mathbf{s}) \boxtimes U(\mathbf{t}) \coloneqq U(\mathbf{s} \boxtimes \mathbf{t}) \in \widetilde{\mathrm{CSP}}(mn).$$

Proof. We have that $i|S_i$ and $j|T_j$ for $i = 1, \dots, m$, $j = 1, \dots, n$ and $S_i, T_j = 0$ for $i \not| m$, $j \not| n$. It follows that

$$(\mathbf{s} \boxtimes \mathbf{t})_k = \sum_{\substack{ij=k \\ i|m, j|n}} S_i T_j,$$

with $k|(\mathbf{s} \boxtimes \mathbf{t})_k$ for $k = 1, \dots, mn$ and $(\mathbf{s} \boxtimes \mathbf{t})_k = 0$ if $k \not| mn$. □

7. Geometry of the CSP cone

The below theorem provides the half-space description of $\mathrm{CSP}(n)$, showing that it is indeed a rational convex polyhedral cone.

Theorem 7.1. *Let $n \in \mathbb{N} \setminus \{0\}$ and $A = (a_{ij}) \in \mathbb{R}^{n \times n}$. Let the divisors of n be given by*

$$1 = c_1 < c_2 < \cdots < c_d = n.$$

Let

$$H_k(\mathbf{x}) \coloneqq \sum_{i=0}^{n-1} \sum_{j=2}^{d} \alpha_{ijk} x_{ij} \in \mathbb{Z}[\mathbf{x}],$$

where

$$\alpha_{ijk} \coloneqq \begin{cases} -n + \frac{n}{c_j}, & \text{if } i = k \text{ and } k \equiv 0 \pmod{\frac{n}{c_j}}, \\ -n & \text{if } i = k \text{ and } k \not\equiv 0 \pmod{\frac{n}{c_j}}, \\ \frac{n}{c_j}, & \text{if } i \neq k \text{ and } k \equiv 0 \pmod{\frac{n}{c_j}}, \\ 0 & \text{if } i \neq k \text{ and } k \not\equiv 0 \pmod{\frac{n}{c_j}}. \end{cases}$$

Then A is a CSP matrix if and only if

$$A = (\mathbf{a}_1|\mathbf{a}_2|\cdots|\mathbf{a}_n),$$

where

$$\mathbf{a}_1 = (x_{01}, H_1(\mathbf{x}), \ldots, H_{n-1}(\mathbf{x}))^t,$$
$$\mathbf{a}_c = \begin{cases} (nx_{0c}, nx_{1c}, \ldots, nx_{(n-1)c})^t, & \text{if } c|n, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

for $c = 2, \ldots, n$ with $H_k(\mathbf{x}) \geq 0$ and $x_{ij} \geq 0$ for all i, j, k.

Proof. For $\mathbf{z} \in \mathbb{C}^{n-1}$, let

$$V(\mathbf{z}) \coloneqq \begin{pmatrix} z_1 & z_1^2 & \ldots & z_1^{n-1} \\ z_2 & z_2^2 & \ldots & z_2^{n-1} \\ \vdots & \vdots & & \vdots \\ z_{n-1} & z_{n-1}^2 & \ldots & z_{n-1}^{n-1} \end{pmatrix}.$$

Let $\boldsymbol{\omega} \coloneqq (\omega_n, \omega_n^2, \ldots, \omega_n^{n-1})$ and set

$$B_j \coloneqq (\mathbf{1}^t|V(\boldsymbol{\omega})) - J_{c_j}$$

for $j = 1, \ldots, d$ where $\mathbf{1} \coloneqq (1, \ldots, 1) \in \mathbb{R}^{n-1}$ and

$$J_{c_j}(k, \ell) \coloneqq \begin{cases} 1 & \text{if } c_j|k, \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq k \leq n-1$ and $1 \leq \ell \leq n$. Consider the matrix

$$B \coloneqq [B_1|B_2|\cdots|B_d].$$

Then $A = (a_{ij}) \in \mathbb{R}_{\geq 0}^{n \times n}$ satisfies (4.1) if and only if

$$B\mathbf{a} = \mathbf{0}, \tag{7.1}$$

where $\mathbf{a} = (\mathbf{a}_1|\cdots|\mathbf{a}_d)^t$ and $\mathbf{a}_j = (a_{1c_j}, \ldots, a_{nc_j})$ for $j = 1, \ldots, d$. Note that the defining CSP-equations (4.1) immediately give that $a_{ij} = 0$ for all $1 \leq i \leq n$ and $j \nmid n$. We claim that the real solutions to (7.1) are of the form

$$\mathbf{a}_1 = \begin{pmatrix} x_{01} \\ H_1(\mathbf{x}) \\ \vdots \\ H_{n-1}(\mathbf{x}) \end{pmatrix}, \quad \mathbf{a}_j = \begin{pmatrix} nx_{0j} \\ nx_{1j} \\ \vdots \\ nx_{(n-1)j} \end{pmatrix} \tag{7.2}$$

where $x_{01}, x_{ij} \in \mathbb{R}$ for $0 \leq i \leq n-1$, $2 \leq j \leq d$ and

$$H_k(\mathbf{x}) = \sum_{i=0}^{n-1} \sum_{j=2}^{d} \alpha_{ijk} x_{ij}$$

for some $\alpha_{ijk} \in \mathbb{Z}$, $k = 1, \dots, n-1$. Since B has full rank $n-1$, the solutions (7.2) make up the whole null space of B for dimensional reasons. Thus we only need to concern ourselves with the existence of solutions of the form (7.2).

Given (7.1) and supposing (7.2) we thus require

$$(V(\boldsymbol{\omega}) - J_1)\boldsymbol{\alpha}^{(ij)} = \mathbf{u}^{(ij)}, \tag{7.3}$$

for $i = 0, \dots, n-1$ and $j = 2, \dots, d$ where

$$\boldsymbol{\alpha}^{(ij)} := \begin{pmatrix} \alpha_{ij1} \\ \alpha_{ij2} \\ \vdots \\ \alpha_{ij(n-1)} \end{pmatrix}, \quad \mathbf{u}^{(ij)} := \begin{pmatrix} u_1^{(ij)} \\ u_2^{(ij)} \\ \vdots \\ u_{n-1}^{(ij)} \end{pmatrix}, \quad u_k^{(ij)} := \begin{cases} -n\omega_n^{ik} + n, & \text{if } c_j | k, \\ -n\omega_n^{ik}, & \text{otherwise} \end{cases}.$$

Note that

$$(V(\boldsymbol{\omega}) - J_1)^{-1} = \frac{1}{n} V(\overline{\boldsymbol{\omega}}).$$

Therefore

$$\boldsymbol{\alpha}^{(ij)} = \frac{1}{n} V(\overline{\boldsymbol{\omega}})\mathbf{u}^{(ij)},$$

which gives

$$\begin{aligned} \alpha_{ijk} &= \sum_{\ell=1}^{n-1} \frac{\overline{\omega}_n^{k\ell}}{n}(-n\omega_n^{i\ell}) + \sum_{\substack{\ell=1 \\ c_j|\ell}}^{n-1} \frac{\overline{\omega}_n^{k\ell}}{n} n \\ &= -\sum_{\ell=0}^{n-1} (\omega_n^{(i-k)})^\ell + \sum_{s=0}^{\frac{n}{c_j}-1} ((\omega_n^{c_j})^k)^s \\ &= \begin{cases} -n + \frac{n}{c_j}, & \text{if } i = k \text{ and } k \equiv 0 \pmod{\frac{n}{c_j}}, \\ -n & \text{if } i = k \text{ and } k \not\equiv 0 \pmod{\frac{n}{c_j}}, \\ \frac{n}{c_j}, & \text{if } i \neq k \text{ and } k \equiv 0 \pmod{\frac{n}{c_j}}, \\ 0 & \text{if } i \neq k \text{ and } k \not\equiv 0 \pmod{\frac{n}{c_j}}. \end{cases} \end{aligned}$$

Hence the theorem follows. □

The following corollary follows immediately from Theorem 7.1.

Corollary 7.2. *Let $n \in \mathbb{N} \setminus \{0\}$ and $d := |\{c \in \mathbb{N} : c|n\}|$ denote the number of divisors of n. Then* $\mathrm{CSP}(n)$ *has dimension* $n(d-1)+1$.

Recall that a polyhedral cone is given by $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \geq \mathbf{b}\}$ for some $n \times n$ matrix A. A non-zero element $\mathbf{x}$ of a polyhedral cone P is called an *extreme ray* if there are $d-1$ linearly independent constraints that are active at $\mathbf{x}$ (*i.e.* hold with equality at $\mathbf{x}$). If $\mathbf{x}$ is an extreme ray, then $\lambda\mathbf{x}$ is also an extreme ray for $\lambda > 0$. Two extreme rays that are positive multiples of each other are called *equivalent*. Equivalent extreme rays correspond to the same $d-1$ active constraints. Extreme rays can also be defined as points in $\mathbf{x} \in P$ that cannot be expressed as a convex combination of two points in the interior of P.

Below we give an explicit description of a subset of the extreme rays of $\mathrm{CSP}(n)$. This subset includes all extreme rays of the universal CSP-cone $\widetilde{\mathrm{CSP}}(n)$ (see Corollary

7.4). When $n = p$ for some prime number p, then we get all the extreme rays (see Corollary 7.5).

Theorem 7.3. *Let $n \in \mathbb{N} \setminus \{0\}$ and suppose*

$$1 = c_1 < c_2 < \cdots < c_{d-1} < c_d = n$$

are the divisors of n. Let $\ell_0 \in [d]$. Then $\mathbf{r} = (r_{ij}) \in \mathbb{R}^{n\times n}$ is an extreme ray of CSP(n) *if*

$$r_{ij} = \begin{cases} 1, & \text{if } (i,j) = (0, c_{\ell_0}), \\ \frac{1}{c_{\ell_0} - |I|}, & \text{if } i \in I \text{ and } j = c_{\ell_0}, \\ 0, & \text{otherwise} \end{cases}$$

for $0 \le i \le n-1$ and $1 \le j \le n$ where $I \subseteq \{t\frac{n}{c_{\ell_0}} \in \mathbb{N} : 1 \le t < c_{\ell_0}\}$. In particular the number of extreme rays of CSP(n) *is at least*

$$\frac{1}{2}\sum_{\ell=1}^{d} 2^{c_\ell}.$$

Proof. By Theorem 7.1, CSP(n) is isomorphic to the polyhedral cone

$$\{\mathbf{x} \in \mathbb{R}^{n(r-1)+1} : n\mathbf{x} \ge 0 \text{ and } H_k(\mathbf{x}) \ge 0 \text{ for all } k = 1, \ldots, n-1\}.$$

Let $\mathbf{r} = (r_{ij}) \in \mathbb{R}^{n\times n}$ be an extremal ray of CSP(n) such that $r_{ij} = 0$ if $j \ne c_{\ell_0}$. Note that the defining inequalities of CSP(n) imply in particular that $r_{ij} \ge 0$ for all $0 \le i < n$ and $1 \le j \le d$.

Suppose first that $r_{0c_{\ell_0}} = 0$. Let $k \in [n-1]$ be such that $r_{kc_{\ell_0}} \ge r_{ic_{\ell_0}}$ for all $i \in [n-1]$. Suppose for a contradiction that $r_{kc_{\ell_0}} > 0$. The maximality of $r_{kc_{\ell_0}}$ implies $\frac{r_{ic_{\ell_0}}}{r_{kc_{\ell_0}}} \le 1$. The defining inequalities of the polyhedral cone CSP(n) gives $-nr_{ic_{\ell_0}} \ge 0$ for $i \not\equiv 0 \pmod{\frac{n}{c_{\ell_0}}}$, which implies that $r_{ic_{\ell_0}} = 0$ for $i \not\equiv 0 \pmod{\frac{n}{c_{\ell_0}}}$. Thus we may assume $k \equiv 0 \pmod{\frac{n}{c_{\ell_0}}}$. Now, $H_k(\mathbf{x}) \ge 0$ gives

$$0 \le -n + \frac{n}{c_{\ell_0}} + \sum_{\substack{i \in [n-1]\setminus k \\ \frac{n}{c_{\ell_0}} | i}} \frac{r_{ic_{\ell_0}}}{r_{kc_{\ell_0}}} \le -n + \frac{n}{c_{\ell_0}} + c_{\ell_0} - 2,$$

which holds if and only if $c_{\ell_0} \le \frac{n+2}{2} - \Delta$ or $c_{\ell_0} \ge \frac{n+2}{2} + \Delta$ where $\Delta = \left(\left(\frac{n+2}{2}\right)^2 - n\right)^{1/2}$. Since $\frac{n+2}{2} - \Delta < 1$ and $\frac{n+2}{2} + \Delta > n$ for $n > 0$ whereas $1 \le c_{j_0} \le n$ this gives a contradiction.

Hence we may assume $r_{0c_{\ell_0}} > 0$. Let

$$M_\ell := \left\{ t\frac{n}{c_\ell} \in \mathbb{N} : 1 \le t < c_\ell \right\}.$$

Suppose $I \subseteq M_{c_{\ell_0}}$ such that $r_{ic_{\ell_0}} > 0$ for $i \in I$ and $r_{ic_{\ell_0}} = 0$ for $i \in M_{c_{\ell_0}} \setminus I$. Since $\mathbf{r}$ is an extreme ray there are by definition $n(d-1)$ linearly independent constraints active at $\mathbf{r}$. Since $r_{ij} = 0$ for $j \ne c_{\ell_0}$ and $r_{ic_{\ell_0}} = 0$ for $i \in [n-1] \setminus I$ there are $n(d-2) + 1 + (n-1) - |I|$ active constraints covered. Note that we have

$(-n+\frac{n}{c_{\ell_0}})r_{kc_{\ell_0}}+\sum_{i\neq k}\frac{n}{c_{\ell_0}}r_{ic_{\ell_0}}>0$ for $k\notin I$ and $nr_{kc_{\ell_0}}>0$ for $k\in M_{c_{\ell_0}}\setminus I$. Hence the remaining $|I|$ inequalities must be active at $\mathbf{r}$ which gives

$$\left(-n+\frac{n}{c_{\ell_0}}\right)r_{kc_{\ell_0}}+\sum_{i\neq k}\frac{n}{c_{\ell_0}}r_{ic_{\ell_0}}=0 \tag{7.4}$$

for $k\in I$. If $I=\emptyset$, then the only non-zero entry of $\mathbf{r}$ is $r_{0c_{\ell_0}}$. Suppose $I\neq\emptyset$. Summing the equations (7.4) and dividing by $\frac{n}{c_{\ell_0}}r_{0c_{\ell_0}}$, we get

$$0=\frac{c_{\ell_0}}{nr_{0c_{\ell_0}}}\sum_{i\in I}\left(\left(-n+\frac{n}{c_{\ell_0}}\right)r_{ic_{\ell_0}}+\sum_{k\neq i}\frac{n}{c_{\ell_0}}r_{kc_{\ell_0}}\right)=(-c_{\ell_0}+|I|)\sum_{i\in I}\frac{r_{ic_{\ell_0}}}{r_{0c_{\ell_0}}}+|I|.$$

Hence we get the average ratio

$$\frac{1}{|I|}\sum_{i\in I}\frac{r_{ic_{\ell_0}}}{r_{0c_{\ell_0}}}=\frac{1}{c_{\ell_0}-|I|} \tag{7.5}$$

Suppose

$$\frac{r_{kc_{\ell_0}}}{r_{0c_{\ell_0}}}>\frac{1}{c_{\ell_0}-|I|}$$

for some $k\in I$. Then by dividing (7.4) with $\frac{n}{c_{\ell_0}}r_{0c_{\ell_0}}$ and using (7.5) we have

$$\begin{aligned}0&=(-c_{\ell_0}+1)\frac{r_{kc_{\ell_0}}}{r_{0c_{\ell_0}}}+1+\sum_{i\in I}\frac{r_{ic_{\ell_0}}}{r_{0c_{\ell_0}}}-\frac{r_{kc_{\ell_0}}}{r_{0c_{\ell_0}}}\\&=-c_{\ell_0}\frac{r_{kc_{\ell_0}}}{r_{0c_{\ell_0}}}+1+\frac{|I|}{c_{\ell_0}-|I|}\\&<\frac{-c_{\ell_0}}{c_{\ell_0}-|I|}+1+\frac{|I|}{c_{\ell_0}-|I|}=0,\end{aligned}$$

which gives a contradiction. Hence by (7.5) we have that

$$r_{ic_{\ell_0}}=\frac{r_{0c_{\ell_0}}}{c_{\ell_0}-|I|}$$

for all $i\in I$ proving the theorem.

□

Corollary 7.4. *Let $n\in\mathbb{N}\setminus\{0\}$ and suppose $1=c_1<c_2<\cdots<c_{d-1}<c_d=n$ are the divisors of n. Let $M_\ell=\{t\frac{n}{c_\ell}:0\leq t<c_\ell\}$ and define $\mathbf{r}^{(\ell)}=(r_{ij}^{(\ell)})\in\mathbb{R}^{n\times n}$ by*

$$r_{ij}^{(\ell)}=\begin{cases}1, & \text{if } i\in M_\ell \text{ and } j=c_\ell,\\ 0, & \text{otherwise,}\end{cases}$$

for $1\leq\ell\leq d$. Then the extreme rays of $\widetilde{\mathrm{CSP}}(n)$ are given by $\{\mathbf{r}^{(\ell)}:1\leq\ell\leq d\}$.

Proof. By Theorem 7.3 the set $\{\mathbf{r}^{(\ell)}:1\leq\ell\leq d\}$ are indeed extreme rays and they clearly generate all universal CSP matrices (cf. Definition 6.2). □

Corollary 7.5. *Let $p \in \mathbb{N}$ be a prime number. Then the extreme rays of* $\mathrm{CSP}(p)$ *are given by $E_{01} \in \mathbb{R}^{p\times p}$ and $\mathbf{r} = (r_{ij}) \in \mathbb{R}^{p\times p}$ such that*

$$r_{ij} = \begin{cases} 1, & \text{if } (i,j) = (0,p), \\ \frac{1}{p-|I|}, & \text{if } i \in I \text{ and } j = p, \\ 0, & \text{otherwise,} \end{cases}$$

where $I \subseteq \{1,\dots,p-1\}$. In particular the number of extreme rays of $\mathrm{CSP}(p)$ *is given by $2^{p-1}+1$.*

By adding a size restriction on the set X we can also talk about a natural family of polytopes associated with cyclic sieving phenomena.

Definition 7.6. Let $m \in \mathbb{N}$. The m^{th} *CSP-polytope* is the convex rational polytope defined by

$$\mathrm{CSP}(n,m) \coloneqq \{A \in \mathrm{CSP}(n) : ||A|| = m\}.$$

Let $\mathrm{CSP}_{\mathbb{Z}}(n,m) \coloneqq \mathrm{CSP}(n,m) \cap \mathbb{Z}^{n\times n}$ denote the set of integer lattice points in $\mathrm{CSP}(n,m)$.

Once again, in the case where $n = p$ for some prime number $p \in \mathbb{N}$ we are able to make explicit computations. In the following two propositions we compute the vertices and the number of integer lattice points of $\mathrm{CSP}(n,m)$.

Proposition 7.7. *Let $p \in \mathbb{N}$ be a prime number and $m \in \mathbb{N}$. Then the vertices of* $\mathrm{CSP}(p,m)$ *are given by $mE_{01} \in \mathbb{R}^{p\times p}$ and $\mathbf{v} = (v_{ij}) \in \mathbb{R}^{p\times p}$ such that*

$$v_{ij} = \begin{cases} C, & \text{if } (i,j) = (0,p), \\ \frac{C}{p-|I|}, & \text{if } i \in I \text{ and } j = p, \\ 0, & \text{otherwise,} \end{cases}$$

where $I \subseteq \{2,\dots,p\}$ and

$$C = \frac{m}{2p-1+\frac{(p-1)|I|}{p-|I|}}.$$

In particular the number of vertices of $\mathrm{CSP}(p,m)$ *is given by $2^{p-1}+1$.*

Proof. Suppose $\mathbf{v} = (v_{ij}) \in \mathbb{R}^{p\times p}$ is a vertex of $\mathrm{CSP}(p,m)$.

If $v_{0p} = 0$, then arguing as in the first part of the proof of Theorem 7.3 gives that $v_{0p} = v_{1p} = \cdots = v_{p-1p} = 0$. Therefore $v_{ij} = 0$, unless $j = 1$ by Lemma 5.1. The additional constraint $||\mathbf{v}|| = m$ thus gives

$$m = \sum_{\substack{0\le i<p \\ 1\le j\le p}} v_{ij} = \sum_{j=1}^{p} v_{0j} = x_{01} + \sum_{k=1}^{p-1} H_k(\mathbf{x}), \tag{7.6}$$

which is the same as

$$x_{01} + (2p-1)x_{0p} + (p-1)x_{1p} + \cdots + (p-1)x_{p-1p} = m. \tag{7.7}$$

Since $x_{ip} = v_{ip}$ for $i = 0, 1\dots, p-1$ we get that $v_{01} = x_{01} = m$, so that $\mathbf{v} = mE_{01}$.

Therefore suppose $v_{0p} > 0$. Moreover suppose $I \subseteq \{1,\dots,p-1\}$ such that $v_{ip} > 0$ for $i \in I$ and $v_{ip} = 0$ for $i \in \{1,\dots,p-1\} \setminus I$. Since $\mathbf{v}$ is a vertex, there are by definition $p+1$ linearly independent constraints active at $\mathbf{v}$. Since p of these constraints arise from the polyhedral description of CSP(p) in Theorem 7.1 it follows, as in **??**, that

$$v_{ip} = \begin{cases} C, & \text{if } (i,j) = (0,p), \\ \frac{C}{p-|I|}, & \text{if } i \in I, \\ 0, & \text{if } i \in \{1,\dots,p-1\} \setminus I, \end{cases}$$

for some $C > 0$. The remaining active constraint is Equation (7.7). Inserting the above into Equation (7.7) and solving for C yields

$$C = \frac{m}{2p-1+\frac{(p-1)|I|}{p-|I|}},$$

from which the proposition follows. ☐

Proposition 7.8. *Let $p, m \in \mathbb{N}$ where p is a prime number. The number of lattice points in* CSP(p,m) *is given by*

$$|\mathrm{CSP}_{\mathbb{Z}}(p,m)| = \sum_{j=0}^{m} \sum_{r \in \left[\frac{2j}{2p-1}, \frac{j}{p-1}\right] \cap \mathbb{Z}} C\left(r(2p-1)-2j, p-1, \lfloor r - j/p \rfloor\right),$$

where

$$C(n,k,w) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{n-jw-1}{k-1}.$$

Proof. Let $x = x_{0p}$, $y = x_{1p} + \cdots + x_{p-1p}$ and $z = x_{01}$. According to the constraint $||A|| = m$ we seek non-negative integer solutions to

$$(2p-1)x + (p-1)y + z = m,$$

(cf. Equation (7.7)) satisfying $H_k(\mathbf{x}) \geq 0$. We therefore consider the Diophantine equations

$$(2p-1)x + (p-1)y = j,$$

for $j = 0, \dots, m$ which have the non-negative integer solutions

$$x = j - r(p-1) \text{ and } y = -2j + r(2p-1),$$

for $r \in \left[\frac{2j}{2p-1}, \frac{j}{p-1}\right] \cap \mathbb{Z}$. The constraints $H_k(\mathbf{x}) \geq 0$ for $k = 1, \dots, p-1$ give

$$x - (p-1)x_{kp} + (y - x_{kp}) \geq 0,$$

which implies

$$x_{kp} \leq r - \frac{j}{p}$$

for $k = 1, \dots, p-1$. Hence the lattice points in CSP(p,m) are in one-to-one correspondence with weak compositions of $y = -2j + r(2p-1)$ into $p-1$ parts of size at most $\lfloor r - j/p \rfloor$. By [Abr76] the number of such compositions are given by $C(r(2p-1)-2j, p-1, \lfloor r - j/p \rfloor)$. ☐

8. Appendix

In this appendix we prove inequalities needed for the estimations in Theorem 3.7. The first inequality below gives a sufficient condition for a Riemann sum to be monotonically increasing. A slightly weaker result appears in [BJ00, Theorem 3A].

Proposition 8.1. *Let $f(x)$ be a* decreasing *convex*[1] *function on $\mathbb{R}_{\geq 0}$, let p be a positive integer and $r \geq 0$. Then*

$$\frac{1}{p}\sum_{\ell=1}^{p} f\left(\frac{\ell+r}{p}\right) \leq \frac{1}{p+1}\sum_{\ell=1}^{p+1} f\left(\frac{\ell+r}{p+1}\right). \tag{8.1}$$

Proof. Let $x_i \coloneqq (i+r)/p$ and $y_i \coloneqq (i+r)/(p+1)$ and note that

$$x_i = \left(1-\frac{i}{p}\right) y_i + \frac{i}{p} y_{i+1} + \frac{r}{p(p+1)}. \tag{8.2}$$

Since f is decreasing and convex, we have that

$$f(x_i) \leq f\left[\left(1-\frac{i}{p}\right) y_i + \frac{i}{p} y_{i+1}\right] \leq \left(1-\frac{i}{p}\right) f(y_i) + \frac{i}{p} f(y_{i+1})$$

Now let $a_i \coloneqq f(x_i)$ and $b_i \coloneqq f(y_i)$ and note that the decreasing property implies

$$a_i \leq \left(1-\frac{i}{p}\right) b_i + \frac{i}{p} b_{i+1} \leq \left(1-\frac{i}{p+1}\right) b_i + \frac{i}{p+1} b_{i+1} \text{ for } i = 1, \ldots, p.$$

We add all these inequalities and obtain

$$\sum_{i=1}^{p} a_i \leq \frac{1}{p+1}\sum_{i=1}^{p}(p+1-i)b_i + \frac{1}{p+1}\sum_{i=1}^{p} i b_{i+1}.$$

We then have

$$\begin{aligned}
(p+1)(a_1+\cdots+a_p) &\leq \sum_{i=1}^{p}(p+1-i)b_i + \sum_{i=2}^{p+1}(i-1)b_i \\
&\leq p\sum_{i=1}^{p} b_i + \sum_{i=1}^{p}(1-i)b_i + p b_{p+1} + \sum_{i=2}^{p}(i-1)b_i \\
&\leq p(b_1+\cdots+b_{p+1}).
\end{aligned}$$

This implies (8.1). ☐

Corollary 8.2. *Let $r, s \geq 0$ and $p \in \mathbb{N}$. Then the expression*

$$g(p) = \frac{1}{p}\sum_{\ell=1}^{p} \frac{1}{s+(r+\ell)/p} \tag{8.3}$$

is increasing with p.

---

[1] For all a, b we have $f(\frac{a+b}{2}) \leq \frac{f(a)+f(b)}{2}$, or equivalently for twice differentiable functions, $f'' \geq 0$.

Proof. Choosing the decreasing convex function $f(x) = 1/(s+x)$ in Proposition 8.1 together with the given r yields

$$\frac{1}{p}\sum_{\ell=1}^{p} \frac{1}{s+(r+\ell)/p} \leq \frac{1}{p+1}\sum_{\ell=1}^{p+1} \frac{1}{s+(r+\ell)/(p+1)}.$$

□

Corollary 8.3. *If a, t, i, j and k are non-negative integers such that $a \leq t$ and $j \leq k$, then*

$$\sum_{\ell=0}^{ka-1} \frac{1}{kt+i-\ell} \geq \sum_{\ell=0}^{ja-1} \frac{1}{jt+i-\ell}. \tag{8.4}$$

Proof. Choosing $p = ak$, $s = (t-a)/a$ and $r = i$ in (8.3) gives that

$$f(ak) = \frac{1}{ka}\sum_{\ell=1}^{ak} \frac{1}{s+(i+\ell)/(ak)} = \sum_{\ell=1}^{ka} \frac{1}{kt-ka+i+\ell} = \sum_{\ell=0}^{ka-1} \frac{1}{kt+i-\ell}.$$

The fact that $f(ka) \geq f(kj)$ if $k \geq j$ now gives the desired inequality. □

Lemma 8.4. *If a, b, i, j and k are non-negative integers such that $a \leq b$ and $j \leq k$, then*

$$\frac{\binom{kb+i}{ka}^{1/k}}{\binom{ka+i}{ka}^{1/k}} \geq \frac{\binom{jb+i}{ja}^{1/j}}{\binom{ja+i}{ja}^{1/j}}. \tag{8.5}$$

Proof. The inequality can be rewritten as $f(b) \geq f(a)$, where

$$f(t) := \frac{\binom{kt+i}{ka}^{j}}{\binom{jt+i}{ja}^{k}}. \tag{8.6}$$

Thus, it suffices to show that $f(t)$ is increasing. Computing the derivative and factoring out positive terms reduces to Equation (8.4). □

Remark 8.5. In the case where $j|k$, the binomial inequality (8.5) admits the following combinatorial interpretation. A certain organization wants ka members to sit on its executive committee and ja members to sit on the committee for each of its k/j factions. Then the number of possible committee constellations with $kb+i$ candidates for the executive committee and $ja+i$ candidates for each of the factions, is greater than the number of committee constellations with $ka+i$ candidates for the executive committee and $jb+i$ candidates for each faction.

Lemma 8.6. *If a, b and k are non-negative integers such that $b > a$, then for each $0 \leq i \leq ka$ we have*

$$\binom{kb+i}{ka}\binom{ka+i}{ka}^{-1} \geq \left(\frac{b+a}{2a}\right)^{ka}$$

Proof. If $B > A$, then the function $f(x) = \frac{B+x}{A+x}$ is decreasing as x increases. Thus for $0 \leq i \leq ka$ we have

$$\binom{kb+i}{ka}\binom{ka+i}{ka}^{-1} = \frac{(kb+i)(kb+i-1)\cdots(kb+i-ka+1)}{(ka+i)(ka+i-1)\cdots(i+1)}$$
$$\geq \left(\frac{kb+i}{ka+i}\right)^{ka}$$
$$\geq \left(\frac{b+a}{2a}\right)^{ka}$$

□

Acknowledgements. The authors would like to thank Darij Grinberg, Gleb Nenashev, Mehtaab Sawhney and Olof Sisak for helpful discussions. The first author is funded by the Knut and Alice Wallenberg Foundation (2013.03.07).

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