NON–REPRESENTABLE HYPERBOLIC MATROIDS

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Abstract. The generalized Lax conjecture asserts that each hyperbolicity cone is a linear slice of the cone of positive semidefinite matrices. Hyperbolic polynomials give rise to a class of (hyperbolic) matroids which properly contains the class of matroids representable over the complex numbers. This connection was used by the second author to construct counterexamples to algebraic (stronger) versions of the generalized Lax conjecture by considering a non–representable hyperbolic matroid. The Vámos matroid and a generalization of it are, prior to this work, the only known instances of non–representable hyperbolic matroids.

We prove that the Non–Pappus and Non–Desargues matroids are non-representable hyperbolic matroids by exploiting a connection between Euclidean Jordan algebras and projective geometries. We further identify a large class of hyperbolic matroids which contains the Vámos matroid and the generalized Vámos matroids recently studied by Burton, Vinzant and Youm. This proves a conjecture of Burton *et al.* We also prove that many of the matroids considered here are non–representable. The proof of hyperbolicity for the matroids in the class depends on proving nonnegativity of certain symmetric polynomials. In particular we generalize and strengthen several inequalities in the literature, such as the Laguerre–Turán inequality and an inequality due to Jensen. Finally we explore consequences to algebraic versions of the generalized Lax conjecture.

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1. Introduction

Although hyperbolic polynomials have their origin in PDE theory, they have during recent years been studied in diverse areas such as control theory, optimization, real algebraic geometry, probability theory, computer science and combinatorics, see [35, 36, 39, 40] and the references therein. To each hyperbolic polynomial is associated a closed convex (hyperbolicity) cone. Over the past 20 years methods have been developed to do optimization over hyperbolicity cones, which generalize semidefinite programming. A problem that has received considerable interest is the generalized Lax conjecture which asserts that each hyperbolicity cone is a linear slice of the cone of positive semidefinite matrices (of some size). Hence if the generalized Lax conjecture is true then hyperbolic programming is the same as semidefinite programming.

Choe *et al.* [11] and Gurvits [20] proved that hyperbolic polynomials give rise to a class of matroids, called hyperbolic matroids or matroids with the weak half–plane property. The class of hyperbolic matroids properly contains the class of matroids which are representable over the complex numbers, see [11, 41]. This fact was used by the second author [7] to construct counterexamples to algebraic (stronger) versions of the generalized Lax conjecture. To better understand, and to identify potential counterexamples to the generalized Lax conjecture, it is therefore of interest to study hyperbolic matroids which are not representable over $\mathbb{C}$, or even better not representable over any (skew) field. However previous to this work essentially just two such matroids were known: The Vámos matroid V_8 [41] and a generalization V_{10} [10]. In this paper we first show that the Non-Pappus and Non-Desargues matroids are hyperbolic (but not representable over any field) by utilizing a known connection between hyperbolic polynomials and Euclidean Jordan algebras. Then, in Theorem 6.5, we construct a family of hyperbolic matroids which are parametrized by uniform hypergraphs, and prove that many of these matroids fail to be representable over any field, and more generally over any modular lattice. The proof of the main result is involved and uses several ingredients. In order to prove that the polynomials coming from our family of matroids are hyperbolic we need to prove that certain symmetric polynomials are nonnegative. The results obtained generalize and strengthen several inequalities in the literature, such as the Laguerre–Turán inequality and an inequality due to Jensen. Finally we explore some consequences to algebraic versions of the generalized Lax conjecture.

2. Hyperbolic and stable polynomials

A homogeneous polynomial $h(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_n]$ is *hyperbolic* with respect to a vector $\mathbf{e} \in \mathbb{R}^n$ if $h(\mathbf{e}) \neq 0$, and if for all $\mathbf{x} \in \mathbb{R}^n$ the univariate polynomial $t \mapsto h(t\mathbf{e} - \mathbf{x})$ has only real zeros. Note that if h is a hyperbolic polynomial of degree d, then we may write

$$h(t\mathbf{e} - \mathbf{x}) = h(\mathbf{e}) \prod_{j=1}^{d} (t - \lambda_j(\mathbf{x})),$$

where

$$\lambda_{\max}(\mathbf{x}) = \lambda_1(\mathbf{x}) \geq \cdots \geq \lambda_d(\mathbf{x}) = \lambda_{\min}(\mathbf{x})$$

are called the *eigenvalues* of $\mathbf{x}$ (with respect to $\mathbf{e}$). The *hyperbolicity cone* of h with respect to $\mathbf{e}$ is the set $\Lambda_+(h, \mathbf{e}) = \{\mathbf{x} \in \mathbb{R}^n : \lambda_{\min}(\mathbf{x}) \geq 0\}$. We usually abbreviate and write Λ_+ if there is no risk for confusion. We denote by Λ_{++} the interior Λ_+.

Example 2.1. An important example of a hyperbolic polynomial is $\det(X)$, where $X = (x_{ij})_{i,j=1}^n$ is a symmetric matrix with $\binom{n+1}{2}$ indeterminate entries. If X is a real symmetric $n \times n$ matrix and I_n is the identity matrix of size $n \times n$, then $t \mapsto \det(tI_n - X)$ is the characteristic polynomial of a real symmetric matrix, so it has only real zeros. Hence $\det(X)$ is a hyperbolic polynomial with respect to I_n, and its hyperbolicity cone is the cone of positive semidefinite matrices.

The real linear space of complex hermitian matrices of size n is parametrized by matrices X in n^2 variables, and as above it follows that $\det(X)$ is a hyperbolic polynomial.

The next theorem follows from a theorem of Helton and Vinnikov [22], see [29]. It proved the Lax conjecture, after Peter Lax [28].

Theorem 2.2. *Suppose that $h(x, y, z)$ is of degree d and hyperbolic with respect to $\mathbf{e} = (e_1, e_2, e_3)^T$. Suppose further that h is normalized such that $h(\mathbf{e}) = 1$. Then there are symmetric $d \times d$ matrices A, B, C such that $e_1A + e_2B + e_3C = I_d$ and*

$$h(x, y, z) = \det(xA + yB + zC).$$

Remark 2.3. The exact analogue of the Helton-Vinnikov theorem fails for $n > 3$ variables. This may be seen by comparing dimensions. The space of degree d polynomials on $\mathbb{R}^n$ of the form $\det(x_1A_1 + \cdots x_nA_n)$ with A_i symmetric for $1 \leq i \leq n$, has dimension at most $n\binom{d+1}{2}$ whereas the space of hyperbolic polynomials on $\mathbb{R}^n$ has dimension $\binom{n+d-1}{d}$.

A convex cone in $\mathbb{R}^n$ is *spectrahedral* if it is of the form

$$\left\{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_iA_i \text{ is positive semidefinite}\right\}$$

where A_i, $i = 1, \ldots, n$ are symmetric matrices such that there exists a vector $(y_1, \ldots, y_n) \in \mathbb{R}^n$ with $\sum_{i=1}^n y_iA_i$ positive definite. It is easy to see that spectrahedral cones are hyperbolicity cones. Indeed if $A_1, \ldots, A_n$ are real symmetric $d \times d$ matrices and $\mathbf{e} \in \mathbb{R}^n$ is a vector such that $\sum_{i=1}^n e_iA_i$ is positive definite, then $h(\mathbf{x}) = \det\left(\sum_{i=1}^n x_iA_i\right) \in \mathbb{R}[x_1, \ldots, x_n]$ is a hyperbolic polynomial with respect to $\mathbf{e}$, since for all $\mathbf{x} \in \mathbb{R}^n$ we have that $\det\left(tI_d - \sum_{i=1}^n x_iA_i\right) \in \mathbb{R}[t]$ is the characteristic polynomial of a real symmetric matrix and hence real-rooted. Therefore the hyperbolicity cone of $h(\mathbf{x})$ is precisely the spectrahedral cone $\{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_iA_i \text{ is positive definite}\}$. A major open question asks if the converse is true.

Conjecture 2.4 (Generalized Lax conjecture (geometric version) [22, 39])**.** *All hyperbolicity cones are spectrahedral.*

We may reformulate Conjecture 2.4 as follows, see [22, 39].

Conjecture 2.5 (Generalized Lax conjecture (algebraic version) [22, 39])**.** *If $h(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ is hyperbolic with respect to $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{R}^n$, then there exists a polynomial $q(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$, hyperbolic with respect to $\mathbf{e}$, such that $\Lambda_{++}(h, \mathbf{e}) \subseteq \Lambda_{++}(q, \mathbf{e})$*

and

$$q(\mathbf{x})h(\mathbf{x}) = \det\left(\sum_{i=1}^{n} x_i A_i\right) \tag{2.1}$$

for some real symmetric matrices $A_1, \ldots, A_n$ of the same size such that $\sum_{i=1}^n e_i A_i$ is positive definite.

Here is an overview of known facts regarding Conjecture 2.4.

- Conjecture 2.4 is true for $n = 3$ by Theorem 2.2,
- Conjecture 2.4 is true for homogeneous cones [13], i.e., cones for which the automorphism group acts transitively on its interior,
- Conjecture 2.4 is true for quadratic polynomials, see e.g. [32],
- Conjecture 2.4 is true for elementary symmetric polynomials, see [8],
- Conjecture 2.4 is true for certain multivariate generalizations of matching and independence polynomials, see [1],
- Conjecture 2.4 is true for the first derivative relaxation of the positive semi-definite cone, see [38],
- Weaker versions of Conjecture 2.4 are true for smooth hyperbolic polynomials, see [27, 31].
- Stronger algebraic versions of Conjecture 2.4 are false, see [7].

A class of polynomials which is intimately connected to hyperbolic polynomials is the class of stable polynomials. Below we will collect a few facts about stable polynomials that will be needed in forthcoming sections. A polynomial $P(\mathbf{x}) \in \mathbb{C}[x_1, \ldots, x_n]$ is *stable* if $P(z_1, \ldots, z_n) \neq 0$ whenever $\mathrm{Im}(z_j) > 0$ for all $1 \leq j \leq n$. Stable polynomials satisfy the following basic closure properties, see e.g. [40].

Lemma 2.6. *Let $P(x_1, \ldots, x_n)$ be a stable polynomial of degree d_i in x_i for $i = 1, \ldots, n$. Then for all $i = 1, \ldots, n$ we have*

(i) *Specialization: $P(x_1, \ldots, x_{i-1}, \zeta, x_{i+1}, \ldots, x_n)$ is stable or identically zero for each $\zeta \in \mathbb{C}$ with $\mathrm{Im}(\zeta) \geq 0$.*
(ii) *Scaling: $P(x_1, \ldots, x_{i-1}, \lambda x_i, x_{i+1}, \ldots, x_n)$ is stable for all $\lambda > 0$.*
(iii) *Inversion: $x_i^{d_i} P(x_1, \ldots, x_{i-1}, -x_i^{-1}, x_{i+1}, \ldots, x_n)$ is stable.*
(iv) *Permutation: $P(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ is stable for all $\sigma \in \mathfrak{S}_n$.*
(v) *Differentiation: $(\partial/\partial x_i)P(x_1, \ldots, x_n)$ is stable.*

Hyperbolic and stable polynomials are related as follows, see [5, Prop. 1.1] and [11, Thm. 6.1].

Lemma 2.7. *Let $P \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogenous polynomial. Then P is stable if and only if P is hyperbolic with $\mathbb{R}_+^n \subseteq \Lambda_+$.*

Moreover all non-zero Taylor coefficients of a homogeneous and stable polynomial have the same phase, i.e., the quotient of any two non-zero coefficients is a positive real number.

Lemma 2.8 (Lemma 4.3 in [7])**.** *If $h \in \mathbb{R}[y_1, \ldots, y_n]$ is a hyperbolic polynomial, $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \Lambda_+$ and $\mathbf{v}_0 \in \mathbb{R}^n$, then the polynomial*

$$P(\mathbf{x}) = h(\mathbf{v}_0 + x_1\mathbf{v}_1 + \cdots + x_m\mathbf{v}_m)$$

is either identically zero or stable.

3. Hyperbolic polymatroids

We refer to [34] for undefined matroid terminology. The connection between hyperbolic/stable polynomials and matroids was first realized in [11]. A polynomial is *multiaffine* provided that each variable occurs at most to the first power. Choe *et al.* [11] proved that if

$$P(\mathbf{x}) = \sum_{B \subseteq [m]} a(B) \prod_{i \in B} x_i \in \mathbb{C}[x_1, \ldots, x_m] \tag{3.1}$$

is a homogeneous, multiaffine and stable polynomial, then its *support*

$$\mathcal{B} = \{B : a(B) \neq 0\}$$

is the set of bases of a matroid, $\mathcal{M}$, on $[m]$. Such matroids are called *weak half–plane property matroids* (abbreviated WHPP–matroids). If further $P(\mathbf{x})$ can be chosen so that $a(B) \in \{0, 1\}$, then $\mathcal{M}$ is called a *half–plane property matroid* (abbreviated HPP–matroid). If so, then $P(\mathbf{x})$ is the *bases generating polynomial* of $\mathcal{M}$. Here are a few known facts regarding WHPP or HPP matroids.

- All matroids representable over $\mathbb{C}$ are WHPP, [11].
- A binary matroid is WHPP if and only if it is HPP, and if and only if it is regular, [9, 11].
- No finite projective geometry PG(r, n) is WHPP, [9, 11].
- The Vámos matroid V_8 is HPP (but not representable over any field), [41].

We shall now see how weak half-plane property matroids may conveniently be described in terms of hyperbolic polynomials.

Let E be a finite set. A *polymatroid* is a function $r : 2^E \to \mathbb{N}$ satisfying

(i) $r(\emptyset) = 0$,
(ii) $r(S) \leq r(T)$ whenever $S \subseteq T \subseteq E$,
(iii) r is *semimodular*, i.e.,

$$r(S) + r(T) \geq r(S \cap T) + r(S \cup T),$$

for all $S, T \subseteq E$.

Recall that rank functions of matroids on E coincide polymatroids r on E with $r(\{i\}) \leq 1$ for all $i \in E$.

Let $\mathcal{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_m)$ be a tuple of vectors in $\Lambda_+(h, \mathbf{e})$, where $\mathbf{e} \in \mathbb{R}^n$. The *(hyperbolic) rank*, $\mathrm{rk}(\mathbf{x})$, of $\mathbf{x} \in \mathbb{R}^n$ is defined to be the number of non-zero eigenvalues of $\mathbf{x}$, i.e., $\mathrm{rk}(\mathbf{x}) = \deg h(\mathbf{e} + t\mathbf{x})$. Define a function $r_\mathcal{V} : 2^{[m]} \to \mathbb{N}$, where $[m] := \{1, 2, \ldots, m\}$, by

$$r_\mathcal{V}(S) = \mathrm{rk}\left(\sum_{i \in S} \mathbf{v}_i\right).$$

It follows from [20] (see also [7]) that $r_\mathcal{V}$ is a polymatroid. We call such polymatroids *hyperbolic polymatroids*. Hence if the vectors in $\mathcal{V}$ have rank at most one, then we obtain the hyperbolic rank function of a *hyperbolic matroid*.

Example 3.1. Let $A_1 = \mathbf{u}_1\mathbf{u}_1^*, \ldots, A_m = \mathbf{u}_m\mathbf{u}_m^*$ be PSD matrices of rank at most one in $\mathbb{C}^n$. By Example 2.1 the function $r : 2^{[m]} \to \mathbb{N}$ defined by

$$r(S) = \mathrm{rk}\left(\sum_{i \in S} A_i\right)$$

is the rank function of a hyperbolic matroid. It is not hard to see that $r(S)$ is equal to the dimension of the subspace of $\mathbb{C}^n$ spanned by $\{\mathbf{u}_i : i \in S\}$. Hence r is the rank function of the linear matroid defined by $\mathbf{u}_1, \ldots, \mathbf{u}_m$.

Proposition 3.2. *A matroid is hyperbolic if and only it has the weak half–plane property.*

Proof. Suppose $\mathcal{B}$ is the set of bases of a matroid, $\mathcal{M}$, with the weak half–plane property realized by (3.1). By Lemma 2.7 we may assume that $a(B)$ is a nonnegative real number for all $B \subseteq [m]$. Then $P(\mathbf{x})$ is hyperbolic with hyperbolicity cone containing the positive orthant by Lemma 2.7. Let $\mathcal{V} = (\delta_1, \ldots, \delta_m)$, be the standard basis of $\mathbb{R}^m$, and let $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^m$ be the all ones vector. Then

$$\begin{aligned} r_{\mathcal{V}}(S) &= \mathrm{rk}\left(\sum_{i \in S} \delta_i\right) = \deg P\left(\mathbf{1} + t\sum_{i \in S} \delta_i\right) \\ &= \deg \sum_{B} a(B)(1+t)^{|B \cap S|} = \max\{|B \cap S| : B \in \mathcal{B}\}, \end{aligned}$$

and hence $r_{\mathcal{V}}$ is the rank function of $\mathcal{M}$.

Conversely, assume that h is hyperbolic, that $\mathcal{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_m) \in \Lambda_+(h, \mathbf{e})^m$, and that $r_{\mathcal{V}}$ is the rank function of a hyperbolic matroid of rank r. We may assume $h(\mathbf{e}) > 0$. The polynomial $g(x_0, x_1, \ldots, x_m) = h(x_0\mathbf{e} + x_1\mathbf{v}_1 + \cdots + x_m\mathbf{v}_m)$ is stable by Lemma 2.8 and has only nonnegative coefficients by Lemma 2.7. Since $\mathbf{v}_i$ has rank at most one for each i we see that g has degree at most one in x_i for all $i \geq 1$. It follows that

$$g(\mathbf{x}) = x_0^{d-r} \sum_{i=0}^{r} g_i(x_1, \ldots, x_m) x_0^{r-i},$$

where $g_i(\mathbf{x})$ is a homogeneous and multiaffine polynomial of degree i for $0 \leq i \leq r \leq d = \deg h$. By dividing by x_0^{d-r} and setting $x_0 = 0$, we see that $g_r(\mathbf{x})$ is stable by Lemma 2.6. Moreover B is a basis of the matroid defined by $\mathcal{V}$ if and only if $|B| = r$ and $g(\delta_0 + t\sum_{i \in B} \delta_i)$ has degree d. This happens if and only if $g_r(\sum_{i \in B} \delta_i) \neq 0$, that is, if and only if B is in the support of $g_r(\mathbf{x})$. $\square$

4. Projections and face lattices of hyperbolicity cones

Let C be a closed convex cone in $\mathbb{R}^n$. If $\mathbf{x}, \mathbf{y} \in C$ and $\mathbf{y} - \mathbf{x} \in C$, we write $\mathbf{x} \leq \mathbf{y}$. Recall that a *face* F of a convex cone C is a convex subcone of C with the property that $\mathbf{x}, \mathbf{y} \in C$, $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{y} \in F$ implies $\mathbf{x} \in F$. Equivalently a face is a convex subcone of C such that for each open line segment in C that intersects F, the closure of the segment is contained in F. The collection of all faces of C is a lattice, $L(C)$, under containment with smallest element $\{0\}$ and largest element C. Clearly $F \wedge G = F \cap G$ and $F \vee G = \bigcap_H H$, where H ranges over all faces containing F and G. The collection of all relative interiors of faces of C partitions C. If $F_{\mathbf{x}}$ is the unique face that contains $\mathbf{x} \in C$ in its relative interior, then $F_{\mathbf{x}} \vee F_{\mathbf{y}} = F_{\mathbf{x}+\mathbf{y}}$. See [37] for more on the face lattices of convex cones.

The *rank* of a face F of the hyperbolicity cone Λ_+ is defined by

$$\mathrm{rk}(F) = \max_{\mathbf{x} \in F} \mathrm{rk}(\mathbf{x}).$$

Note that if $L(\Lambda_+)$ is a graded lattice, then the above hyperbolic rank function is not necessarily the rank function of $L(\Lambda_+)$.

Lemma 4.1 (Thm 26, [36])**.** *Let F be a face of Λ_+ and let $\mathbf{x} \in F$. Then* $\mathrm{rk}(\mathbf{x}) = \mathrm{rk}(F)$ *if and only if $\mathbf{x}$ is in the relative interior of F.*

By Lemma 4.1 and the semimodularity of hyperbolic polymatroids we see that $\mathrm{rk} : L(\Lambda_+) \to \mathbb{N}$ is *semimodular*, that is,

$$\mathrm{rk}(F \vee G) + \mathrm{rk}(F \wedge G) \leq \mathrm{rk}(F) + \mathrm{rk}(G)$$

for all $F, G \in L(\Lambda_+)$. We may therefore equivalently define a hyperbolic polymatroid in terms of the face lattice of the hyperbolicity cone as follows: If $\mathcal{F} = (F_1, \ldots, F_m)$ is a tuple of elements of the face lattice $L(\Lambda_+)$, then the function $r_{\mathcal{F}} : 2^{[m]} \to \mathbb{N}$ defined by

$$r_{\mathcal{F}}(S) = \mathrm{rk}\left(\bigvee_{i \in S} F_i\right)$$

is a hyperbolic polymatroid.

The following theorem collects a few fundamental facts about hyperbolic polynomials and their hyperbolicity cones. For proofs see [21, 36].

Theorem 4.2 (Gårding, [21])**.** *Suppose h is hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^n$.*

(i) *$\Lambda_+(\mathbf{e})$ and $\Lambda_{++}(\mathbf{e})$ are convex cones.*
(ii) *$\Lambda_{++}(\mathbf{e})$ is the connected component of*

$$\{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) \neq 0\}$$

which contains $\mathbf{e}$.
(iii) *$\lambda_{\min} : \mathbb{R}^n \to \mathbb{R}$ is a concave function, and $\lambda_{\max} : \mathbb{R}^n \to \mathbb{R}$ is a convex function.*
(iv) *If $\mathbf{e}' \in \Lambda_{++}(\mathbf{e})$, then h is hyperbolic with respect to $\mathbf{e}'$ and $\Lambda_{++}(\mathbf{e}') = \Lambda_{++}(\mathbf{e})$.*

Recall that the *lineality space* of a convex cone C is $C \cap -C$, i.e., the largest linear space contained in C. It follows that the lineality space of a hyperbolicity cone is $\{\mathbf{x} : \lambda_i(\mathbf{x}) = 0 \text{ for all } i\}$, see e.g. [36]. Also if $\mathbf{x}$ is in the lineality space, then $\lambda_i(\mathbf{x} + \mathbf{y}) = \lambda_i(\mathbf{y})$ for all $1 \leq i \leq d$ and $\mathbf{y} \in \mathbb{R}^n$ [36].

By homogeneity of h

$$\lambda_j(s\mathbf{x} + t\mathbf{e}) = \begin{cases} s\lambda_j(\mathbf{x}) + t & \text{if } s \geq 0 \text{ and} \\ s\lambda_{d-j+1}(\mathbf{x}) + t & \text{if } s \leq 0 \end{cases}, \tag{4.1}$$

for all $s, t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.

In analogy with the eigenvalue characterization of matrix projections we define projections in Λ_+ as follows.

Definition 4.3. An element in Λ_+ is a *projection* if its eigenvalues are contained in $\{0, 1\}$.

Remark 4.4. Note that $\mathbf{0}$, $\mathbf{e}$ and appropriate multiples of rank one vectors in Λ_+ are always projections.

Lemma 4.5. *Suppose $\mathbf{x}, \mathbf{y} \in \Lambda_+$ are such that $F_{\mathbf{x}} \leq F_{\mathbf{y}}$ and* $\mathrm{rk}(\mathbf{y}) = r$. *If $\lambda_1(\mathbf{x}) \leq \lambda_r(\mathbf{y})$, then $\mathbf{x} \leq \mathbf{y}$.*

In particular if $\mathbf{x}, \mathbf{y} \in \Lambda_+$ are projections, then $F_{\mathbf{x}} \leq F_{\mathbf{y}}$ if and only if $\mathbf{x} \leq \mathbf{y}$.

Proof. Suppose $\mathbf{x}, \mathbf{y} \in \Lambda_+$ are such that $F_\mathbf{x} \leq F_\mathbf{y}$, $\mathrm{rk}(\mathbf{y}) = r$ and $\lambda_1(\mathbf{x}) \leq \lambda_r(\mathbf{y})$. Consider the polynomial

$$g(u, s, t) = h(u\mathbf{e} + s\mathbf{y} + t\mathbf{x}),$$

which is hyperbolic with respect to $(1, 0, 0)$ and whose hyperbolicity cone contains the positive orthant. Since $\mathbf{x} \in F_\mathbf{y}$ we know that $\mathrm{rk}(a\mathbf{x} + b\mathbf{y}) = r$ for all $a, b > 0$. Since all non-zero Taylor coefficients of $g(u, s, t)$ have the same sign, by Lemma 2.7, we may write

$$g(u, s, t) = u^{d-r} g_0(u, s, t), \quad d = \deg h,$$

where $g_0(u, s, t)$ is hyperbolic with respect to $(1, 0, 0)$ and also $(0, 1, 0)$, and its hyperbolicity cone contains the positive orthant. Let $\lambda'_j(a, b, c)$, $j = 1, \ldots, r$, denote the eigenvalues of g_0 (with respect to $(1, 0, 0)$). Then by (4.1) and the concavity of λ'_r (Theorem 4.2):

$$\lambda'_r(0, 1, -1) \geq \lambda'_r(0, 1, 0) + \lambda'_r(0, 0, -1) = \lambda_r(\mathbf{y}) - \lambda_1(\mathbf{x}) \geq 0.$$

By construction $\lambda_{\min}(\mathbf{y} - \mathbf{x}) = \min\{0, \lambda'_r(0, 1, -1)\}$, and the lemma follows. □

Lemma 4.6. *If $\mathbf{x}, \mathbf{y} \in \Lambda_+$ are projections with $\mathbf{x} \leq \mathbf{y}$, then $\mathbf{y} - \mathbf{x}$ is a projection with*

$$\mathrm{rk}(\mathbf{y} - \mathbf{x}) = \mathrm{rk}(\mathbf{y}) - \mathrm{rk}(\mathbf{x}).$$

Proof. Suppose first that $F_\mathbf{y} = \Lambda_+ = F_\mathbf{e}$. Then $\mathbf{y} - \mathbf{e}, \mathbf{e} - \mathbf{y} \in \Lambda_+$ by Lemma 4.5, and hence $\mathbf{y} - \mathbf{e}$ is in the lineality space of Λ_+. Then

$$\lambda_i(\mathbf{y} - \mathbf{x}) = \lambda_i(\mathbf{e} - \mathbf{x}) = 1 - \lambda_{d-i+1}(\mathbf{x}),$$

for all $1 \leq i \leq d = \deg h$, and hence $\mathbf{y} - \mathbf{x}$ is a projection of rank $d - \mathrm{rk}(\mathbf{x})$.

If $F_\mathbf{y} \neq F_\mathbf{e}$, then $r := \mathrm{rk}(\mathbf{y}) < d$. Consider the hyperbolic polynomial

$$g(u, s, t) = h(u\mathbf{e} + s\mathbf{x} + t\mathbf{y}) = u^{d-r} g_0(u, s, t),$$

where g_0 is hyperbolic with respect to $\mathbf{e}' = (1, 0, 0)$. It follows that $\mathbf{x}' = (0, 1, 0)$ and $\mathbf{y}' = (0, 0, 1)$ are projections with $F_{\mathbf{e}'} = F_{\mathbf{y}'}$. The lemma now follows from the first case considered. □

Remark 4.7. Note that if $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{y} \leq \mathbf{x}$, then $\mathbf{y} - \mathbf{x}$ is in the lineality space of Λ_+. Moreover $\mathbf{x} \leq \mathbf{y}$ if and only if $\mathbf{e} - \mathbf{y} \leq \mathbf{e} - \mathbf{x}$. Since $F_\mathbf{e} = \Lambda_+$ we have by Lemma 4.5 that $\mathbf{x} \leq \mathbf{e}$ for all projections $\mathbf{x} \in \Lambda_+$. Hence by Lemma 4.6 it follows that $\mathbf{x}$ is a projection if and only if $\mathbf{e} - \mathbf{x}$ is a projection.

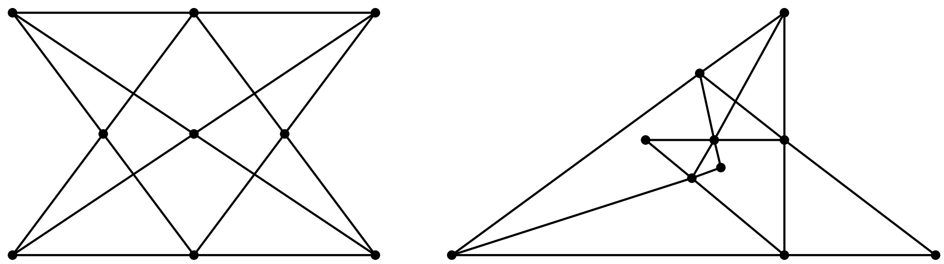
The following proposition gives a sufficient condition for two faces in Λ_+ to be modular with respect to the hyperbolic rank function.

Proposition 4.8. *If $\mathbf{x}, \mathbf{y} \in \Lambda_+$ are projections such that $F_\mathbf{x} \wedge F_\mathbf{y}$, $F_\mathbf{x} \vee F_\mathbf{y}$, $F_{\mathbf{e}-\mathbf{x}} \wedge F_{\mathbf{e}-\mathbf{y}}$ and $F_{\mathbf{e}-\mathbf{x}} \vee F_{\mathbf{e}-\mathbf{y}}$ all contain a projection in their relative interiors, then*

$$\mathrm{rk}(F_\mathbf{x}) + \mathrm{rk}(F_\mathbf{y}) = \mathrm{rk}(F_\mathbf{x} \wedge F_\mathbf{y}) + \mathrm{rk}(F_\mathbf{x} \vee F_\mathbf{y}).$$

Proof. Let $\mathbf{v}, \mathbf{w}, \mathbf{v}', \mathbf{w}'$ be the projections in the relative interiors of $F_\mathbf{x} \wedge F_\mathbf{y}$, $F_\mathbf{x} \vee F_\mathbf{y}$, $F_{\mathbf{e}-\mathbf{x}} \wedge F_{\mathbf{e}-\mathbf{y}}$ and $F_{\mathbf{e}-\mathbf{x}} \vee F_{\mathbf{e}-\mathbf{y}}$, respectively. Then $\mathbf{e} - \mathbf{w} \leq \mathbf{e} - \mathbf{x}$ and $\mathbf{e} - \mathbf{w} \leq \mathbf{e} - \mathbf{y}$, so that $\mathbf{e} - \mathbf{w} \in F_{\mathbf{e}-\mathbf{x}} \wedge F_{\mathbf{e}-\mathbf{y}}$ by Lemma 4.5. By Lemma 4.5 again, $\mathbf{e} - \mathbf{w} \leq \mathbf{v}'$. We also have $\mathbf{e} - \mathbf{v}' \geq \mathbf{x}$ and $\mathbf{e} - \mathbf{v}' \geq \mathbf{y}$ so that $\mathbf{e} - \mathbf{v}' \geq \mathbf{w}$, that is, $\mathbf{e} - \mathbf{w} \geq \mathbf{v}'$. Thus $F_{\mathbf{v}'} = F_{\mathbf{e}-\mathbf{w}}$ and analogously $F_{\mathbf{w}'} = F_{\mathbf{e}-\mathbf{v}}$. Since $\mathrm{rk} : L(\Lambda_+) \to \mathbb{N}$ is semimodular we have

$$\mathrm{rk}(\mathbf{x}) + \mathrm{rk}(\mathbf{y}) \geq \mathrm{rk}(\mathbf{v}) + \mathrm{rk}(\mathbf{w}),$$



(A) Non-Pappus (B) Non-Desargues

FIGURE 1. The Non-Pappus and Non-Desargues configurations.

and also

$$\mathrm{rk}(\mathbf{e}-\mathbf{x})+\mathrm{rk}(\mathbf{e}-\mathbf{y}) \geq \mathrm{rk}(\mathbf{e}-\mathbf{w})+\mathrm{rk}(\mathbf{e}-\mathbf{v}),$$

and so the proposition follows from Lemma 4.6. □

Corollary 4.9. *Let $\Lambda_+(h,\mathbf{e})$ be a hyperbolicity cone with trivial lineality space. Suppose all extreme rays of Λ_+ have the same hyperbolic rank, and that each face of Λ_+ contains a projection in its relative interior. Then $L(\Lambda_+)$ is a modular geometric lattice.*

Proof. Since each face of $L(\Lambda_+)$ except $\{0\}$ is generated by extreme rays, see e.g. [37, Cor. 18.5.2], it follows that $L(\Lambda_+)$ is atomic with all atoms (extreme rays) having the same hyperbolic rank by hypothesis. Suppose $\mathrm{rk}(\mathbf{a}) = c$ for all atoms $\mathbf{a} \in L(\Lambda_+)$. By modularity of the hyperbolic rank function (Proposition 4.8) and induction we see that c divides $\mathrm{rk}(F)$ for all $F \in L(\Lambda_+)$. It follows that the function defined by $\mathrm{rk}(F)/c$ is the proper rank function of $L(\Lambda_+)$, since it is modular and equal to one on each atom. □

5. Hyperbolic matroids and Euclidean Jordan algebras

In light of the generalized Lax conjecture it is of interest to find hyperbolic but non-linear (poly-) matroids. Until present the only known instances of non-linear hyperbolic matroids are the Vámos matroid [41] and a generalization of it [10]. The generalized Vámos matroids introduced in the following section provide an infinite family of such matroids. In this section we identify two further types of matroids that are hyperbolic but not linear through a connection with Euclidean Jordan algebras and projective geometry.

Some classical examples of non-linear matroids are obtained by relaxing a circuit hyperplane in a matroid that comes from a geometric configuration. In fact the Non-Fano, Non-Pappus and Non-Desargues matroids (see Fig 1) are all derived from the family n_3 of symmetric configurations on n points and n lines, arranged such that 3 lines pass through each point and 3 points lie on each line [19]. Note that such configurations need not be unique up to incidence isomorphism for given n. The Non-Fano, Non-Pappus and Non-Desargues matroids are all rank three matroids corresponding respectively to instances of the configurations $7_3, 9_3$ and 10_3 after removing one line. It is interesting to note how representability diminishes as we move upwards in this hierarchy: The Non-Fano matroid is representable over all fields that do not have characteristic 2 [34]. The Non-Pappus matroid is skew-linear but not linear [23], which is to say that it only admits representations over non-commutative division rings e.g. the quaternions $\mathbb{H}$. Moreover it is known that the Non-Desargues matroid is not even skew-linear [23]. On the other hand, it is known that the Non-Desargues matroid can be coordinatized by rank one projections over the octonions $\mathbb{O}$, see e.g. [18]. The octonions form a non-commutative and non-associative division ring over the reals.

An algebra $(A, \circ)$ over a field $\mathbb{K}$ is said to be a *Jordan algebra* if for all $a, b \in A$

$$a \circ b = b \circ a \qquad \text{and} \qquad a \circ ((a \circ a) \circ b) = (a \circ a) \circ (a \circ b).$$

Note in particular that Jordan algebras are not necessarily associative. A Jordan algebra is *Euclidean* if

$$a_1^2 + \cdots + a_k^2 = 0 \text{ implies } a_1 = \cdots = a_k = 0$$

for all $a_1, \ldots, a_k \in A$. By a theorem of Jordan, von Neumann and Wigner [25] the simple finite dimensional real Euclidean Jordan algebras classify into four infinite families and one exceptional algebra (the Albert algebra) as follows:

(i) $H_n(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) - the algebra of Hermitian $n \times n$ matrices over $\mathbb{K}$ with Jordan product $a \circ b = \frac{1}{2}(ab + ba)$.
(ii) $\mathbb{R}^n \oplus \mathbb{R}$ - the real inner product space with inner product $(u \oplus \lambda, v \oplus \mu) = (u, v)_{\mathbb{R}^n} + \lambda\mu$ and Jordan product $(u \oplus \lambda) \circ (v \oplus \mu) = (\mu u + \lambda v) \oplus ((u, v)_{\mathbb{R}^n} + \lambda\mu)$.
(iii) $H_3(\mathbb{O})$ - the algebra of octonionic Hermitian 3×3 matrices with Jordan product $a \circ b = \frac{1}{2}(ab + ba)$.

Let A be a real Euclidean Jordan algebra of rank r with identity e. A *Jordan frame* is a complete system of orthogonal idempotents of rank one, that is, rank one elements $c_1, \ldots, c_r \in A$ such that $c_i^2 = c_i$, $c_i \circ c_j = 0$ for $i \neq j$ and $c_1 + \cdots + c_r = e$. A characteristic property of finite dimensional real Euclidean Jordan algebras is the following spectral theorem, see [17, Theorem III.1.2].

Theorem 5.1. *Let A be a real Euclidean Jordan algebra of rank r. Then for each $x \in A$ there exists a Jordan frame $c_1, \ldots, c_r \in A$ and unique real numbers $\lambda_1(x), \ldots, \lambda_r(x)$ (the eigenvalues) such that*

$$x = \sum_{j=1}^{r} \lambda_j(x) c_j.$$

Moreover

$$\sum_{j : \lambda_j = \lambda} c_j$$

is uniquely determined for each eigenvalue λ.

A finite dimensional real Euclidean Jordan algebra is equipped with a hyperbolic determinant polynomial $\det : A \to \mathbb{R}$ given by

$$\det(x) = \prod_{j=1}^{r} \lambda_j(x).$$

Let P be a set of *points* and L a set of *lines*. Recall that a pair $G = (P, L)$ is a *projective geometry* if the following axioms are satisfied:

(i) For any two distinct points $a, b \in P$ there is a unique line $ab \in L$ containing a and b.
(ii) Any line contains at least three points.
(iii) If $a, b, c, d \in P$ are distinct points such that $ab \cap cd \neq \emptyset$ then $ac \cap bd \neq \emptyset$.

Each projective geometry is a (simple) modular geometric lattice, and each modular geometric lattice is a direct product of a Boolean algebra with projective geometries, see [2, p. 93]. The following proposition is essentially a known connection between Jordan algebras and projective geometries, which we prove here in the theory of hyperbolic polynomials.

Proposition 5.2. *Let A be a finite dimensional real Euclidean Jordan algebra and let Λ_+ denote the hyperbolicity cone of $\det : A \to \mathbb{R}$. Then $L(\Lambda_+)$ is a modular geometric lattice.*

In particular if A is simple, then $L(\Lambda_+)$ is a projective geometry.

Proof. By Theorem 5.1 the extreme rays of Λ_+ are multiples of rank one idempotents. Also, a face F_x contains the projection

$$c = \sum_{j:\lambda_j(x)\neq 0} c_j$$

in its relative interior. The proposition now follows from Corollary 4.9. □

The Non-Pappus and Non-Desargues configurations are depicted in Fig 1. The configurations give rise to rank 3 matroids where three points are dependent if and only if they are collinear. The Non-Pappus and Non-Desargues matroids are not linear but may be represented over the projective geometries associated to the Euclidean Jordan algebras $H_3(\mathbb{H})$ and $H_3(\mathbb{O})$, respectively. This may be deduced from the coordinatizations in [34, Example 1.5.14] and [18]. Hence by Proposition 5.2 we have

Theorem 5.3. *The Non-Pappus and Non-Desargues matroids are hyperbolic.*

6. Generalized Vámos Matroids with the (weak) half–plane property

In this section we provide an infinite family of hyperbolic matroids that do not arise from modular geometric lattices. Suppose that L is a lattice with a smallest element $\hat{0}$, and $f : L \to \mathbb{N}$ is a function satisfying

(i) $f(\hat{0}) = 0$,
(ii) if $x \leq y$, then $f(x) \leq f(y)$,
(iii) for any $x, y \in L$,

$$f(x) + f(y) \geq f(x \vee y) + f(x \wedge y).$$

If $x_1, \dots, x_m \in L$, then the function $r : 2^{[m]} \to \mathbb{N}$ defined by

$$r(S) = f\left(\bigvee_{i \in S} x_i\right)$$

defines a polymatroid. All polymatroids arise in this manner. Indeed if $r : 2^{[m]} \to \mathbb{N}$ is a polymatroid, we may take $L = 2^{[m]}$, $f = r$, and $x_i = \{i\}$ for each $i \in [m]$. However if f is modular, i.e.,

$$f(x) + f(y) = f(x \vee y) + f(x \wedge y), \qquad \text{for all } x, y \in L,$$

we say that r is *modularly represented*. Hence all linear matroids as well as all projective geometries are modularly represented. Although Ingleton's proof [23] of the next lemma only concerns linear matroids it extends verbatim to modularly represented matroids.

Lemma 6.1 (Ingleton's Inequality, [23])**.** *Suppose $r : 2^E \to \mathbb{N}$ is a modularly represented polymatroid and $A, B, C, D \subseteq E$. Then*

$$\begin{aligned} &r(A \cup B) + r(A \cup C \cup D) + r(C) + r(D) + r(B \cup C \cup D) \leq \\ &r(A \cup C) + r(A \cup D) + r(B \cup C) + r(B \cup D) + r(C \cup D). \end{aligned}$$

The *Vámos matroid* V_8 is the rank-four matroid on $E = [8]$ having set of bases

$$\mathcal{B}(V_8) = \binom{E}{4} \setminus \{\{1,2,3,4\},\{1,2,5,6\},\{1,2,7,8\},\{3,4,5,6\},\{5,6,7,8\}\}.$$

The rank function of the Vámos matroid fails to satisfy Ingleton's inequality (see [23]), and hence it is not modularly represented. Nevertheless Wagner and Wei [41] proved that V_8 has the half-plane property, and hence V_8 is hyperbolic. This was used in [7] to provide counterexamples to stronger algebraic versions of the generalized Lax conjecture.

Burton, Vinzant and Youm [10] studied an infinite family of generalized Vámos matroids, $\{V_{2n}\}_{n\geq 4}$, and conjectured that all members of the family have the half-plane property. They proved their conjecture for $n = 5$. Below we generalize their construction and construct a family of matroids; one matroid for each uniform hypergraph. We prove that all matroids corresponding to simple graphs are HPP, and that all matroids corresponding to uniform hypergraphs are WHPP. In particular this will prove the conjecture of Burton *et al.*

Recall that a rank r *paving matroid* is a matroid such that all its circuits have size at least r. Paving matroids may be characterized in terms of d-partition. A *d-partition* of a set E is a collection $\mathcal{S}$ of subsets of E all of size at least d, such that every d-subset of E lies in a unique member of $\mathcal{S}$. The d-partition $\mathcal{S} = \{E\}$ is the *trivial d-partition*. For a proof of the next proposition see [34, Prop. 2.1.21].

Proposition 6.2. *The hyperplanes of any rank $d + 1 \geq 2$ paving matroid form a non-trivial d-partition.*

Conversely, the elements of a non-trivial d-partition form the set of hyperplanes of a paving matroid of rank $d + 1$.

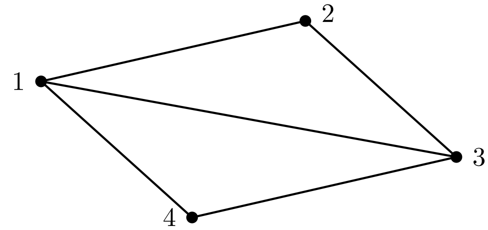
A paving matroid of rank r is *sparse* if its hyperplanes all have size $r - 1$ or r.

Recall that a *hypergraph* H consists of a set $V(H)$ of *vertices* together with a set $E(H) \subseteq 2^{V(H)}$ of *hyperedges*. We say that a hypergraph H is *d-uniform* if all hyperedges have size d.

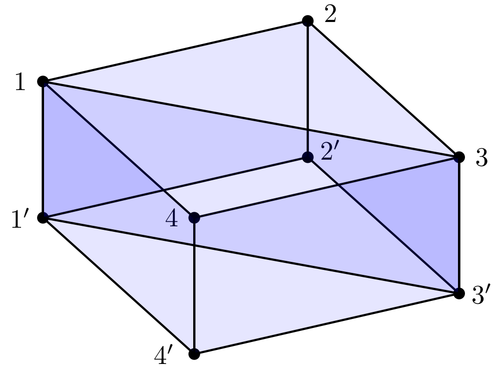
Theorem 6.3. *Let H be an d-uniform hypergraph on $[n]$, and let $E = \{1, 1', \dots, n, n'\}$. Let*

$$\mathcal{B}(V_H) = \binom{E}{2d} \setminus \{e \cup e' : e \in E(H)\},$$

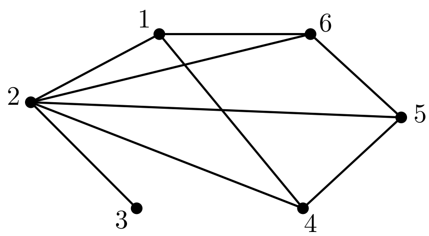
in which $e' := \{i' : i \in e\}$ for each $e \in E(H)$. Then $\mathcal{B}(V_H)$ is the set of bases of a sparse paving matroid V_H of rank $2d$.



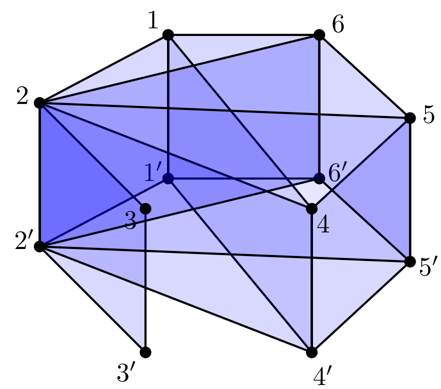
(A) G (Diamond graph)



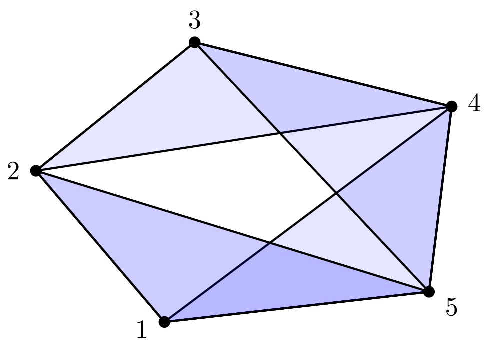
(B) $V_G \cong V_8$ (Vámos matroid)



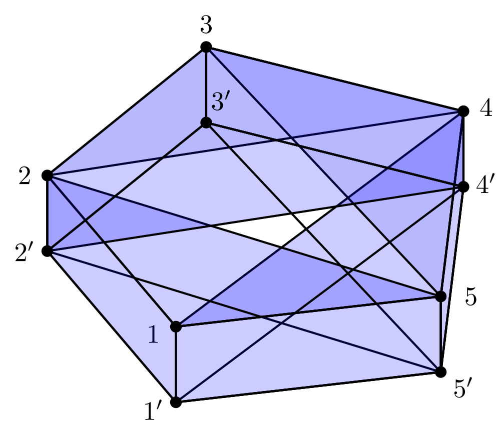
(A) A simple graph G



(B) The matroid V_G



(A) A 3-uniform hypergraph H



(B) The matroid V_H

Proof. Let

$$\mathcal{S} = \{e \cup e' : e \in E(H)\} \cup \left\{ S \in \binom{E}{2d-1} : S \subset e \cup e' \text{ for no } e \in E(H) \right\}.$$

Then $\mathcal{S}$ is a $(2d-1)$-partition, and so it defines a sparse paving matroid with set of bases $\binom{E}{2d} \setminus \{e \cup e' : e \in E(H)\}$ by Proposition 6.2. $\square$

Let $\mathcal{V} = \{V_H : H$ is a d-uniform hypergraph on $[n]$ with $0 < d \leq n, n \in \mathbb{N}\}$.

Example 6.4. If G is the diamond graph (Fig 2a) then $V_G = V_8$, the Vámos matroid. Moreover $V_{\bar{K}_n} = U_{4,2n}$, where $\bar{K}_n$ denotes the complement of the complete graph on n vertices and $U_{4,2n}$ denotes the uniform rank 4 matroid on $2n$ elements. The family $\{V_{2n}\}_{n\geq 4}$ studied by Burton *et al.* [10] corresponds to V_{G_n} where G_n is an n-cycle with edges $\{1,i\}$, $i = 2, \dots, n$, adjoined.

We postpone the proofs of the next two theorems until Section 9.

Theorem 6.5. *All matroids in $\mathcal{V}$ are hyperbolic, i.e., they all have the weak half-plane property.*

Theorem 6.6. *For each simple graph G, V_G has the half–plane property.*

If G contains the Diamond graph as an induced subgraph then the rank function of V_G fails to satisfy Ingleton's inequality, and thus V_G is hyperbolic but not modularly represented.

There is no full analogue of Theorem 6.6 in the hypergraph setting. To see this let H be the complete 3-uniform hypergraph on [6]. The bases generating polynomial of V_H is given by

$$h_{V_H}(\mathbf{x}) = e_6(x_1, x_{1'}, \dots, x_6, x_{6'}) - e_3(x_1x_{1'}, \dots, x_6x_{6'}).$$

By Lemma 2.7 we have that $h_{V_H}(\mathbf{x})$ is stable if and only if $h_{V_H}(\mathbf{x})$ is hyperbolic with $\mathbb{R}^{12}_+ \subseteq \Lambda_+(h_{V_H})$. Take $\mathbf{e} = (1,1,0,\dots,0) \in \mathbb{R}^{12}_+$ and $\mathbf{x} \in \mathbb{R}^{12}$ with $x_1 = x_{1'} = 0$, $x_2 = x_{2'} = x_3 = x_{3'} = 2$ and $x_i = x_{i'} = -1$ for $i > 3$. Then

$$h_{V_H}(t\mathbf{e} - \mathbf{x}) = -4t^2 - 36t - 160$$

is a quadratic polynomial with non-real zeros $t = \frac{9}{2} \pm \frac{1}{2}\sqrt{79}i$. Hence V_H does not have the half-plane property. Clearly if V_8 is a minor of V_H then V_H cannot be representable. Below we give an example of a non-representable matroid V_H with no Vámos minor. Hence this constitutes a genuinely new instance of a hyperbolic matroid in the family which is not representable.

Example 6.7. The following linear rank inequality in six variables was identified by Dougherty *et al.* [15]

$$\begin{aligned}
&r(A\cup D) + r(B\cup C) + r(C\cup E) + r(E\cup F) + r(B\cup D\cup F) + r(A\cup B\cup C\cup D) + \\
&r(A\cup B\cup C\cup E) + r(A\cup C\cup E\cup F) + r(A\cup D\cup E\cup F) \leq \\
&r(A\cup B\cup C) + r(A\cup B\cup D) + r(A\cup C\cup E) + r(A\cup D\cup F) + r(A\cup E\cup F) + \\
&r(B\cup C\cup D) + r(B\cup C\cup E) + r(C\cup E\cup F) + r(D\cup E\cup F).
\end{aligned}$$

This inequality is satisfied by all polymatroids r representable over some field, where $r : 2^{[n]} \to \mathbb{N}$, $n \in \mathbb{N}$ and $A, B, C, D, E, F \subseteq [n]$. We proceed by designing a 3-uniform hypergraph H on [6] such that V_H violates the above inequality. Let

$$A = \{1,1'\},\ B = \{2,2'\},\ C = \{3,3'\},\ D = \{4,4'\},\ E = \{5,5'\},\ F = \{6,6'\}.$$

By taking the hypergraph H with edges

$$\{1,2,3\}, \{1,2,4\}, \{1,3,5\}, \{1,4,6\}, \{1,5,6\}, \{2,3,4\}, \{2,3,5\}, \{3,5,6\}, \{4,5,6\}$$

we see that V_H violates the above inequality. One checks that V_8 is not a minor of V_H.

7. Consequences for the generalized Lax conjecture

Helton and Vinnikov [22] conjectured that if a polynomial $h \in \mathbb{R}[x_1, \dots, x_n]$ is hyperbolic with respect to $\mathbf{e} = (e_1, \dots, e_n) \in \mathbb{R}^n$, then there exist positive integers M, N and a linear polynomial $\ell(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_n]$ which is positive on $\Lambda_{++}(h, \mathbf{e})$ such that

$$\ell(\mathbf{x})^{M-1} h(\mathbf{x})^N = \det\left(\sum_{i=1}^{n} x_i A_i\right)$$

for some symmetric matrices $A_1, \dots, A_n$ such that $e_1 A_1 + \cdots + e_n A_n$ is positive definite. In [7] the second author used the bases generating polynomial h_{V_8} of the Vámos matroid to prove that there is no linear polynomial $\ell(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_n]$ which is nonnegative on the hyperbolicity cone of h_{V_8} and positive integers M, N such that

$$\ell(\mathbf{x})^{M-1} h_{V_8}(\mathbf{x})^N = \det\left(\sum_{i=1}^{8} x_i A_i\right)$$

for some symmetric matrices $A_1, \dots, A_8$ with $e_1 A_1 + \cdots + e_8 A_8$ positive definite. We will here construct further "counterexamples" that preclude more general factors $q(\mathbf{x})$ in (2.1). First we prove two lemmata of matroid theoretic nature. If $r : 2^E \to \mathbb{N}$ is a polymatroid and $A \subseteq E$, we say that A is *spanning* if $r(A) = r(E)$. Moreover $A \subset E$ is a *hyperplane* if it is a maximal non–spanning set.

Lemma 7.1. *For $n, r, c \geq 1$, let $\mathcal{P}(n, r, c)$ be the family of all polymatroids of rank at most r on n elements such that each hyperplane has at most $r - 1 + c$ elements. If $\alpha(n, r, c)$ denotes the maximal number of non-spanning sets of size r taken over all matroids in $\mathcal{P}(n, r, c)$, then*

$$\alpha(n, r, c) \leq c\binom{n}{r-1}. \tag{7.1}$$

Proof. If $r = 1$, then each hyperplane has at most c elements, i.e., there are at most c loops so that $\alpha(n, r, c) = c$ as desired. The proof is by induction over $n \geq 1$ where $r \geq 1$. The lemma is trivially true for $n = 1$.

Let $\mathcal{P} \in \mathcal{P}(n, r, c)$, where $n, r \geq 2$. If $n \leq r$, then (7.1) is trivially true. Assume $n > r$. Let i be a non-loop of $\mathcal{P}$. If $r(E \setminus i) < r(E)$, then $E \setminus i$ is a hyperplane and hence $n - 1 \leq r - 1 + c$, so that $\binom{n}{r} \leq c\binom{n}{r-1}$. Hence we may assume $r(E \setminus i) = r(E) > 0$.

If S is a non-spanning r-set of $\mathcal{P}$, then either S is a non-spanning r-set of $\mathcal{P} \setminus i$, or $S \setminus i$ is a non-spanning $(r-1)$-set of $\mathcal{P}/i$. Hence $\mathcal{P} \setminus i \in \mathcal{P}(n-1, r, c)$ and $\mathcal{P}/i \in \mathcal{P}(n-1, r-1, c)$, and thus

$$\begin{aligned} \alpha(n, r, c) &\leq \alpha(n-1, r, c) + \alpha(n-1, r-1, c) \\ &\leq c\binom{n-1}{r-1} + c\binom{n-1}{r-2} = c\binom{n}{r-1}, \end{aligned}$$

by induction. □

Lemma 7.2. *Let $\mathcal{P}_i$, $i = 1, \dots, s$, be polymatroids on $[n]$ of rank at most $k-1$ such that no hyperplane has more than k elements. If $n \geq (2s+1)k-1$, then there is a set S of size k such that there are at least two $(k-1)$-subsets of S that are spanning in all $\mathcal{P}_i$, $i = 1, \dots, s$.*

Proof. Suppose the conclusion is not true. Let

$$A = \left\{ (S,T) \in \binom{[n]}{k-1} \times \binom{[n]}{k} : S \subset T \text{ and } S \text{ is not spanning in } \mathcal{P}_i \text{ for some } i \in [s] \right\}.$$

Then

$$|A| \geq (k-1)\binom{n}{k}.$$

Furthermore by Lemma 7.1 we have

$$\begin{aligned} |A| &= \#\left\{ S \subseteq \binom{[n]}{k-1} : S \text{ is not spanning in } \mathcal{P}_i \text{ for some } i \in [s] \right\} \cdot (n-k+1) \\ &\leq s\alpha(n, k-1, 2)(n-k+1) \\ &\leq 2s\binom{n}{k-2}(n-k+1). \end{aligned}$$

Hence

$$(k-1)\binom{n}{k} \leq 2s\binom{n}{k-2}(n-k+1).$$

Solving for n gives $n \leq (2s+1)k-2$, which proves the lemma. □

Given positive integers n and k, consider the k-uniform hypergraph $H_{n,k}$ on $[n+2]$ containing all hyperedges $e \in \binom{[n+2]}{k}$ except those for which $\{n+1, n+2\} \subseteq e$. By Theorem 6.5 the matroid $V_{H_{n,k}}$ is hyperbolic and therefore has a stable weighted bases generating polynomial $h_{V_{H_{n,k}}}$ by Proposition 3.2. The polynomial $h_{n,k} \in \mathbb{R}[x_1, \dots, x_{n+2}]$ obtained from the multiaffine polynomial $h_{V_{H_{n,k}}}$ by identifying the variables x_i and $x_{i'}$ pairwise for all $i \in [n+2]$ is stable. Hence by Lemma 2.7 we have $\mathbb{R}_+^{n+2} \subseteq \Lambda_+(h_{n,k})$ so $h_{n,k}$ is hyperbolic with respect to $\mathbf{1}$.

Theorem 7.3. *Let n and k be a positive integers. Suppose there exists a positive integer N and a hyperbolic polynomial $q(\mathbf{x})$ such that*

$$q(\mathbf{x})h_{n,k}(\mathbf{x})^N = \det\left(\sum_{i=1}^{n+2} x_i A_i\right) \tag{7.2}$$

with $\Lambda_+(h_{n,k}) \subseteq \Lambda_+(q)$ for some symmetric matrices $A_1, \dots, A_{n+2}$ such that $A_1 + \cdots + A_{n+2}$ is positive definite and

$$q(\mathbf{x}) = \prod_{i=1}^{s} p_j(\mathbf{x})^{\alpha_i}$$

for some irreducible hyperbolic polynomials $p_1, \dots, p_s \in \mathbb{R}[x_1, \dots, x_{n+2}]$ of degree at most $k-1$ where $\alpha_1, \dots, \alpha_s$ are positive integers. Then

$$n < (2s+1)k-1.$$

Proof. Suppose the hypotheses are satisfied and $n \geq (2s+1)k-1$. Let $r_0 : 2^{[n]} \to \mathbb{N}$ be the hyperbolic polymatroid defined by $h_{n,k}$ and $\mathcal{V} = (\delta_1, \ldots, \delta_n)$, where δ_i, $i \in [n]$ are the standard basis vectors. Hence $r_0(S)$ is the rank of $S \cup \{i' : i \in S\}$ in the matroid $V_{H_{n,k}}$. Moreover, for $i \in [s]$, let $r_i : 2^{[n]} \to \mathbb{N}$ be the hyperbolic polymatroid defined by p_i and $\mathcal{V} = (\delta_1, \ldots, \delta_n)$. Any subset S of $[n]$ of size at least $k+1$ is spanning for r_0, and thus $\sum_{i \in S} \delta_i \in \Lambda_{++}(h_{n,k})$. Hence $\sum_{i \in S} \delta_i \in \Lambda_{++}(p_i)$, and thus S is spanning with respect to r_i for all $i \in [s]$. By Lemma 7.2, since $n \geq (2s+1)k-1$, there exists a subset $T \subseteq [n]$ of size k containing at least two distinct subsets X, Y of size $k-1$ with full rank with respect to all hyperbolic polymatroids r_i, $i = 1, \ldots, s$. Let $x, y \in T$ be the unique elements in X, Y, respectively, not contained in $Z = X \cap Y$. Define

$$A = Z \cup \{n+1\}, \quad B = Z \cup \{n+2\}, \quad C = Z \cup \{x\}, \quad D = Z \cup \{y\}.$$

Now $A \cup B, A \cup C \cup D$ and $B \cup C \cup D$ have full rank with respect to r_0. Since $\Lambda_{++}(h_{n,k}) \subseteq \Lambda_{++}(p_i)$ for all i, we see that $A \cup B, A \cup C \cup D$ and $B \cup C \cup D$ have full rank with respect to r_i for all i. Hence the rank of each set to the left in the Ingleton inequality have full rank with respect to r_i, so that

$$\begin{aligned} &r_i(A \cup B) + r_i(A \cup C \cup D) + r_i(C) + r_i(D) + r_i(B \cup C \cup D) \geq \\ &r_i(A \cup C) + r_i(A \cup D) + r_i(B \cup C) + r_i(B \cup D) + r_i(C \cup D) \end{aligned}$$

for $i = 1, \ldots, s$. Note also that

$$\begin{aligned} &r_0(A \cup B) + r_0(A \cup C \cup D) + r_0(C) + r_0(D) + r_0(B \cup C \cup D) = \\ &2k + 2k + (2k-2) + (2k-2) + 2k > (2k-1) + (2k-1) + (2k-1) + (2k-1) + (2k-1) = \\ &r_0(A \cup C) + r_0(A \cup D) + r_0(B \cup C) + r_0(B \cup D) + r_0(C \cup D). \end{aligned}$$

Thus r_0 violates the Ingleton inequality. Let $\mathcal{R}$ denote the representable polymatroid with rank function

$$r_{\mathcal{R}}(S) = \operatorname{rank}\left(\sum_{i \in S} A_i\right).$$

for all $S \subseteq [n]$. Then, by (7.2),

$$r_{\mathcal{R}}(S) = \operatorname{rank}\left(\sum_{i \in S} A_i\right) = \sum_{i=1}^{s} \alpha_i r_i(S) + N r_0(S).$$

Hence $r_{\mathcal{R}}$ violates Ingleton's inequality, a contradiction. □

Hence, for n sufficiently large, q in (2.1) either has an irreducible factor of large degree or is the product of many factors of low degree.

Example 7.4. Consider

$$h_{2,2} = x_1^2 x_2^2 + 4(x_1 + x_2 + x_3 + x_4)(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4).$$

The polynomial $h_{2,2}$ comes from the bases generating polynomial of the Vámos matroid under the restriction $x_i = x_{i'}$ for $i = 1, \ldots, 4$. Kummer [26] found real symmetric matrices A_i, $i = 1, \ldots, 4$ with $A_1 + A_2 + A_3 + A_4$ positive definite and a hyperbolic polynomial q of degree 3 with $\Lambda_+(h_{2,2}) \subseteq \Lambda_+(q)$ such that

$$q(\mathbf{x}) h_{2,2}(\mathbf{x}) = \det\left(x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4\right),$$

where

$$q(\mathbf{x}) = 32(2x_1 + 3x_2 + 3x_3 + 4x_4)(x_1x_2 + x_1x_3 + 2x_1x_4 + x_2x_4 + x_3x_4).$$

If $s = 2$ and $k = 3$ in Theorem 7.3 it follows that there exists no linear and quadratic hyperbolic polynomials $\ell(\mathbf{x}), q(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_{16}]$ respectively such that $h_{14,3}(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_{16}]$ has a positive definite representation of the form

$$\ell(\mathbf{x})q(\mathbf{x})h_{14,3}(\mathbf{x}) = \det\left(\sum_{i=1}^{16} x_i A_i\right)$$

with $\Lambda_+(h_{14,3}) \subseteq \Lambda_+(\ell q)$.

8. Nonnegative symmetric polynomials

Recall that a polynomial $P(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_n]$ is *nonnegative* if $P(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, and it is *symmetric* if it is invariant under the action (permuting the variables) of the symmetric group of order n. In this section we prove that certain symmetric polynomials are nonnegative. This is needed for the proof of Theorem 6.5. The results are interesting in their own right, and they generalize several well known inequalities in the literature.

Recall that a *partition* of a natural number d is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of natural numbers such that $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\lambda_1 + \lambda_2 + \cdots = d$. We write $\lambda \vdash d$ to denote that λ is a partition of d. The *length*, $\ell(\lambda)$, of λ is the number of nonzero entries of λ. If λ is a partition and $\ell(\lambda) \leq n$, then the *monomial symmetric polynomial*, m_α, is defined as

$$m_\lambda(x_1, \ldots, x_n) = \sum x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n},$$

where the sum is over all distinct permutations $(\beta_1, \beta_2, \ldots, \beta_n)$ of $(\lambda_1, \ldots, \lambda_n)$. If $\ell(\lambda) > n$, we set $m_\lambda(\mathbf{x}) = 0$. If $k_1, \ldots, k_\ell$ are distinct positive integers and $a_1, \ldots, a_\ell \in \mathbb{N}$ we denote by $k_1^{a_1} k_2^{a_2} \cdots k_\ell^{a_\ell}$ the unique partition of $a_1 + \cdots + a_\ell$ with exactly a_j coordinates equal to k_j for $1 \leq j \leq \ell$. The dth *elementary symmetric polynomial* is $e_d(\mathbf{x}) = m_{1^d}(\mathbf{x})$, and the dth *power symmetric polynomial* is $p_d(\mathbf{x}) = m_d(\mathbf{x})$.

Nonnegative symmetric polynomials have been studied in several areas of mathematics, see [3, 12, 16] and the references therein. We will initially concentrate on nonnegative polynomials of the form

$$\sum_{k=0}^{2r} a_k e_k(\mathbf{x}) e_{2r-k}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^m, \tag{8.1}$$

where r is a positive integer and $a_k \in \mathbb{R}$ for $0 \leq k \leq 2r$. Hence these are the nonnegative symmetric polynomials spanned by $\{m_{2^k 1^{2(r-k)}} : 0 \leq k \leq r\}$. A classical family of such nonnegative and symmetric polynomials was found already by Newton [33]:

$$\frac{e_r(\mathbf{x})^2}{\binom{n}{r}^2} - \frac{e_{r-1}(\mathbf{x})}{\binom{n}{r-1}} \frac{e_{r+1}(\mathbf{x})}{\binom{n}{r+1}} \geq 0,$$

for $\mathbf{x} \in \mathbb{R}^m$ with $m \leq n$ and $1 \leq r \leq n-1$. Letting $n \to \infty$ in Newton's inequalities we obtain the Laguerre–Turán inequalities (see e.g. [14]):

$$re_r(\mathbf{x})^2 - (r+1)e_{r-1}(\mathbf{x})e_{r+1}(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \mathbb{R}^m, m \geq 1.$$

A different but equivalent view on nonnegative symmetric polynomial is that of inequalities satisfied by the derivatives of a real–rooted polynomial: Let $\{a_k\}_{k=0}^{2m}$ be a sequence of real numbers. Then the polynomial (8.1) is nonnegative if and only if

$$\sum_{k=0}^{2r} a_k \binom{2r}{k} f^{(k)}(t) f^{(2r-k)}(t) \geq 0, \quad t \in \mathbb{R}, \tag{8.2}$$

holds for all real–rooted polynomials f of degree at most m. Indeed by translation invariance (8.2) holds for all real–rooted polynomials f of degree at most m if and only if (8.2) holds at $t = 0$ for all real–rooted polynomials f of degree at most m. Hence if $f(t) = \prod_{j=1}^{m}(1 + x_j t)$, then the left–hand–side of (8.2) at $t = 0$ is the same as (8.1) up to a constant factor $(2r)!$. The following inequality is due to Jensen.

Theorem 8.1 (Jensen [24])**.**

$$\sum_{k=0}^{2r} (-1)^{r+j} \binom{2r}{k} f^{(k)}(t) f^{(2r-k)}(t) \geq 0, \quad t \in \mathbb{R}, \tag{8.3}$$

for all real–rooted polynomials f.

The inequality (8.3) follows easily from a symmetric function identity as follows

$$\begin{aligned}
\sum_{r=0}^{n} m_{2^r}(\mathbf{x}) t^{2r} &= \sum_{r=0}^{n} e_r(x_1^2, \dots, x_n^2) t^{2r} \\
&= \prod_{j=1}^{n} (1 + x_j^2 t^2) = \prod_{j=1}^{n} (1 + i x_j t) \prod_{j=1}^{n} (1 - i x_j t) \\
&= \left(\sum_{k=0}^{n} i^k e_k(\mathbf{x}) t^k \right) \left(\sum_{k=0}^{n} (-i)^k e_k(\mathbf{x}) t^k \right) \\
&= \sum_{r=0}^{n} \left(\sum_{k=0}^{2r} (-1)^{k+r} e_k(\mathbf{x}) e_{2r-k}(\mathbf{x}) \right) t^{2r}.
\end{aligned}$$

Clearly $m_{2^r}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, so that inequality (8.3) follows from

$$m_{2^r}(\mathbf{x}) = \sum_{k=0}^{2r} (-1)^{k+r} e_k(\mathbf{x}) e_{2r-k}(\mathbf{x}). \tag{8.4}$$

Lemma 8.2. *If r is a positive integer and $0 \leq t \leq 2/r$, then*

$$m_{2^r}(\mathbf{x}) + t m_{2^{r-1}1^2}(\mathbf{x})$$

is a sum of squares (sos for short), and in particular nonnegative.

Proof. Since $m_{2^r}(\mathbf{x})$ is a sos it suffices to consider $t = 2/r$, by convexity. Note that

$$m_{2^{r-1}1^2}(\mathbf{x}) = \sum_{|S|=r-1} e_2(\mathbf{x}^S) \prod_{i \in S} x_i^2,$$

where $\mathbf{x}^S = \mathbf{x} \setminus \{x_i : i \in S\}$. Using $e_2(\mathbf{x}) = e_1(\mathbf{x})^2/2 - p_2(\mathbf{x})/2$

$$\begin{aligned}
m_{2^{r-1}1^2}(\mathbf{x}) &= \frac{1}{2} \sum_{|S|=r-1} e_1(\mathbf{x}^S)^2 \prod_{i \in S} x_i^2 - \frac{1}{2} \sum_{|S|=r-1} p_2(\mathbf{x}^S) \prod_{i \in S} x_i^2 \\
&= S(\mathbf{x}) - \frac{r}{2} m_{2^r}(\mathbf{x}),
\end{aligned}$$

where $S(\mathbf{x})$ is a sum of squares. Indeed

$$\frac{1}{2}\sum_{|S|=r-1} p_2(\mathbf{x}^S)\prod_{i\in S} x_i^2 = Cm_{2^r}(\mathbf{x})$$

for some C, and setting $\mathbf{x} = (1, \dots, 1)$ one sees that

$$\frac{1}{2}\binom{n}{r-1}(n-r+1) = C\binom{n}{r},$$

so that $C = r/2$. The lemma follows. □

Let $P(\mathbf{x})$ be a symmetric polynomial. Suppose $P(\mathbf{x}) = Q(e_1(\mathbf{x}), \dots, e_m(\mathbf{x}))$ is the unique expression of P in terms of the elementary symmetric polynomials. If Q is of degree d, let $H(x_0, x_1, \dots, x_m) = x_0^d Q(x_1/x_0, \dots, x_m/x_0)$ be its homogenization, and let

$$L(P) := H(e_1(\mathbf{x}), 2e_2(\mathbf{x}), \dots, (m+1)e_{m+1}(\mathbf{x}))$$

be the *lift* of P. This operation enables us to lift symmetric nonnegative polynomial inequalities to higher degrees.

Lemma 8.3. *If P is a symmetric and nonnegative polynomial, then so is its lift $L(P)$.*

Proof. Note first that if P is nonnegative and symmetric, then the degree of Q above is even. Indeed if $\mathbf{x}(t) = (t, x_2, \dots, x_n)$ where $x_2, \dots, x_n \in \mathbb{R}$ are generic and t is a variable, then we obtain a univariate nonnegative polynomial $t \mapsto P(\mathbf{x}(t))$ of degree d. Hence d is even. Now if $\mathbf{x} \in \mathbb{R}^n$ is such that $e_1(\mathbf{x}) \neq 0$, then there is a $\mathbf{y} \in \mathbb{R}^n$ such that $e_k(\mathbf{y}) = (k+1)e_{k+1}(\mathbf{x})/e_1(\mathbf{x})$ for all k. Indeed

$$\frac{1}{e_1(\mathbf{x})}\frac{d}{dt}\prod_{k=1}^{n}(1+x_kt) = \prod_{k=0}^{n}(1+y_kt), \qquad \text{where } y_n = 0,$$

since the operator d/dt preserves real–rootedness. Thus

$$L(P)(\mathbf{x}) = P(\mathbf{y})$$

and the proof follows. □

Lemma 8.4. *The lift of $m_{2^{r-1}}(\mathbf{x})$ is*

$$r^2 m_{2^r}(\mathbf{x}) + 2m_{2^{r-1}1^2}(\mathbf{x}).$$

Proof. By (8.4), the lift of $m_{2^{r-1}}(\mathbf{x})$ is

$$\begin{aligned} L &:= \sum_{j=0}^{2r-2}(-1)^{j+r-1}(j+1)(2r-1-j)e_{j+1}(\mathbf{x})e_{2r-1-j}(\mathbf{x}) \\ &= \sum_{j=0}^{2r}(-1)^{j+r}j(2r-j)e_j(\mathbf{x})e_{2r-j}(\mathbf{x}). \end{aligned}$$

The coefficient in front of $m_{2^k1^{2(r-k)}}(\mathbf{x})$ in the expansion of $e_j(\mathbf{x})e_{2r-j}(\mathbf{x})$ in the monomial bases is seen to be $\binom{2r-2k}{j-k}$. (Look at how many times we get the monomial $x_1^2x_2^2\cdots x_k^2x_{k+1}x_{k+2}\cdots$ in the expansion of the $e_j(\mathbf{x})e_{2r-j}(\mathbf{x})$.) Hence the coefficient infront of $m_{2^k1^{2(r-k)}}(\mathbf{x})$ in the expansion of L in the monomial basis is

$$a_k = \sum_{j=0}^{2r}(-1)^{j+r}j(2r-j)\binom{2r-2k}{j-k}.$$

Now $a_r = r^2, a_{r-1} = 2$, and $a_k = 0$ otherwise. This follows from the fact that if p is a polynomial of degree d, then

$$\sum_{j=0}^{n}(-1)^j p(j)\binom{n}{j} = 0$$

whenever $n > d$. □

Our next lemma is a refinement of the Laguerre–Turán inequalities and may be formulated as *the Laguerre–Turán inequalities beat Jensen's inequalities* (8.3). Lemma 8.5 is also a generalization of [16, Theorem 3], where the case $r = 2$ was proved. If $P, Q \in \mathbb{R}[\mathbf{x}]$, we write $P \leq Q$ if $Q - P$ is a nonnegative polynomial.

Lemma 8.5. *If $r \geq 1$, then*

$$m_{2^r}(\mathbf{x}) \leq re_r(\mathbf{x})^2 - (r+1)e_{r-1}(\mathbf{x})e_{r+1}(\mathbf{x}).$$

Proof. The proof is by induction over r. For $r = 1$ we have equality. Let $r \geq 2$ and assume the inequality in the case $r - 1$ by induction. Taking the lift of this inequality and applying Lemmas 8.3 and 8.4, we find that

$$r^2 m_{2^r}(\mathbf{x}) + 2m_{2^{r-1}1^2}(\mathbf{x}) \leq (r-1)r^2 e_r(\mathbf{x})^2 - r(r-1)(r+1)e_{r-1}(\mathbf{x})e_{r+1}(\mathbf{x}).$$

By Lemma 8.2

$$(r^2 - r)m_{2^r}(\mathbf{x}) \leq r^2 m_{2^r}(\mathbf{x}) + 2m_{2^{r-1}1^2}(\mathbf{x}),$$

and the lemma follows. □

Lemma 8.6. *If $r \geq 2$ is an integer, then*

$$(a_r e_{r-1}(\mathbf{x})e_r(\mathbf{x}) - e_{r-2}(\mathbf{x})e_{r+1}(\mathbf{x}))^2 \geq C_r e_{r-2}(\mathbf{x})e_r(\mathbf{x})m_{2^r}(\mathbf{x}), \tag{8.5}$$

where

$$a_r = 3\frac{r-1}{r+1} \quad \textit{and} \quad C_r = 9\frac{r-1}{(r+1)^2}.$$

Proof. We prove the inequality by induction over $r \geq 2$. Assume $r = 2$. The polynomial $(t, \mathbf{x}) \mapsto e_4(t, t, x_1, x_1, \ldots, x_n, x_n)$ is stable, this may for instance be deduced from the Grace-Walsh-Szegő theorem (see Remark 9.3). It specializes to a real-rooted (or identically zero) polynomial when we set $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$:

$$(4e_2(\mathbf{x}) + p_2(\mathbf{x}))t^2 + 4(e_1(\mathbf{x})e_2(\mathbf{x}) + e_3(\mathbf{x}))t + m_{2^2}(\mathbf{x}) + 4e_1(\mathbf{x})e_3(\mathbf{x}).$$

Hence its discriminant is nonnegative, which gives

$$(e_3(\mathbf{x}) + e_1(\mathbf{x})e_2(\mathbf{x}))^2 \geq (e_2(\mathbf{x}) + p_2(\mathbf{x})/4)(m_{2^2}(\mathbf{x}) + 4e_1(\mathbf{x})e_3(\mathbf{x})).$$

To prove (8.5) for $r = 2$ we may assume $e_2(\mathbf{x}) > 0$. Rewriting (8.5) as

$$(e_3(\mathbf{x}) + e_1(\mathbf{x})e_2(\mathbf{x}))^2 \geq e_2(\mathbf{x})(m_{2^2}(\mathbf{x}) + 4e_1(\mathbf{x})e_3(\mathbf{x})),$$

we may assume also $m_{2^2}(\mathbf{x}) + 4e_1(\mathbf{x})e_3(\mathbf{x}) > 0$. Then, since $p_2(\mathbf{x}) \geq 0$,

$$\begin{aligned}(e_3(\mathbf{x}) + e_1(\mathbf{x})e_2(\mathbf{x}))^2 &\geq (e_2(\mathbf{x}) + p_2(\mathbf{x})/4)(m_{2^2}(\mathbf{x}) + 4e_1(\mathbf{x})e_3(\mathbf{x})) \\ &\geq e_2(\mathbf{x})(m_{2^2}(\mathbf{x}) + 4e_1(\mathbf{x})e_3(\mathbf{x}))\end{aligned}$$

which proves the lemma for $r=2$.

Assume that the inequality is true for a given $r \geq 2$. We lift the inequality for r and use Lemma 8.4 to get

$$\begin{aligned}&(a_r r(r+1)e_r(\mathbf{x})e_{r+1}(\mathbf{x}) - (r-1)(r+2)e_{r-1}(\mathbf{x})e_{r+2}(\mathbf{x}))^2 \geq \\ &C_r(r-1)(r+1)^3 e_{r-1}(\mathbf{x})e_{r+1}(\mathbf{x})\left(m_{2^{r+1}(\mathbf{x})} + \frac{2}{(r+1)^2}m_{2^r1^2}(\mathbf{x})\right)\end{aligned}$$

We may exchange the factor $m_{2^{r+1}} + (2/(r+1)^2)m_{2^r1^2}$ by something nonnegative and smaller and still get a valid inequality. By Lemma 8.2 we obtain the inequality

$$\begin{aligned}&(a_r r(r+1)e_r(\mathbf{x})e_{r+1}(\mathbf{x}) - (r-1)(r+2)e_{r-1}(\mathbf{x})e_{r+2}(\mathbf{x}))^2 \geq \\ &C_r(r-1)(r+1)^3 e_{r-1}(\mathbf{x})e_{r+1}(\mathbf{x})\frac{r}{r+1}m_{2^{r+1}}(\mathbf{x}).\end{aligned}$$

Dividing through by $(r-1)^2(r+2)^2$ we obtain

$$\begin{aligned}&\left(a_r\frac{r(r+1)}{(r-1)(r+2)}e_r(\mathbf{x})e_{r+1}(\mathbf{x}) - e_{r-1}(\mathbf{x})e_{r+2}(\mathbf{x})\right)^2 \geq \\ &C_r\frac{r(r+1)^2}{(r-1)(r+2)^2}e_{r-1}(\mathbf{x})e_{r+1}(\mathbf{x})m_{2^{r+1}}(\mathbf{x}),\end{aligned}$$

which simplifies to the desired inequality for $r+1$. □

9. Proof of Theorem 6.6

The next tool for the proof of Theorem 6.6 is a lemma that enables us to prove hyperbolicity of a polynomial by proving real-rootedness along a few (degenerate) directions.

Lemma 9.1. *Let $h \in \mathbb{C}[x_1,\dots,x_n]$ and $\mathbf{v}_1,\mathbf{v}_2 \in \mathbb{C}^n$. Define d to be the maximum degree of the polynomial $t \mapsto h(t\mathbf{v}_2+\mathbf{y})$, where the maximum is taken over all $\mathbf{y} \in \mathbb{C}^n$. Let further*

$$P(\mathbf{x}) := \lim_{t\to\infty} t^{-d}h(t\mathbf{v}_2+\mathbf{x}) \in \mathbb{C}[x_1,\dots,x_n].$$

Suppose $S \subseteq \mathbb{C}^n$ and $\mathbf{x}_0 \in S$ are such that

- (i) *$S + \mathbb{R}\mathbf{v}_2 = S$, i.e., S is closed under translations by real multiples of $\mathbf{v}_2$.*
- (ii) *S is pathwise connected.*
- (iii) *The polynomial $(s,t) \mapsto h(s\mathbf{v}_1+t\mathbf{v}_2+\mathbf{x}_0)$ is stable and not identically zero.*
- (iv) *For each $\mathbf{x} \in S$, the polynomials $s \mapsto h(s\mathbf{v}_1+\mathbf{x})$ and $s \mapsto P(s\mathbf{v}_1+\mathbf{x})$ are stable and not identically zero.*

Then the polynomial $(s,t) \mapsto h(s\mathbf{v}_1+t\mathbf{v}_2+\mathbf{x})$ is stable for all $\mathbf{x} \in S$.

Proof. The proof is by contradiction. Suppose $\mathbf{x}_1 \in S$ and $\xi,\eta \in \mathbb{C}$ are such that $\mathrm{Im}(\xi) > 0$, $\mathrm{Im}(\eta) > 0$ and

$$h(\xi\mathbf{v}_1+\eta\mathbf{v}_2+\mathbf{x}_1) = 0.$$

Let $\mathbf{x}(\theta) : [0,1] \to S$ be a continuous path such that $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(1) = \mathbf{x}_1$ and let

$$p_\theta(t) = h(\xi\mathbf{v}_1+t\mathbf{v}_2+\mathbf{x}(\theta)) = t^dP(\xi\mathbf{v}_1+\mathbf{x}(\theta)) + O(t^{d-1}),$$

where $P(\xi\mathbf{v}_1+\mathbf{x}(\theta)) \neq 0$ by (iv). By assumption all zeros of $p_0(t)$ are in the closed lower half-plane, while $p_1(\eta) = 0$ where $\mathrm{Im}(\eta) > 0$. Hence, by continuity using

Hurwitz' theorem on the continuity of zeros (see e.g., [11, Footnote 3, p. 96]), a zero will cross the real axis as θ runs from 0 to 1. In other words

$$0 = p_\theta(\alpha) = h(\xi \mathbf{v}_1 + \alpha \mathbf{v}_2 + \mathbf{x}(\theta)),$$

for some $\alpha \in \mathbb{R}$ and $\theta \in [0,1]$. Since $\alpha \mathbf{v}_2 + \mathbf{x}(\theta) \in S$, by (i), this contradicts (iv). □

The next theorem is a version of the Grace-Walsh-Szegő coincidence theorem, see [4, Prop. 3.4].

Theorem 9.2 (Grace-Walsh-Szegő)**.** *Suppose $P(x_1, \dots, x_n) \in \mathbb{C}[\mathbf{x}]$ is a polynomial of degree at most d in the variable x_1:*

$$P(x_1, \dots, x_n) = \sum_{k=0}^{d} P_k(x_2, \dots, x_n) x_1^k.$$

Let Q be the polynomial in the variables $x_2, \dots, x_n, y_1, \dots, y_n$

$$Q = \sum_{k=0}^{d} P_k(x_2, \dots, x_n) \frac{e_k(y_1, \dots, y_d)}{\binom{d}{k}}.$$

Then P is stable if and only if Q is stable.

Remark 9.3. Note that $e_k(x_1, \dots, x_n)$ is stable, by the Grace–Walsh–Szegő theorem applied to the polynomial x_1^k considered as a polynomial of degree at most n.

Lemma 9.4. *If $\mathbf{x} \in \mathbb{R}^n$ and $e_k(\mathbf{x}) = e_{k+1}(\mathbf{x}) = 0$ where $0 < k < n$, then $\mathbf{x}$ has at most $k-1$ nonzero coordinates.*

Proof. It is well known that if $e_k(\mathbf{x}) = e_{k+1}(\mathbf{x}) = 0$, then $e_j(\mathbf{x}) = 0$ for all $k \le j \le n$, see e.g., [6, Example 3.6]. Hence the number of non-zero coordinates of $\mathbf{x}$ is equal to

$$\max\{0 \le i \le n : e_i(\mathbf{x}) \neq 0\} < k.$$

□

The following theorem provides families of stable polynomials which are closed under convex sums.

Theorem 9.5. *Let $r \ge 2$ be an integer, and let*

$$M(\mathbf{x}) = \sum_{|S|=r} a(S) \prod_{i \in S} x_i^2 \in \mathbb{R}[x_1, \dots, x_n],$$

where $0 \le a(S) \le 1$ for all $S \subseteq [n]$, where $|S| = r$. Then the polynomial

$$4e_{r+1}(\mathbf{x})e_{r-1}(\mathbf{x}) + \frac{3}{r+1} M(\mathbf{x}) \tag{9.1}$$

is stable.

Proof. Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{x}' = (x_2, \dots, x_n)$ where $n \ge 2r+4$. Suppose M is of the form described in the statement of the theorem and that additionally no x_i, $i = 1, \dots, 2r+3$ appears in M. We will prove, by applying Lemma 9.1, that polynomials of the form

$$h(\mathbf{x}) := 4(a_r x_1 e_r(\mathbf{x}') + e_{r+1}(\mathbf{x}'))(x_1 e_{r-2}(\mathbf{x}') + e_{r-1}(\mathbf{x}')) + \frac{3}{r+1} M \tag{9.2}$$

are stable, where a_r is defined as in Lemma 8.6. Since any polynomial of the form (9.1) may be obtained from some polynomial of the form (9.2) by setting variables to zero and relabelling the indices, the stability of polynomials of the form (9.1) follows from Lemma 2.6.

Recall the notation of Lemma 9.1. Let $\mathbf{v}_1 = \delta_1$, $\mathbf{v}_2 = \delta_2 + \delta_3 + \cdots + \delta_{r+2}$, and let S be the set of all $\mathbf{x} \in \mathbb{R}^n$ such that at least $r+1$ of the coordinates $\{x_{r+3}, \ldots, x_n\}$ are nonzero. Note that S is pathwise connected and $S + \mathbb{R}\mathbf{v}_2 = S$. Let $\mathbf{x}_0 = \delta_{r+3} + \cdots + \delta_{2r+3}$. The polynomial

$$\begin{aligned} q(\mathbf{x}) &:= 4(a_r x_1 e_r(\mathbf{x}') + e_{r+1}(\mathbf{x}'))(x_1 e_{r-2}(\mathbf{x}') + e_{r-1}(\mathbf{x}')) \\ &= 4e_{r+1}(a_r x_1, x_2, \ldots, x_n) e_{r-1}(x_1, x_2, \ldots, x_n) \end{aligned}$$

is stable by Remark 9.3 and Lemma 2.6. Since $q(\mathbf{x}_0) > 0$ we know, by Lemma 2.8, that the bivariate polynomial $h(s\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{x}_0) = q(s\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{x}_0)$ is stable and not identically zero. This verifies (iii) of Lemma 9.1. Note that $P(\mathbf{x})$ is a non-zero constant. Consider

$$\begin{aligned} h(s\mathbf{v}_1 + \mathbf{x}) = {} & 4a_r e_r(\mathbf{x}')e_{r-2}(\mathbf{x}')(s + x_1)^2 \\ & + 4(a_r e_r(\mathbf{x}')e_{r-1}(\mathbf{x}') + e_{r+1}(\mathbf{x}')e_{r-2}(\mathbf{x}'))(s + x_1) \\ & + 4e_{r+1}(\mathbf{x}')e_{r-1}(\mathbf{x}') + \frac{3}{r+1}M. \end{aligned}$$

We now prove that $h(s\mathbf{v}_1 + \mathbf{x}) \not\equiv 0$ for each $\mathbf{x} \in S$, as a polynomial in s. Assume $h(s\mathbf{v}_1 + \mathbf{x}) \equiv 0$ for some $\mathbf{x} \in S$. Then $e_r(\mathbf{x}')e_{r-2}(\mathbf{x}') = 0$, so suppose first $e_r(\mathbf{x}') = 0$. If $e_{r+1}(\mathbf{x}')e_{r-1}(\mathbf{x}') = 0$, then either $e_{r-1}(\mathbf{x}') = e_r(\mathbf{x}') = 0$ or $e_{r+1}(\mathbf{x}') = e_r(\mathbf{x}') = 0$, which implies $\mathbf{x}'$ has at most $r-1$ non-zero coordinates by Lemma 9.4. This contradicts $\mathbf{x} \in S$. Hence $e_{r+1}(\mathbf{x}')e_{r-1}(\mathbf{x}') < 0$ by Lemma 8.5. Then $h(-x_1\mathbf{v}_1 + \mathbf{x})$ is equal to

$$\begin{aligned} 4e_{r+1}(\mathbf{x}')e_{r-1}(\mathbf{x}') + \frac{3}{r+1}M &< 3e_{r+1}(\mathbf{x}')e_{r-1}(\mathbf{x}') + \frac{3}{r+1}M \\ &\leq 3e_{r+1}(\mathbf{x}')e_{r-1}(\mathbf{x}') + \frac{3}{r+1}m_{2^r}(\mathbf{x}') \leq 0, \end{aligned}$$

by Lemma 8.5, a contradiction. If $e_r(\mathbf{x}') \neq 0$, then $e_{r-2}(\mathbf{x}') = 0$. But then also $e_{r-1}(\mathbf{x}') = 0$, since $h(s\mathbf{v}_1 + \mathbf{x}) \equiv 0$. Hence $\mathbf{x}'$ has at most $r-3$ non-zero coordinates Lemma 9.4, which contradicts $\mathbf{x} \in S$. We conclude that $h(s\mathbf{v}_1 + \mathbf{x}) \not\equiv 0$ for $\mathbf{x} \in S$.

To apply Lemma 9.1 and prove that $h(s\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{x})$ is stable for all $\mathbf{x} \in S$ it remains to prove that $h(s\mathbf{v}_1 + \mathbf{x})$ is real–rooted. However $h(s\mathbf{v}_1 + \mathbf{x})$ is of degree at most two so it suffices to show that its discriminant Δ is nonnegative. Now

$$\frac{\Delta}{16} = (a_r e_{r-1}(\mathbf{x}')e_r(\mathbf{x}') - e_{r-2}(\mathbf{x}')e_{r+1}(\mathbf{x}'))^2 - \frac{3}{r+1}a_r e_r(\mathbf{x}')e_{r-2}(\mathbf{x}')M.$$

If $e_r(\mathbf{x}')e_{r-2}(\mathbf{x}') < 0$, then clearly $\Delta \geq 0$, so assume $e_r(\mathbf{x}')e_{r-2}(\mathbf{x}') \geq 0$. Then, since $M(\mathbf{x}') \leq m_{2^r}(\mathbf{x}')$, it follows that $\Delta \geq 0$ by Lemma 8.6.

Since S is dense in $\mathbb{R}^n$ we have by Hurwitz' theorem that $h(s\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{x})$ is stable or identically zero for all $\mathbf{x} \in \mathbb{R}^n$. However $h(\mathbf{v}_2) \neq 0$ so that $h(s\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{x})$ is stable for all $\mathbf{x} \in \mathbb{R}^n$. In particular h is hyperbolic with respect to $\mathbf{v}_2$. Since all Taylor coefficients of h are nonnegative we see that the hyperbolicity cone contains the positive orthant, i.e., h is stable, by Lemma 2.7.

□

Lemma 9.6. *Let $r \geq 2$. Then*

$$\begin{aligned}&e_{2r}(x_1, x_1, \ldots, x_n, x_n) - m_{2^r}(\mathbf{x}) = \\ &4(e_{r-1}(\mathbf{x})e_{r+1}(\mathbf{x}) + e_{r-3}(\mathbf{x})e_{r+3}(\mathbf{x}) + e_{r-5}(\mathbf{x})e_{r+5}(\mathbf{x}) + \cdots).\end{aligned}$$

Proof. Note that

$$\sum_{k=0}^{2n} e_k(x_1, x_1, \ldots, x_n, x_n)t^k = \prod_{j=1}^{n}(1 + x_j t)^2 = \left(\sum_{k=0}^{2n} e_k(\mathbf{x})t^k\right)^2.$$

The coefficient of t^{2r} is

$$e_{2r}(x_1, x_1, \ldots, x_n, x_n) = \sum_{j=0}^{2r} e_j(\mathbf{x})e_{2r-j}(\mathbf{x}).$$

The proof follows by combining this with (8.4). □

We are now in a position to prove Theorem 6.6 and Theorem 6.5.

Proof of Theorem 6.5. By definition the bases generating polynomial of $V_H \in \mathcal{V}$ is given by

$$h_{V_H} = \sum_{B \in \mathcal{B}(V_H)} \prod_{i \in B} x_i = e_{2r}(x_1, x_{1'}, \ldots, x_n, x_{n'}) - e_r(x_1x_{1'}, \ldots, x_nx_{n'}) + N(\mathbf{x}).$$

where

$$N(\mathbf{x}) = \sum_{(i_1,\ldots,i_r) \notin E(H)} \prod_{j=1}^{r} x_{i_j}x_{i'_j}.$$

The polynomial h_{V_H} is clearly multiaffine and symmetric pairwise in $x_i, x_{i'}$ for all $i \in [n]$. Set $x_{i'} = x_i$ for all $1 \leq i \leq n$ and obtain the polynomial

$$f_{V_H} = e_{2r}(x_1, x_1, \ldots, x_n, x_n) - e_r(x_1^2, \ldots, x_n^2) + N(x_1, x_1, \ldots, x_n, x_n).$$

By Lemma 9.6

$$f_{V_H} = 4\sum_{j=0}^{\lceil r/2 \rceil - 1} e_{r+2j+1}(\mathbf{x})e_{r-2j-1}(\mathbf{x}) + N(x_1, x_1, \ldots, x_n, x_n).$$

The support of $e_{r+j}(\mathbf{x})e_{r-j}(\mathbf{x})$ is contained in the support of $e_{r+1}(\mathbf{x})e_{r-1}(\mathbf{x})$ for each $1 \leq j \leq r$. Hence f_{V_H} has the same support as the polynomial

$$W_{V_H} = 4e_{r+1}(\mathbf{x})e_{r-1}(\mathbf{x}) + \frac{3}{r+1}N(x_1, x_1, \ldots, x_n, x_n)$$

which in turn is stable by Theorem 9.5. Hence if we replace x_i^k for $k = 0, 1, 2,$ in W_{V_H} with $e_k(x_i, x_{i'})/\binom{2}{k}$, we obtain a polynomial which is stable by the Grace-Walsh-Szegő theorem (Theorem 9.2), and has the same support as h_{V_H}. Hence V_H is a WHPP-matroid so V_H is hyperbolic by Proposition 3.2. □

Proof of Theorem 6.6. Recall the notation in the proof of Theorem 6.5. If $r = 2$, then $W_{V_G} = f_{V_G}$, so that V_G has the half-plane property by the proof of Theorem 6.5. □

10. Representability and minor closure

The following amended section is not part of the published research article. We begin the section by proving some facts related to the representability of the matroids in the class $\mathcal{V}$. The class $\mathcal{V}$ is not closed under taking minors. Therefore in the final part of the section we determine the minor closure of $\mathcal{V}$.

Let H be a d-uniform hypergraph on $[n]$. Let H_j denote the hypergraph with vertex set $V(H_j) = V(H)\backslash j$ and edges $E(H_j) = \{e\backslash j : e \in E(H),\, j \in e\}$. Moreover let H^* denote the $(n-d)$-uniform hypergraph on $[n]$ with $E(H^*) = \{[n] \setminus e : e \in E(H)\}$. Recall that the *dual* of a matroid M on E, denoted M^*, is the matroid on E whose bases are the complements of the bases of M. The *free extension* of a matroid M with rank function $r_M : 2^{E(M)} \to \mathbb{N}$, by an element $e \notin E(M)$, is given by the matroid, denoted $M + e$, on $E(M) \sqcup e$ with rank function

$$r_{M+e}(S) = \begin{cases} r_M(S), & \text{if } e \notin S, \\ r_M(S \setminus e) + 1, & \text{if } e \in S \text{ and } r_M(S \setminus e) < d, \\ d, & \text{if } r_M(S \setminus e) = d. \end{cases}$$

The *free coextension* of M by e is given by the matroid $M \times e = (M^* + e)^*$. For $i \in [n]$ we shall also use the convention that $(i')' = i$. Below we list some basic properties of the matroid class $\mathcal{V}$.

Proposition 10.1. *If H, H_1 and H_2 are d-uniform hypergraphs on $[n]$, then*

(i) $V_{H_1} \cong V_{H_2}$ *if and only if* $H_1 \cong H_2$,
(ii) $([n] \cup [n]', \mathcal{I}(V_{H_1}) \cap \mathcal{I}(V_{H_2})) = V_{H_1 \cup H_2}$,
(iii) $([n] \cup [n]', \mathcal{I}(V_{H_1}) \cup \mathcal{I}(V_{H_2})) = V_{H_1 \cap H_2}$,
(iv) $V_H^* = V_{H^*}$. *In particular V_H is self-dual if and only if $H \cong H^*$,*
(v) $V_H \setminus j = V_{H \backslash j} + j'$,
(vi) $V_H / j = V_{H_j} \times j'$,
(vii) V_H *has Tutte-polynomial,*

$$T_{V_H}(x, y) = |E(H)|(xy - x - y) + \sum_{i=1}^{2d} \binom{2n-i-1}{2n-2d-1} x^i + \sum_{j=1}^{2n-2d} \binom{2n-j-1}{2d-1} y^j.$$

Proof. For (i), suppose that $\Phi : E(V_{H_1}) \to E(V_{H_2})$ is a an isomorphism between V_{H_1} and V_{H_2}. Then we have $\Phi(i') = \Phi(i)'$ for all $i \in [n]$ since i is included in a circuit hyperplane of V_{H_1} if and only if i' is included in a circuit hyperplane of V_{H_1}. Hence

$$\begin{aligned} \phi : V(H_1) &\to V(H_2) \\ i &\mapsto \begin{cases} j, & \text{if } \Phi(i) = j, \\ j, & \text{if } \Phi(i) = j', \end{cases} \end{aligned}$$

is a hypergraph isomorphism between H_1 and H_2 since $\{i_1, \dots, i_r\} \in E(H_1)$ if an only if $\{i_1, i_1', \dots, i_r, i_r'\}$ is a circuit hyperplane in V_{H_1} if and only if $\{\Phi(i_1), \Phi(i_1'), \dots, \Phi(i_r), \Phi(i_r')\}$ is a circuit hyperplane in V_{H_2} if and only if $\{\phi(i_1), \dots, \phi(i_r)\} \in E(H_2)$.

Conversely if $\phi : V(H_1) \to V(H_2)$ is a hypergraph isomorphism, then clearly

$$\begin{aligned} \Phi : E(V_{H_1}) &\to E(V_{H_2}) \\ i &\mapsto \phi(i) \\ i' &\mapsto \phi(i)' \end{aligned}$$

is an isomorphism between the sparse paving matroids V_{H_1} and V_{H_2} since Φ bijectively maps circuit hyperplanes to circuit hyperplanes.

The statements (ii) and (iii) are clear since both sides are readily seen to have the same independent sets.

For (iv), we use the circuit-hyperplane correspondence between a matroid M on E and its dual M^*. Namely, C is a circuit in M if and only if $E-C$ is a hyperplane in M^* and H is a hyperplane in M if and only if $E \setminus C$ is a circuit in M^* (see [34, Prop. 2.16]). This shows that sparse paving matroids are closed under duality. In particular C is a circuit hyperplane in V_H if and only if $E(V_H) \setminus C$ is a circuit hyperplane in the sparse paving matroid V_H^*. Hence $V_H^* = V_{H^*}$ which proves (iv).

For (v), let $S \subseteq E(V_H) \setminus j$. If $j' \notin S$, then

$$r_{V_{H \setminus j}+j'}(S) = r_{V_{H\setminus j}}(S) = r_{V_H \setminus j}(S).$$

Suppose therefore $j' \in S$. If $r_{V_{H\setminus j}+j'}(S \setminus j') < k$, then

$$r_{V_{H\setminus j}+j'}(S) = r_{V_{H\setminus j}}(S \setminus j') + 1 = r_{V_H \setminus j}(S \setminus j') + 1 = r_{V_H\setminus j}(S).$$

Otherwise if $r_{V_{H\setminus j}+j'}(S \setminus j') = k$, then

$$k = r_{V_{H\setminus j}+j'}(S) = r_{V_{H\setminus j}+j'}(S \setminus j') = r_{V_{H\setminus j}}(S \setminus j') = r_{V_H\setminus j}(S \setminus j'),$$

so $r_{V_H\setminus j}(S) = k$. Hence $r_{V_{H\setminus j}+j'}(S) = r_{V_H\setminus j}(S)$ for all $S \subseteq E(V_H) \setminus j$ which establishes (v). Statement (vi) now follows from (iv) and (v) via

$$V_H/j = (V_H^* \setminus j)^* = (V_{H^*} \setminus j)^* = (V_{H^*\setminus j} + j')^* = V^*_{H^*\setminus j} \times j' = V_{(H^*\setminus j)^*} \times j' = V_{H_j} \times j'.$$

For (vii), note that the uniform matroid $U_{2n,2d}$ on $2n$ elements of rank $2d$ is obtained from V_H via a sequence of $|E(H)|$ circuit hyperplane relaxations. Hence (vii) follows from the fact that

$$T_M(x, y) = xy - x - y + T_{M'}(x, y)$$

where M' is the matroid obtained from M by relaxing a circuit hyperplane, and

$$T_{U_{n,d}}(x, y) = \sum_{i=1}^{d} \binom{n-i-1}{n-d-1} x^i + \sum_{j=1}^{n-d} \binom{n-j-1}{d-1} y^j,$$

see [30]. □

Remark 10.2. The class $\mathcal{V}$ is seen not to be closed under minors, direct sums and matroid unions.

Although V_H does not have the half-plane property for hypergraphs H in general, a simple consequence of Proposition 10.1 (iv) is the following.

Proposition 10.3. *Let H be a d-uniform hypergraph on $[n]$. If $n \leq d + 2$, then V_H has half-plane property.*

Proof. Suppose $n = d + 2$. By Proposition 10.1 (iv) we have that $V_H^* = V_{H^*}$ where H^* is a 2-uniform hypergraph on $[n]$ (i.e. a graph). Thus V_H has half-plane property if and only if V_{H^*} has half-plane property by closure under duality [11]. Hence V_H has half-plane property by Theorem 6.6. Finally if $n < d + 2$ then via free extension V_H is a minor of a matroid $V_{H'}$ with $|V(H')| = d + 2$. Since the half–plane property is closed under taking minors [11], the result follows. □

Clearly any matroid $V_H \in \mathcal{V}$ in which the Vámos matroid V_8 is a minor cannot be representable (and necessarily fails to satisfy Ingleton's inequality). Hence Theorem 6.5 provides an infinite family of hyperbolic matroids which are not representable. By Proposition 10.1 we have that $V_H \setminus j$ and V_H/j are representable if and only if $V_{H\setminus j}$ and V_{H_j} are respectively representable. It follows that every non-representable matroid V_H has a minimal excluded minor for representability of the form $V_{H'}$ in its minor hierarchy for some hypergraph H'.

A natural question is which matroids in $\mathcal{V}$ are representable/non-representable? Below we identify a class of matroids in $\mathcal{V}$ which are guaranteed to be representable over any infinite field.

Theorem 10.4. *Let H be a d-uniform hypergraph on $[n]$ and let $\mathbb{F}$ be an infinite field. Suppose $j \in [n]$ such that $j \in e$ for at most one $e \in E(H)$. Then V_H is $\mathbb{F}$-representable if and only if $V_{H\setminus j}$ is $\mathbb{F}$-representable.*

Proof. If V_H is $\mathbb{F}$-representable, then $V_{H\setminus j}$ is $\mathbb{F}$-representable by Proposition (v) since representability is closed under taking minors. Conversely suppose $V_{H\setminus j}$ is representable over $\mathbb{F}$. If $j \notin e$ for all $e \in E(H)$, then $V_H = (V_{H\setminus j} + j) + j'$ so the statement follows since representability is closed under taking free extensions. Therefore suppose $j \in [n]$ belongs to a unique $e \in E(H)$. If $n \leq d$ then the proposition is clear so we may assume $n > d$. By relabelling if necessary we may assume $j = n$. Let $u_1, u_{1'} \ldots, u_{n-1}, u_{(n-1)'}$ be vectors in a $2d$-dimensional vector space V over $\mathbb{F}$ representing the elements $E_{n-1} = \{1, 1', \ldots, n-1, (n-1)'\}$ of $V_{H\setminus j}$. Since $\mathbb{F}$ is infinite, V cannot be a union of finitely many proper subspaces, so we may choose

$$u_n \in V \setminus \bigcup_{i_1,\ldots,i_{2d-2} \in E_{n-1}} \langle u_{i_1}, \ldots, u_{i_{2d-2}} \rangle,$$

where $\langle u_{i_1}, \ldots, u_{i_{2d-2}} \rangle$ denotes the linear span of the vectors $u_{i_1}, \ldots, u_{i_{2d-2}}$ over $\mathbb{F}$. Let

$$U = \langle u_n \rangle \oplus \langle u_i, u_{i'} : i \in e \setminus n \rangle.$$

By modularity we have

$$\begin{aligned} \dim\left(U \cap \langle u_{i_1}, \ldots, u_{i_{2d-1}} \rangle\right) &= \dim(U) + \dim(\langle u_{i_1}, \ldots, u_{i_{2d-1}} \rangle) - \dim(U + \langle u_{i_1}, \ldots, u_{i_{2d-1}} \rangle) \\ &\leq (2d-1) + (2d-1) - 2d < 2d - 1. \end{aligned}$$

for all $i_1, \ldots, i_{2d-1} \in E$. Thus $U \neq \langle u_{i_1}, \ldots, u_{i_{2d-1}} \rangle$ for all $\{i_1, \ldots, i_{2d-1}\} \neq \{n\} \cup \{i, i' : i \in e \setminus n\}$. Therefore we may choose

$$u_{n'} \in U \setminus \bigcup_{\substack{i_1,\ldots,i_{2d-1} \in E_{n-1} \cup \{n\} \\ \{i_1,\ldots,i_{2d-1}\} \neq \{n\} \cup \{i,i' : i \in e \setminus n\}}} \langle u_{i_1}, \ldots, u_{i_{2d-1}} \rangle.$$

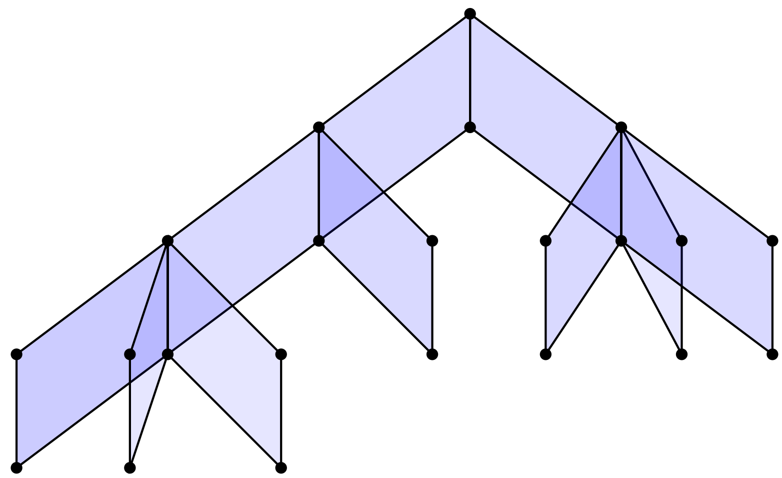


FIGURE 5. The matroid V_H in the figure is representable since H is a tree graph.

It follows that $\langle u_i, u_{i'} : i \in e\rangle$ has rank $2d-1$, and for any other subset containing u_n or $u_{n'}$ of size at most $2r$ the rank is equal to the cardinality of the spanning set. Hence $u_1, u_{1'}, \dots, u_n, u_{n'}$ are vectors in V representing V_H. □

Corollary 10.5. *If G is a forest then V_G is $\mathbb{F}$-representable for any infinite field $\mathbb{F}$.*

Proof. If G is the empty graph forest then $V_G \cong U_{2n,4}$ which is representable over $\mathbb{F}$. If G is non-empty then the statement follows by removing a leaf in any tree-component and arguing by induction applying Theorem 10.4. □

If $\mathcal{A}$ is a class of matroids, then denote by $\overline{\mathcal{A}}$ the *minor closure* of the class $\mathcal{A}$, that is, the smallest class that contains all minors of matroids in $\mathcal{A}$. Consider the following class:

Definition 10.6. Let $\mathcal{C}$ denote the class of sparse paving matroids M with $E(M) \subseteq [n] \cup [n]'$ having circuit hyperplanes $C_1, \dots, C_k$ and a set $S \subseteq \bigcap_{t=1}^k C_t$ such that

(i) $i \in S$ implies $i' \notin E(M)$,

(ii) $i \in C_t \setminus S$ if and only if $i' \in C_t \setminus S$ for all $t = 1, \dots, k$.

Remark 10.7. Note that the definition implies that S is unique. Indeed suppose $S_1, S_2 \subseteq \bigcap_{t=1}^k C_t$ both satisfy conditions (i) and (ii). If $j \in S_1 \setminus S_2$, then on one hand $j' \notin E(M)$, and on the other hand $j \in C_t \setminus S_2$ for all $t = 1, \dots, k$, so $j' \in C_t \setminus S_2 \subseteq E(M)$ for all $t = 1, \dots, k$. This gives a contradiction. Hence $S_1 = S_2$.

Lemma 10.8. *The matroid class $\mathcal{C}$ is minor closed, i.e., $\mathcal{C} = \overline{\mathcal{C}}$.*

Proof. Let $M \in \mathcal{C}$ be a matroid with circuit hyperplanes $C_1, \dots, C_k$ and let $i \in E(M)$. We distinguish between three cases:

Case 1. Suppose $i \in E(M) \setminus \bigcup_{t=1}^k C_t$. Clearly $M \setminus i$ is a matroid with the same set of circuits and therefore belongs to $\mathcal{C}$. The matroid M/i has no circuit hyperplanes since i belongs to no circuit hyperplane of M. Hence M/i is a uniform matroid which again belongs to $\mathcal{C}$.

Case 2. Suppose $i \in S$. Then $M \setminus i$ has no circuit hyperplanes since i belongs to every circuit hyperplane of M. Thus $M \setminus i$ is a uniform matroid and therefore belongs to $\mathcal{C}$. Since $i \in S$ the circuit hyperplanes of M/i are given by $C_t \setminus i$ for $t = 1, \ldots, k$. Hence the axioms of $\mathcal{C}$ remain intact with $S \setminus i \subseteq \bigcap_{t=1}^{k} C_t \setminus i$.

Case 3. Suppose $i \in C_t \setminus S$ for $t \in I$ where $I \subseteq [k]$. The remaining circuit hyperplanes in $M \setminus i$ are given by $\{C_t : t \in [k] \setminus I\}$ which are easily seen to satisfy the axioms of the class $\mathcal{C}$. The circuit hyperplanes in M/i are given by $\{C_t \setminus i : t \in I\}$ and the axioms of $\mathcal{C}$ are satisfied with $S \cup \{i'\} \subseteq \bigcap_{i \in I} C_t \setminus i$.

Hence $\mathcal{C}$ is minor closed. □

Theorem 10.9. *The minor closure of $\mathcal{V}$ is $\mathcal{C}$, i.e., $\overline{\mathcal{V}} = \mathcal{C}$.*

Proof. Certainly $\mathcal{V} \subseteq \mathcal{C}$ since the matroids in $\mathcal{V}$ are instances of matroids in $\mathcal{C}$ with $S = \emptyset$. Hence by Lemma 10.8 we have $\overline{\mathcal{V}} \subseteq \overline{\mathcal{C}} = \mathcal{C}$. Conversely let $M \in \mathcal{C}$ and suppose M has circuit hyperplanes $C_1, \ldots, C_k$. Let $T = (S \cup S') \cap [n]$ and $e_t = (C_t \cup C_t') \cap [n]$ for $t = 1, \ldots, k$. By definition $|C_t \setminus S| = 2l$ for all $t = 1, \ldots, k$ for some $l \in \mathbb{N}$. Thus $|e_1| = \cdots = |e_k| = |T| + l$. Consider the matroid $V_H \in \mathcal{V}$ where H is the $(|T|+l)$-uniform hypergraph on $[n]$ with edges $e_1, \ldots, e_k$. It follows that M is a minor of V_H. Indeed since $S \cup S' \subseteq \bigcap_{t=1}^{k}(e_t \cup e_t')$, we find that the circuit hyperplanes of V_H/S' are given by $C_1, \ldots, C_k$. Finally delete all elements in V_H/S' belonging to $U = [n] \cup [n]' \setminus E(M)$. Then $M = (V_H/S') \setminus U$. Hence $\overline{\mathcal{V}} \supseteq \mathcal{C}$. □

Remark 10.10. We remark that $\mathcal{C}$ is also closed under taking duals. Indeed if $M \in \mathcal{C}$ with circuit hyperplanes $C_1, \ldots C_k$, then M^* has circuit hyperplanes $E(M) \setminus C_t$ for $t = 1, \ldots, k$ which together with the unique maximal subset $S \subseteq E(M) \setminus \bigcup_{t=1}^{k} C_t$ such that $i \in S \Rightarrow i' \notin E(M)$ satisfies the axioms of $\mathcal{C}$.

Corollary 10.11. *The class $\mathcal{C}$ consists of hyperbolic matroids.*

Proof. By Theorem 6.5 the class $\mathcal{V}$ consists of hyperbolic matroids. Since the class of hyperbolic matroids is minor closed the statement follows by Theorem 10.9. □

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