Geometry of zeros and applications

Abstract

This set of personal notes is an introduction to the geometry of zeros of multivariate polynomials. It is based on a set of lecture notes composed by Petter Brändén and contains expanded arguments and solved exercises. The author of this document (Nima Amini) takes full responsibility for any typos or errors that may have been caused by misrepresenting the original text.

Contents

1 Stable Polynomials

Definition 1.1. (Stable Polynomial).
Let Ω be a subset of $\mathbb{C}^n$ and $P(z) \in \mathbb{C}[z_1, \ldots, z_n]$. We say P is **Ω-stable** if $P(\zeta) \neq 0$. Let $H = \{\zeta \in \mathbb{C} : Im(\zeta) > 0\}$. We refer to H^n-stable polynomials as **stable** and H^n-stable polynomials with real coefficients as **real stable**.

Proposition 1.2. A univariate real polynomial is real-rooted iff it is real stable.

Proof.
Let $f = \sum_{k=0}^{n} a_k x^k$ be a polynomial with real coefficients. Suppose $f(a+bi) = 0$ where $a, b \in \mathbb{R}$, $b \neq 0$. Then

$$0 = \overline{f(a+bi)} = \sum_{k=0}^{n} \overline{a_k}\,\overline{(a+bi)}^k = \sum_{k=0}^{n} a_k (a-bi)^k = f(a-bi).$$

Hence $a - bi$ is also a root and so a univariate real polynomial has an imaginary root iff it is not real stable. □

Proposition 1.3. $P \in \mathbb{R}[z_1, \ldots, z_n]$ is real stable iff $P(\alpha + \beta t)$ is a univariate real-rooted polynomial in t for all $\alpha, \beta \in \mathbb{R}^n$ with $\beta \in \mathbb{R}^n_+$.

Proof.
Let $P \in \mathbb{R}[z_1, \ldots, z_n]$ and suppose $P(\alpha + \beta t)$ is not real-rooted for some $\alpha, \beta \in \mathbb{R}^n$ with $\beta \in \mathbb{R}^n_+$. By Lemma 1.2 $P(\alpha + \beta t)$ is not real stable, so there exists $\zeta \in H$ such that $P(\alpha + \beta\zeta) = 0$. Hence $P(\alpha_1 + \beta_1\zeta, \ldots, \alpha_n + \beta_n\zeta) = 0$ with $Im(\alpha_i + \beta_i\zeta) = \beta_i Im(\zeta) > 0$ for $1 \leq i \leq n$ showing that $P(z_1, \ldots, z_n)$ is not stable. Conversely suppose $P(\alpha + \beta t)$ is real-rooted for all $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^n_+$, but P is not real stable. Then there exists $\zeta = (\zeta_1, \ldots, \zeta_n) \in H^n$ such that $P(\zeta) = 0$. Let $a := Re(\zeta)$ and $b := Im(\zeta)$. Since $\zeta \in H^n$ it follows that $b_i > 0$ for all $1 \leq i \leq n$. But then $P(\zeta) = P(a + bi) = 0$ so that i is a root of $P(a + bt)$ contradicting its real rootedness.
□

We next concern ourselves with Hurwitz theorem associating the zeros of a sequence of uniformly converging analytic functions (on locally compact subsets of a connected open set) with that of its limit. Hurwitz theorem is often useful when proving stability for a function that can be realized as the uniform limit of more easily shown stable functions. Before we state and prove Hurwitz theorem in the multivariate setting we recall a few classical results from complex analysis.

Proposition 1.4. (Uniform limit of analytics is analytic)
Suppose (f_n) is a sequence of analytic functions $f_n : D \longrightarrow \mathbb{C}$ converging to f uniformly on compact subsets of D. Then f is analytic.

Proof. (Sketch)
Continuity of f follows from a standard $\epsilon/3$ argument. Given a compact subset $S \subset D$ and a triangle γ in S we have $\oint_\gamma f_n(z)dz = 0$ by Cauchy integral theorem since f_n is analytic in D. By uniform convergence it follows that

$$0 = \lim_{n\to 0} \oint_\gamma f_n(z)dz = \oint_\gamma \lim_{n\to 0} f_n(z)dz = \oint_\gamma f(z)dz.$$

Since f is continuous and the integral of f is zero for all triangles $\gamma \subset S$ it follows by Morera's theorem that f is analytic in S. □

Theorem 1.5. (Rouchés theorem)
Let f and g be holomorphic inside and on a contour γ. Suppose that $|f(z)| > |g(z)|$ on the image γ^* of γ. Then f and $f+g$ have the same number of zeros inside γ.

Proof. (Sketch)
The assumption that $|f(z)| > |g(z)|$ for all $z \in \gamma^*$ implies $f(z) \neq 0$ for all $z \in \gamma^*$. Moreover by triangle inequality it follows that

$$|f(z) + g(z)| \geq |f(z)| - |g(z)| > 0 \text{ for all } z \in \gamma^*.$$

Thus $F(z) := (f(z) + g(z))/f(z)$ has no zeros nor poles. By the Argument Principle (in turn following from the Cauchy Residue Theorem) we have

$$Ind(F \circ \gamma, 0) = \#\text{Zeros of } F \text{ inside } \gamma - \#\text{Poles of } F \text{ inside } \gamma.$$

(where $Ind(F \circ \gamma, 0)$ is the number of times F winds around the origin i.e the number of times the argument of F increases by a multiple of 2π).
Now

$$|F(z) - 1| = |(f(z) + g(z))/f(z) - 1| = |g(z)/f(z)| < 1 \text{ for all } z \in \gamma^*.$$

Thus on γ^*, $F(z)$ only takes values in $D(1,1)$. In particular it never winds around zero so $Ind(F \circ \gamma, 0) = 0$ Therefore

$$\#\text{Zeros of } F \text{ inside } \gamma = \#\text{Poles of } F \text{ inside } \gamma.$$

Hence

$$\begin{aligned} \#\text{Zeros of } f+g \text{ inside } \gamma &= \#\text{Zeros of } F \text{ inside } \gamma \\ &= \#\text{Poles of } F \text{ inside } \gamma \\ &= \#\text{Zeros of } f \text{ inside } \gamma \end{aligned}$$

□

Theorem 1.6. (Identity theorem [Disc version])
Let f be analytic in the open disc $D(a;r)$ and suppose $f(a) = 0$. Then either $f \equiv 0$ in $D(a;r)$ or there exists $\epsilon > 0$ such that the punctured disc $D'(a;\epsilon)$ contains no zeros of f.

Proof.
Consider the Taylor expansion $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ for $z \in D(a;r)$. Suppose f is not identically zero in $D(a;r)$. Then there exists a smallest index $m > 0$ such that $c_m \neq 0$. We may therefore write

$$f(z) = (z-a)^m g(z) \text{ where } g(z) := \sum_{k=0}^{\infty} c_{k+m}(z-a)^k.$$

Since g has radius of convergence at least r (apply e.g ratio test) it is continuous on $D(a;r)$. Since $g(a) = c_m \neq 0$ and g continuous at a there is some $\epsilon > 0$ such that $g(z) \neq 0$ in $D'(a;\epsilon)$. Throughout this punctured disc it follows that $f(z) \neq 0$. $\square$

Theorem 1.7. (Hurwitz' theorem)
Let $D \subset \mathbb{C}^n$ be a non-empty connected open set, and let $\{f_k\}$ be a sequence of analytic functions on D that are nonvanishing on D, and converges to f uniformly on compact subsets of D. Then f is either identically zero or nonvanishing on D.

Proof. We first prove the statement for $n = 1$ and subsequently extend to the multivariate case. By Proposition 1.4 it follows that f is analytic on D being the uniform limit of analytic functions over D. Suppose for a contradiction that f is not identically zero on D but $f(a) = 0$ for some $a \in D$. Since $f \not\equiv 0$ on D it follows by Identity Theorem (Theorem 1.6) that there exists $\epsilon > 0$ such that $f(z) \neq 0$ on the punctured disc $D'(a;\epsilon)$. Since f is continuous and non-zero on the closure of $D'(a;\epsilon)$ then so is the function $1/|f|$. By standard analysis a continuous function on a closed bounded subset attains its supremum and so there exists $M > 0$ such that $1/|f(z)| \leq M$ for all $z \in D'(a;\epsilon)$. This holds in particular on the circle boundary $\partial D'(a;\epsilon)$. Therefore

$$|f(z)| \geq 1/M > 0 \text{ for all } z \in \partial D'(a;\epsilon).$$

On the other hand since $f_k \to f$ uniformly there exists $N \in \mathbb{N}$ such that $k > N$ implies

$$|f_k(z) - f(z)| < 1/M \leq |f(z)| \text{ for all } z \in \partial D'(a;\epsilon)$$

Now apply Rouchés theorem (Theorem 1.5) with $g = f_k - f$ to see that f and $f + g = f + (f_k - f) = f_k$ have the same number of zeros inside the circular contour $\partial D'(a;\epsilon)$. But by assumption $f(a) = 0$ whereas f_k is nonvanishing on whole of D by hypothesis. This is a contradiction. Hence the $n = 1$ case follows. For $n > 1$ suppose $f(w_1, \ldots, w_n) = 0$ for some $(w_1, \ldots, w_n) \in D$. By the $n = 1$ case the functions $g_i(z) := f(w_1, \ldots, w_{i-1}, z, w_{i+1}, \ldots, w_n) = 0$ whenever $|z - w_i| < \epsilon_i$ for some $\epsilon_i > 0$, $1 \leq i \leq n$. Let $\epsilon := \min\{\epsilon_1, \ldots, \epsilon_n\}$. Then $f(z_1, \ldots, z_n) \equiv 0$ in $D((w_1, \ldots, w_n); \epsilon)$. Thus the restriction of f and the zero function to $D((w_1, \ldots, w_n); \epsilon)$ coincide, so they coincide on the whole of D by uniqueness of analytic continuation. $\square$

Next we prove some basic closure properties in the class of stable functions.

Proposition 1.8. (Basic closure properties)
Let $P(z_1, \ldots, z_n)$ be a stable polynomial of degree d_j in z_j for $1 \leq j \leq n$. Then for any $1 \leq i \leq n$:
(1 - Specialization) $P(z_1, \ldots, z_{i-1}, \zeta, z_{i+1}, \ldots, z_n)$ is stable or identically zero for each $\zeta \in \mathbb{C}$ with $Im(\zeta) \geq 0$.
(2 - Scaling) $P(z_1, \ldots, z_{i-1}, \lambda z_i, z_{i+1}, \ldots, z_n)$ is stable for all $\lambda > 0$.
(3 - Inversion) $z_i^{d_i} P(z_1, \ldots, z_{i-1}, -z_i^{-1}, z_{i+1}, \ldots, z_n)$ is stable.
(4 - Permutation) $P(z_1, \ldots, z_{i-1}, z_j, z_{i+1}, \ldots, z_{j-1}, z_i, z_{j+1}, \ldots, \ldots, z_n)$ stable.
(5 - Differentiation) $\partial z_i P(z_1, \ldots, z_n)$ is stable.

Proof.
(1) Suppose $P(z_1, \dots, z_{i-1}, \zeta, z_{i+1}, \dots, z_n)$ is not identically zero. If $Im(\zeta) > 0$ then the statement is clear since instability of $P(z_1, \dots, z_{i-1}, \zeta, z_{i+1}, \dots, z_n)$ immediately implies instability of $P(z_1, \dots, z_n)$. Thus suppose $Im(\zeta) = 0$. Then $P(z_1, \dots, z_{i-1}, \zeta + i\frac{1}{k}, z_{i+1}, \dots, z_n)$ is stable for all $k \in \mathbb{N}$ for the same reason as above. Hence by Hurwitz' theorem $\lim_{k\to\infty} P(z_1, \dots, z_{i-1}, \zeta + i\dfrac{1}{2^k}, z_{i+1}, \dots, z_n) = P(z_1, \dots, z_{i-1}, \zeta, z_{i+1}, \dots, z_n)$ is stable.
(2) Obvious.
(3) Suppose $(\zeta_1, \dots, \zeta_n) \in H^n$ with $\zeta_i^{d_i} P(\zeta_1, \dots, \zeta_{i-1}, -\zeta_i^{-1}, \zeta_{i+1}, \dots, \zeta_n) = 0 \implies P(\zeta_1, \dots, \zeta_{i-1}, -\zeta_i^{-1}, \zeta_{i+1}, \dots, \zeta_n) = 0$. Note that $Im(\zeta_i^{-1}) < 0$ so $Im(-\zeta_i^{-1}) > 0$. This implies $P(z_1, \dots, z_n)$ is not stable, a contradiction.
(4) Obvious.
(5) W.l.o.g consider $i = 1$. Let $\zeta_2, \dots, \zeta_n \in H$ and consider the degree d_1 polynomial $Q(z_1) := P(z_1, \zeta_2, \dots, \zeta_n)$. Then $Q'(z_1) = \partial z_1 P(z_1, \zeta_2, \dots, \zeta_n)$. Write $Q(z_1) = c\prod_{h=1}^{d_1}(z_1 - \xi_h)$ where $c, \xi_h \in \mathbb{C}$. Note that $Im(\xi_h) \le 0$ for all $h = 1, \dots, d_1$, for otherwise if $Im(\xi_h) > 0$ for some $1 \le h \le d_1$ then $P(\xi_h, \zeta_2, \dots, \zeta_n) = 0$ contradicting stability of P. Now,

$$\frac{Q'(z_1)}{Q(z_1)} = \frac{d}{dz_1}\log Q(z_1) = \frac{d}{dz_1}\left(\log c + \sum_{h=1}^{d_1}\log(z_1 - \xi_h)\right) = \sum_{h=1}^{d_1}\frac{1}{z_1 - \xi_h}.$$

If $Im(z_1) > 0$ then $Im(\frac{1}{z_1-\xi_h}) < 0$ for all $h = 1, \dots, d_1$ since $Im\,(z_1 - \xi_h) = \underbrace{Im(z_1)}_{>0} - \underbrace{Im(\xi_h)}_{\le 0} > 0$ for $h = 1, \dots, d_1$. Thus $Q'(z_1) \ne 0$ so if $\zeta_1 \in H$ then $\partial z_1 P(\zeta_1, \dots, \zeta_n) = Q'(\zeta_1) \ne 0$. Hence $\partial z_1 P$ is stable. □

Proposition 1.9.
Suppose that A_0 is a Hermitian $m \times m$, and that $A_1, \dots, A_n$ are positive semidefinite Hermitian $n \times n$ matrices. Then the polynomial

$$P = det\left(A_0 + \sum_{j=1}^{n} z_j A_j\right)$$

is either identically zero or real stable.

Proof. Let $A_j^{(k)} := A_j + \frac{1}{2^k}I$ for all $k \in \mathbb{N}$ so that $A_j^{(k)}$ is positive definite for all k (as eigenvalues are strictly positive). Define $P^{(k)} := det\left(A_0 + \sum_{j=1}^{n} z_j A_j^{(k)}\right)$ for all $k \in \mathbb{N}$. Then $P^{(k)} \to P$ as $k \to \infty$. Thus by Hurwitz' theorem it suffices to show $P^{(k)}$ is stable for all $k \in \mathbb{N}$. To this end let $\zeta_j = x_j + iy_j \in \mathbb{C}$ where $x_j \in \mathbb{R}$ and $y_j > 0$ for all j. We must show $P^{(k)}(\zeta_1, \dots, \zeta_n) \ne 0$. We have

$$P^{(k)}(\zeta_1, \dots, \zeta_n) = det\left(A_0 + \sum_{j=1}^{n} x_j A_j^{(k)} + i\sum_{j=1}^{n} y_j A_j^{(k)}\right) = det(A^{(k)} + iQ^{(k)})$$

where $A^{(k)}$ is Hermitian and $Q^{(k)}$ is positive definite (since a sum of positive definites is positive definite) and Hermitian. Thus $Q^{(k)}$ has a (Hermitian) square root and so

$$\begin{aligned}P^{(k)}(\zeta_1,\dots,\zeta_n) &= det\left(Q^{(k)}\right) det\left(Q^{(k)}\right)^{-1} det\left(A^{(k)}+iQ^{(k)}\right)\\ &= det\left(Q^{(k)}\right) det\left(\left(Q^{(k)}\right)^{-1/2}\right) det\left(A^{(k)}+iQ^{(k)}\right) det\left(\left(Q^{(k)}\right)^{-1/2}\right)\\ &= det\left(Q^{(k)}\right) det\left(\left(Q^{(k)}\right)^{-1/2} A^{(k)} \left(Q^{(k)}\right)^{-1/2}+iI\right).\end{aligned}$$

Now $\left(Q^{(k)}\right)^{-1/2} A^{(k)} \left(Q^{(k)}\right)^{-1/2}$ is Hermitian as

$$\begin{aligned}\left(\left(Q^{(k)}\right)^{-1/2} A^{(k)} \left(Q^{(k)}\right)^{-1/2}\right)^* &= \left(\left(Q^{(k)}\right)^{-1/2}\right)^* \left(A^{(k)}\right)^* \left(\left(Q^{(k)}\right)^{-1/2}\right)^*\\ &= \left(Q^{(k)}\right)^{-1/2} A^{(k)} \left(Q^{(k)}\right)^{-1/2}\end{aligned}$$

Claim: If M is a Hermitian matrix then it has only real eigenvalues.
With the claim it immediately follows that $det\left(\left(Q^{(k)}\right)^{-1/2} A^{(k)} \left(Q^{(k)}\right)^{-1/2}+iI\right)$ must be non-zero and hence $P^{(k)}(\zeta_1,\dots,\zeta_n)$ is stable for all $k\in\mathbb{N}$ as required.
Proof of claim:
Suppose λ is an eigenvalue of M with eigenvector v. Then

$$\begin{aligned}Mv=\lambda v &\implies (Mv)^*=(\lambda v)^*\\ &\implies v^*M^*=\overline{\lambda}v^*\\ &\implies v^*M^*v=\overline{\lambda}v^*v\\ &\implies v^*Mv=\overline{\lambda}v^*v\\ &\implies \lambda v^*v=\overline{\lambda}v^*v\\ &\implies \lambda=\overline{\lambda}\end{aligned}$$

Hence $\lambda\in\mathbb{R}$. □

Exercise 1: Let A be a normal matrix and let $Z = diag(z_1,\dots,z_n)$ be the diagonal matrix with variables on the diagonal. Prove that the polynomial $P(z)=det(A+Z)$ is stable if and only if all eigenvalues of A lie in the closed upper half plane.

Proof. [Find a better argument?]
The statement is straightforward in one direction. Suppose $P(z)=det(A+Z)$ is stable and A has an eigenvalue $\lambda\in\mathbb{C}$ with $Im\lambda<0$. Then $P(-\lambda,\dots,-\lambda)=det(A-\lambda I)=0$ where $Im(-\lambda)>0$. This contradicts stability of P. Conversely suppose all eigenvalues of A lie in the closed upper half plane and write $Z=diag(z_1,\dots,z_n)=\sum_{i=1}^n z_iE_{ii}$ where E_{ii} is the matrix with entry 1 at (i,i) and zeros elsewhere. Put $A_i^{(k)} := E_{ii}+\frac{1}{2^k}I$ for all $k\in\mathbb{N}$ and $1\le i\le n$. Then $A_i^{(k)}$ is clearly positive definite and $P^{(k)} := det\left(A+\sum_{j=1}^n z_jA_j^{(k)}\right)\to P$ as $k\to\infty$. Hence by Huruwiz theorem it suffices to show that $P^{(k)}$ is stable for all $k\in\mathbb{N}$. We employ much the same tactics as in the previous proposition, except we now borrow some basic results from matrix analysis to make corresponding statements about the eigenvalues of normal matrices. Indeed if $\zeta_j=x_j+iy_j\in\mathbb{C}$

where $x_j \in \mathbb{R}$ and $y_j > 0$ for all j we want to show $P^{(k)}(\zeta_1, \dots, \zeta_n) \neq 0$. We have

$P^{(k)}(\zeta_1, \dots, \zeta_n) = det\left(A + \sum_{j=1}^n x_j A_j^{(k)} + i \sum_{j=1}^n y_j A_j^{(k)}\right) = det(A^{(k)} + iQ^{(k)})$

where $Q^{(k)} = \sum_{j=1}^n y_j A_j^{(k)}$ is positive definite (since a sum of positive definites is positive definite) and $A^{(k)} := A + \sum_{j=1}^n x_j A_j^{(k)}$. Now

$$P^{(k)}(\zeta_1, \dots, \zeta_n) = det(Q^{(k)}) det\left((Q^{(k)})^{-1} A^{(k)} + iI\right)$$

where $det(Q^{(k)}) = (1+1/2^k)(1/2^{(n-1)k}) \neq 0$. We want to show that the eigenvalues of $(Q^{(k)})^{-1}A^{(k)}$ lie in the closed upper half-plane. Then $det\left((Q^{(k)})^{-1}A^{(k)} - (-i)I\right) \neq 0$ proving that $P^{(k)}$ is stable. To show this let $\sigma(M)$ denote the spectrum of eigenvalues of matrix M and let $F(M) := \{z^* M z : z \in \mathbb{C}^n, z^* z = 1\}$ denote the field of values (aka numerical range) of M. There are a few facts relating these two sets, namely:

- (Spectral containment) $\sigma(M) \subseteq F(M)$.
- (Subadditivity) $F(M+N) \subseteq F(M) + F(N)$,
- (Normality) If M is normal then $F(M) =$ Convex hull of$\sigma(M)$.
- (Positive definiteness) If Q is positive definite then $F(QM) \subseteq F(Q)F(M)$.

Denote by Conv(S) the convex hull of the subset S of the complex plane. With above facts in mind we first deduce that $A^{(k)}$ has all eigenvalues in the closed upper half-plane since

$$\begin{aligned}
\sigma(A^{(k)}) &= \sigma\left(A + \sum_{j=1}^n x_j A_j^{(k)}\right) \\
&\subseteq F\left(A + \sum_{j=1}^n x_j A_j^{(k)}\right) \\
&\subseteq F(A) + \sum_{j=1}^n F(x_j A_j^{(k)}) \\
&\subseteq \mathrm{Conv}(\sigma(A)) + \sum_{j=1}^n \mathrm{Conv}(\sigma(x_j A_j^{(k)})) \\
&\subseteq \mathrm{Conv}(\sigma(A)) + \sum_{j=1}^n \mathrm{Conv}(\{x_j(1+1/2^k), x_j(1/2^k)\}) \\
&\subseteq \overline{H}.
\end{aligned}$$

since the spectrum of A lies in the upper half-plane by hypothesis and therefore so does its convex hull, and the convex hull of real numbers lie on the real axis. Moreover since $(Q^{(k)})^{-1}$ is positive definite it follows that

$$\begin{aligned}
\sigma((Q^{(k)})^{-1}A^{(k)}) &= F((Q^{(k)})^{-1})F(A^{(k)}) \\
&\subseteq \mathrm{Conv}(\sigma((Q^{(k)})^{-1}))\mathrm{Conv}(\sigma(A^{(k)})) \\
&\subseteq \overline{H}.
\end{aligned}$$

Note that the first inclusion follows since $A^{(k)}$ is normal (since A is normal) and has spectrum in the closed upper half plane by above. Moreover again the convex hull of real numbers lie on the real axis giving the final inclusion. Hence the eigenvalues of $(Q^{(k)})^{-1}A^{(k)}$ all lie in the closed upper half-plane $\overline{H}$ giving the required conclusion.

$\square$

Lemma 1.10. *Let $P(z) + wQ(z) \in \mathbb{C}[z_1, \ldots, z_n]$, where $Q(z)$ is not identically zero. Then $P(z) + wQ(z)$ is stable if and only if $Q(z)$ is stable and*

$$Im\left(\frac{P(z)}{Q(z)}\right) \geq 0.$$

Proof. Suppose $P(z) + wQ(z)$ is stable. Then by scaling and specialization $n^{-1}P(z) + iQ(z)$ is also stable. Moreover

$$sup_z|(n^{-1}P(z) + iQ(z)) - iQ(z)| = sup_z|(n^{-1}P(z)| \to 0 \text{ as } n \to \infty.$$

Thus $n^{-1}P(z)+iQ(z) \to 0$ uniformly on compact subsets as $n \to \infty$. Therefore $iQ(z)$ is stable by Hurwitz theorem and hence so is $Q(z)$. Finally if $\zeta \in H^n$ and $P(\zeta) + wQ(\zeta) = 0$ then $Im(w) \leq 0$ since $P(z) + wQ(z)$ is stable. Thus

$$-w = \frac{P(z)}{Q(z)} \implies Im\left(\frac{P(z)}{Q(z)}\right) = -Im(w) \geq 0 \text{ for all } \zeta \in H^n.$$

Conversely suppose $P(z) + wQ(z)$ is not stable and $Q(z)$ is stable. Then there exists $\zeta \in H^n, w \in H$ such that $P(\zeta) + wQ(\zeta) = 0$. Since $Q(z)$ is stable $Q(\zeta) \neq 0$. Hence

$$Im\left(\tfrac{P(\zeta)}{Q(\zeta)}\right) = -Im(w) < 0.$$

$\square$

Lemma 1.11. *(Lieb-Sokal)*
Let $P(z) + wQ(z) \in \mathbb{C}[z_1, \ldots, z_n, w]$ be stable. If the degree in variable z_j is at most one then the polynomial

$$P(z) - \tfrac{\partial Q(z)}{\partial z_j}$$

is either identically zero or stable.

Proof. Suppose $Q(z) \not\equiv 0$ and w.l.o.g assume $j = 1$. Since $P(z) + wQ(z)$ is stable so is $Q(z)$ by Proposition 1.10. Moreover $Im(w) > 0$ iff $Im(-w^{-1}) > 0$. Thus

$$wQ(z) - \frac{\partial Q(z)}{\partial z_1} = wQ(z, -w^{-1}, z_2, \ldots, z_n)$$

is stable. By Proposition 1.10

$$Im\left(\frac{P(z) - \partial Q(z)/\partial z_1}{Q(z)}\right) = Im\left(\frac{P(z)}{Q(z)}\right) + Im\left(\frac{-\partial Q(z)/\partial z_1}{Q(z)}\right) \geq 0$$

for all $z \in H^n$. Thus by Proposition 1.10 it follows that below function is stable

$$P(z) - \frac{\partial Q(z)}{\partial z_1} + wQ(z)$$

In particular the sequence $\{P(z) - \frac{\partial Q(z)}{\partial z_1} + \frac{1}{2^k}Q(z)\}_{k\in\mathbb{N}}$ is stable and uniformly convergent to $P(z) - \frac{\partial Q(z)}{\partial z_1}$ so the statement follows from Hurwitz theorem. $\square$

Definition 1.12. *(Multiaffine polynomial)*
*A polynomial $P(z_1, \dots, z_n)$ is called **multiaffine** if*

$$P = \sum_{S\subseteq[n]} a(S)z^S, \text{ where } z^S = \prod_{j\in S} z_j$$

and $a(S) \in \mathbb{C}$ for all $S \subseteq [n]$. We denote the space of complex multiaffine polynomials by $\mathbb{C}_1[z_1, \dots, z_n]$.

Definition 1.13. *(Symbol)*
*Let $T : \mathbb{C}_1[z_1, \dots, z_n] \to \mathbb{C}[z_1, \dots, z_n]$ be a linear transformation. The **symbol** of T is the polynomial in $\mathbb{C}[z_1, \dots .z_m, w_1, \dots, w_n]$ defined by*

$$G_T = \sum_{S\subseteq[n]} T(z^S)w^{[n]\setminus S}$$

Proposition 1.14. *Let $T : \mathbb{C}_1[z_1, \dots, z_n] \to \mathbb{C}[z_1, \dots, z_n]$ be a linear transformation. If the symbol G_T is stable, then T preserves stability.*

Proof. Since $Im(w) > 0$ if and only if $Im(-w^{-1}) > 0$ we have that

$$(-1)^n w^{[n]} G_T(z, -w^{-1}) \text{ is stable iff } G_T(z, w) \text{ is stable.}$$

Therefore if $P \in \mathbb{C}_1[v_1, \dots, v_n]$ is stable then the polynomial

$$\begin{aligned}
(-1)^n w^{[n]} G_T(z, -w^{-1})P(v) &= (-1)^n w^{[n]} \sum_{S\subseteq[n]} T(z^S)(-w^{-1})^{[n]\setminus S} P(v) \\
&= (-1)^n \sum_{S\subseteq[n]} T(z^S) \frac{\prod_{j\in[n]} w_j}{(-1)^{n-|S|} \prod_{j\in[n]\setminus S} w_j} P(v) \\
&= \sum_{S\subseteq[n]} T(z^S)(-1)^{|S|} \prod_{j\in S} w_j P(v) \\
&= \sum_{S\subseteq[n]} T(z^S)(-w)^S P(v)
\end{aligned}$$

is stable, where $v = (v_1, \dots, v_n)$. Write

$$\sum_{S\subseteq[n]} T(z^S)(-w)^S P(v) = \sum_{S\subseteq[n-1]} T(z^S)(-w)^S P(v) - w_n \sum_{S\subseteq[n-1]} T(z^{S\cup\{n\}})(-w)^S P(v).$$

Then by Lieb-Sokal (Lemma 1.11) it follows that

$$\sum_{S\subseteq[n-1]} T(z^S)(-w)^S P(v) + \sum_{S\subseteq[n-1]} T(z^{S\cup\{n\}})(-w)^S \frac{\partial}{\partial v_n} P(v)$$

is stable. Using the lemma inductively, at each step replacing $-w_j$ with $\frac{\partial}{\partial v_j}$ for $j = 1, \dots, n$ we get

$$\sum_{S\subseteq[n]} T(z^S)P^{(S)}(v), \text{ where } P^{(S)} = \prod_{j\in S} \frac{\partial}{\partial v_j} P$$

is stable or identically zero. Letting $v \to 0$ in H^n and invoking Hurwitz theorem it finally follows that

$$T(P) = \sum_{S \subseteq [n]} T(z^S) a(S) = \sum_{S \subseteq [n]} T(z^S) P^{(S)}(0)$$

is stable. ☐

2 Partial Symmetrization And Grace-Walsh-Szegö Coincidence Theorem

Theorem 2.1.
Let $1 \leq i < j \leq n$, $\tau = (i, j)$ a transposition and $0 \leq p \leq 1$.
Define a linear operator $T_{\tau,p}$ on $\mathbb{C}[z_1, \dots, z_n]$ by

$$T_{r,p}(P) = (1-p)\tau(P) + pP.$$

Then $T_{\tau,p}$ preserves stability on $\mathbb{C}_{\mathbf{1}}[z_1, \dots, z_n]$.

Proof. The trick is to identify the relevant part of the symbol $G_{T_{\tau,p}}$ with a determinant satisfying the hypotheses of Proposition 1.9 and then invoke Proposition 1.14 to conclude stability for $T_{\tau,p}$. Assume w.l.o.g that $\tau = (1\,2)$. Then

$$\begin{aligned}
G_{T_{\tau,p}}(z, w) &= \sum_{S \subseteq [n]} T_{\tau,p}(z^S) w^{[n] \setminus S} \\
&= \sum_{S \subseteq [n]} ((1-p)\tau(z^S) + pz^S) w^{[n] \setminus S} \\
&= (1-p) \sum_{S \subseteq [n]} \tau(z^S) w^{[n] \setminus S} + p \sum_{S \subseteq [n]} z^S w^{[n] \setminus S} \\
&= (1-p)(z_1 + w_2)(z_2 + w_1) \prod_{k=3}^{n} (z_k + w_k) + p \prod_{k=1}^{n} (z_k + w_k) \\
&= ((1-p)(z_1 + w_2)(z_2 + w_1) + p(z_1 + w_1)(z_2 + w_2)) \prod_{k=3}^{n} (z_k + w_k)
\end{aligned}$$

(Informally the computation follows from the fact that each term in $\sum_{S \subseteq [n]} z^S w^{[n] \setminus S}$ comes from choosing a factor z_k or w_k from each bracket in the product $(z_1 + w_1) \dots (z_n + w_n)$. Letting τ permute the z_i's we no longer have terms with z_1 and w_2 appearing together, and symmetrically no terms with z_2 and w_1 together, hence the factor $(z_1 + w_2)(z_2 + w_1)$).

Now, $\prod_{k=3}^{n} (z_k + w_k)$ is clearly stable given that for $k = 3, \dots, n$,

$$Im(z_k), Im(w_k) > 0 \implies z_k + w_k \neq 0 \implies \prod_{k=3}^{n} (z_k + w_k) \neq 0.$$

Thus it suffices to show stability of the polynomial

$$G(z, w) = (1-p)(z_1 + w_2)(z_2 + w_1) + p(z_1 + w_1)(z_2 + w_2)$$

where $z = (z_1, z_2)$ and $w = (w_1, w_2)$. We can realize $G(z, w)$ as the determinant $\det(Z + C(w_1, w_2))$ where

$$Z = diag(z_1, z_2),\ C(w_1, w_2) = \begin{pmatrix} (1-p)w_1 + pw_2 & \sqrt{p(1-p)}(w_2 - w_1) \\ \sqrt{p(1-p)}(w_2 - w_1) & pw_1 + (1-p)w_2 \end{pmatrix}$$

[How does $C(w_1, w_2)$ arise?]

Now if $w_i, z_i \in H$ where $w_i = x_i + iy_i$ and $z_i = a_i + ib_i$ for $x_i, y_i, a_i, b_i \in \mathbb{R}$, $y_i, b_i > 0$, $i = 1, 2$ then

$$G(z, w) = det\left((diag(a_1, a_2) + C(x)) + i(diag(b_1, b_2) + C(y))\right)$$

Note that $C(x)$ and $C(y)$ are clearly real symmetric (and therefore Hermitian) so $diag(a_1, a_2) + C(x)$ is Hermitian. Moreover $diag(b_1, b_2)$ is clearly positive definite, so $diag(b_1, b_2) + C(y)$ is positive definite if $C(y)$ is (since a sum of positive definite matrices is again positive definite). Thus in order to apply Proposition 1.9 we must show $C(y)$ is positive definite. It is already symmetric so we must show its eigenvalues are positive. Indeed letting

$$a = (1-p)y_1 + py_2\ ,\ b = py_1 + (1-p)y_2\ ,\ c = \sqrt{p(1-p)}(y_2 - y_1)$$

we have

$$\begin{aligned} det(C(y_1, y_2) - \lambda I) = 0 &\implies \begin{vmatrix} a - \lambda & c \\ c & b - \lambda \end{vmatrix} = 0 \\ &\implies \lambda^2 - (a+b)\lambda + ab - c^2 = 0 \\ &\implies \lambda = \frac{a+b}{2} \pm \sqrt{\left(\frac{a+b}{2}\right)^2 - ab + c^2}. \end{aligned}$$

By straightforward calculation $\frac{a+b}{2} = \frac{y_1 + y_2}{2} > 0$ and $-ab + c^2 = -y_1 y_2 < 0$. Moreover

$$\left(\frac{a+b}{2}\right)^2 - ab + c^2 = \left(\frac{y_1 + y_2}{2}\right)^2 - y_1 y_2 = \frac{(y_1 - y_2)^2}{4} \geq 0$$

so both eigenvalues are real. Finally both eigenvalues are positive since

$$\sqrt{\left(\frac{a+b}{2}\right)^2 - ab + c^2} = \sqrt{\left(\frac{y_1 + y_2}{2}\right)^2 - y_1 y_2} \leq \frac{y_1 + y_2}{2} = \frac{a+b}{2}.$$

Thus $C(y)$ is positive definite and hence the symbol $G_{T_{\tau,p}}(z, w)$ is stable by Proposition 1.9 so $T_{\tau,p}$ preserves stability by Proposition 1.14 □

Definition 2.2. *(Symmetrization operator)*
*The **symmetrization operator** $Sym_n : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ is defined by*

$$Sym_n(f) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma(f).$$

Corollary 2.3. *The symmetrization operator Sym_n preserves stability on $\mathbb{C}_{\mathbf{1}}[z_1, \ldots, z_n]$*

Proof. Let $P \in \mathbb{C}_{\mathbf{1}}[z_1, \dots, z_n]$ be stable. We claim that $Sym_n(P)$ is the uniform limit on compact sets, of a sequence $\{P_k\}$ where $P_0 = P$ and $P_k = T_{\tau_k,1/2}(P_{k-1})$ for some sequence of transpositions $\{\tau_k\}$. The sequence $\{P_k\}$ is stable by Theorem 2.1 whence the corollary follows from Hurwitz theorem.
If $P(z) = \sum_{S\subseteq[n]} a(S)z^S$ and τ is a transposition, let

$$||P||_\tau = \sum_{S\subseteq[n]} |a(S) - a(\tau(S))| \text{ and } ||P|| = \textstyle\sum_\tau ||P||_\tau$$

where the latter sum runs over the set of all transpositions $\tau \in \mathfrak{S}_n$. Then

$$\begin{aligned}
||P|| = 0 &\iff ||P||_\tau = 0 \text{ for all } \tau \\
&\iff a(S) = a(\tau(S)) \text{ for all } \tau \text{ and } S \subseteq [n] \\
&\iff a(S) = a(\sigma(S)) \text{ for all } \sigma \in \mathfrak{S}_n \text{ and } S \subseteq [n] \text{ (since } \mathfrak{S}_n \text{ is generated by transpositions)} \\
&\iff P \text{ is symmetric.}
\end{aligned}$$

Note that

$$\begin{aligned}
||T_{\tau,1/2}(P)||_\tau &= \frac{1}{2}||\tau(P) + P||_\tau \\
&= \frac{1}{2}\sum_{S\subseteq[n]} |(a(\tau(S)) + a(S)) - (\underbrace{a(\tau^2(S)))}_{=a(S)} + a(\tau(S))| \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
||T_{\tau,1/2}(P)||_\sigma &= \frac{1}{2}\sum_{S\subseteq[n]} |a(S) + a(\tau(S)) - a(\sigma(S)) - a(\tau\sigma(S))| \\
&\le \frac{1}{2}\sum_{S\subseteq[n]} |a(S) - a(\sigma(S))| + \frac{1}{2}\sum_{S\subseteq[n]} |a(\tau(S)) - a(\tau\sigma\tau\tau(S))| \quad \text{[by } \Delta\text{-inequality and } \tau\tau = id] \\
&= \frac{1}{2}||P||_\sigma + \frac{1}{2}\sum_{S\subseteq[n]} |a(S) - a(\tau\sigma\tau(S))| \quad \text{[since } \tau \text{ is a bijection]} \\
&= \frac{1}{2}||P||_\sigma + \frac{1}{2}||P||_{\tau\sigma\tau}.
\end{aligned}$$

Hence

$$\begin{aligned}
||T_{\tau,1/2}(P)|| &= \sum_\sigma ||T_{\tau,1/2}(P)||_\sigma \\
&= \sum_{\sigma\neq\tau} ||T_{\tau,1/2}(P)||_\sigma \\
&\le \frac{1}{2}\sum_{\sigma\neq\tau} ||P||_\sigma + \frac{1}{2}\sum_{\sigma\neq\tau} ||P||_{\tau\sigma\tau} \\
&= \frac{1}{2}\sum_{\sigma\neq\tau} ||P||_\sigma + \frac{1}{2}\sum_{\sigma\neq\tau} ||P||_\sigma \quad \text{[since } \phi(\sigma) = \tau\sigma\tau = \tau\sigma\tau^{-1} \text{ is an automorphism of } \mathfrak{S}_n] \\
&= ||P|| - ||P||_\tau.
\end{aligned}$$

Out of the $\binom{n}{2}$ transpositions in $\mathfrak{S}_n$ choose a transposition τ_1 for which $||P||_{\tau_1}$ is maximal. Then

$$||P||_{\tau_1} \geq \frac{\sum_\tau ||P||_\tau}{\binom{n}{2}} = \frac{||P||}{\binom{n}{2}} \quad \text{[the max is at least the average]}$$

so that

$$||T_{\tau^*,1/2}(P)|| \leq ||P|| - ||P||_{\tau^*} \leq ||P|| - \frac{||P||}{\binom{n}{2}} = ||P|| \left(1 - \frac{1}{\binom{n}{2}}\right).$$

Now inductively given $P_k = T_{\tau_k,1/2}(P_{k-1})$ choose a transposition τ_{k+1} maximizing $||P_k||_{\tau_{k+1}}$. Then by induction

$$||P_{k+1}|| = ||T_{\tau_{k+1},1/2}(P_k)|| \leq ||P_k|| \left(1 - \frac{1}{\binom{n}{2}}\right) \leq ||P|| \left(1 - \frac{1}{\binom{n}{2}}\right)^{k+1}.$$

Now by triangle inequality and invariance under permutation

$$\begin{aligned}
\sup_{z\in\mathbb{C}^n,|z|=r} |T_{\tau,1/2}(P_k(z))| &= \sup_{z\in\mathbb{C}^n,|z|\leq r} \frac{1}{2}|\tau(P_k(z)) + P_k(z)| \\
&\leq \frac{1}{2} \sup_{z\in\mathbb{C}^n,|z|\leq r} |\tau(P_k(z))| + \frac{1}{2} \sup_{z\in\mathbb{C}^n,|z|\leq r} |P_k(z)| \\
&= \sup_{z\in\mathbb{C}^n,|z|\leq r} |P_k(z)|.
\end{aligned}$$

Hence

$$\sup_{z\in\mathbb{C}^n,|z|\leq r} |P_k(z)| \leq \sup_{z\in\mathbb{C}^n,|z|\leq r} |P(z)| < \infty \text{ for all } k \in \mathbb{N}$$

so the sequence $\{P_k\}_{k\in\mathbb{N}}$ is locally bounded. Hence by Montel's theorem from complex analysis there exists a subsequence $\{P_{n_k}\}_{k\in\mathbb{N}}$ of $\{P_k\}_{k\in\mathbb{N}}$ converging uniformly on compact subsets of $\mathbb{C}^n$. Note that $||.|| : \mathbb{C}_{\mathbf{1}}[z_1, \ldots, z_n] \to \mathbb{R}$ is a continuous function since it is a seminorm. Thus

$$0 = \lim_{k\to\infty} ||P_{n_k}|| = || \lim_{k\to\infty} P_{n_k}||.$$

Therefore by our earlier observation $\lim_{k\to\infty} P_{n_k}$ is symmetric.

[Why is $\lim_{k\to\infty} P_{n_k} = Sym_n(P)$?] □

Definition 2.4. *(Elementary Symmetric Polynomial)*
The k^{th} ***elementary symmetric polynomial*** *$e_k(z_1, \ldots, z_n)$ is defined by*

$$\prod_{j=1}^{n} (z_j + t) = \sum_{k=0}^{n} e_k(z_1, \ldots, z_n) t^{n-k},$$

that is

$$e_k(z_1, \ldots, z_n) = \sum_{S\subseteq[n],|S|=k} z^S.$$

Corollary 2.5. *Let C be a circular domain, and $P(z) = \sum_{k=0}^{n} a_k z^k$ a C-stable polynomial of degree at most n (and of degree exactly n if C is non-convex). Then the polynomial*

$$\sum_{k=0}^{n} a_k e_k(z_1, \ldots, z_n) / \binom{n}{k}$$

is C^n-stable.

Proof. We first prove the corollary for the case when $C = H$. Write $P(z) = a_d \prod_{j=1}^{d}(z - \zeta_j)$. Here $Im(\zeta_j) \leq 0$ for all $1 \leq j \leq d$ since $P(z)$ is H-stable. The polynomial $Q(z_1, \ldots, z_d) = a_d \prod_{j=1}^{d}(z_j - \zeta_j)$ is then multiaffine and H^n-stable. Now

$$Sym_n(Q) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma(Q) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \prod_{j=1}^{n} (z_{\sigma(j)} - \zeta_j).$$

The term $z^S \zeta^{[n] \setminus \sigma(S)}$ appears $|\sigma(S)|!(n - |\sigma(S)|)! = |S|!(n - |S|)!$ times in $Sym_n(Q)$ (the number of permutations that permute $\sigma(S)$ and $[n] \setminus \sigma(S)$ internally). Thus summing over all permutations, z^S appears with factor $a_{|S|}|S|!(n - |S|)!$ in $Sym_n(Q)$ for all $S \subseteq [n]$ since $\sum_{\sigma \in \mathfrak{S}_n} \zeta^{[n] \setminus \sigma(S)} = a_{|S|}$. Hence

$$\begin{aligned} Sym_n(Q) &= \frac{1}{n!} \sum_{S \subset [n]} a_{|S|} |S|!(n - |S|)! z^S \\ &= \frac{1}{n!} \sum_{k=0}^{n} \sum_{S \subseteq [n], |S| = k} a_k k!(n-k)! z^S \\ &= \sum_{k=0}^{n} a_k e_k(z_1, \ldots, z_n) / \binom{n}{k}. \end{aligned}$$

By Corollary 2.3, Q stable implies $Sym_n(Q)$ stable so the statement follows. Now let C be an arbitrary open circular domain. Möbius transformations map circular domains bijectively onto circular domains so there exists a Möbius transformation $\phi : z \mapsto \dfrac{az + b}{cz + d}$ mapping H onto $C \cup \{\infty\}$. If C is convex it cannot contain ∞ so we may assume $-d/c \notin H$. Then $Q(z) = (cz + d)^n P(\phi(z))$ is H-stable so

$$(cz_1 + d) \ldots (cz_n + d) a_d \prod_{j=1}^{n} (\phi(z_j) - \zeta_j)$$

is H^n-stable and so by Corollary 2.3 $Sym_n(a_d \prod_{j=1}^{n}(\phi(z_j) - \zeta_j))$ is H^n-stable. Thus

$$\sum_{k=0}^{n} a_k e_k(\phi(z_1), \ldots, \phi(z_n)) / \binom{n}{k}$$

is H^n-stable. But ϕ maps H onto C so if $(b_1, \ldots, b_n) \in C^n$ then there exists $(a_1, \ldots, a_n) \in H^n$ such that $\phi(a_k) = b_k$. Therefore for $(b_1, \ldots, b_n) \in C^n$ we have

$$\sum_{k=0}^{n} a_k e_k(b_1,\ldots,b_n)/\binom{n}{k} = \sum_{k=0}^{n} a_k e_k(\phi(a_1),\ldots,\phi(a_n))/\binom{n}{k} \neq 0$$

Hence $\sum_{k=0}^{n} a_k e_k(z_1,\ldots,z_n)/\binom{n}{k}$ is C^n-stable. □

Theorem 2.6. *(Grace-Walsh-Szegö)*
Let $P \in \mathbb{C}[z_1,\ldots,z_n]$ be a multiaffine and symmetric polynomial, and let C be a circular region. Assume that either C is convex or that the degree of P is n. For any $\zeta_1,\ldots,\zeta_n \in C$ there exists a $\zeta \in C$ such that $P(\zeta_1,\ldots,\zeta_n) = P(\zeta,\ldots,\zeta)$.

Proof. Suppose there exists $K \in \mathbb{C}$ such that $P(\zeta,\ldots,\zeta) \neq K$ for all $\zeta \in C$. We must show $P(\zeta_1,\ldots,\zeta_n) \neq K$ for any $\zeta_1,\ldots,\zeta_n \in C$. By assumption $P(z,\ldots,z) - K$ is never zero so it is C-stable. By Corollary 2.5 it follows that $P(z_1,\ldots,z_n) - K$ is C^n-stable proving the theorem. □

Definition 2.7. *(Apolar polynomial)*
Two polynomials $P(z) = \sum_{k=0}^{n} a_k z^k$ and $Q(z) = \sum_{k=0}^{n} b_k z^k$ of degree n are ***apolar*** *if*

$$\sum_{k=0}^{n}(-1)^k a_k b_{n-k}/\binom{n}{k} = 0.$$

Theorem 2.8. *(Graces's theorem)*
Let P and Q be apolar polynomials. Then every circular domain containing all the zeros of one of them contains at least one zero of the other.

Proof. Suppose C is a circular region containing all the zeros of P and write $Q(z) = b_n \prod_{j=1}^{n}(z - \zeta_j)$. Then the coefficient b_{n-k} of z^{n-k} is given by the sum of all products $(-1)^{|S|}\zeta^S$ where $S \subseteq [n]$ and $|S| = k$, in other words by $(-1)^k e_k(\zeta_1,\ldots,\zeta_n)$. Hence apolarity is equivalent to

$$\sum_{k=0}^{n} a_k e_k(\zeta_1,\ldots,\zeta_n)/\binom{n}{k} = 0.$$

By assumption P has no zeros in the complement of C. By Corollary 2.5

$$\sum_{k=0}^{n} a_k e_k(z_1,\ldots,z_n)/\binom{n}{k}$$

is C^n-stable. Thus at least one of the roots ζ_j must lie in C in order to not contradict stability of above polynomial. □

Theorem 2.9. *(Schur-Szegö composition theorem)*
Let $P = \sum_{k=0}^{n} z^k$ and $Q = \sum_{k=0}^{n} b_k z^k$ be polynomials of degree n. Suppose that the circular region C contains all the zeros of P, then each zero ζ of

$$P * Q = \sum_{k=0}^{n} a_k b_k z^k/\binom{n}{k}$$

is of the form $\zeta = -\alpha\beta$, where $\alpha \in C$ and $Q(\beta) = 0$.

Proof. Suppose first that C is closed and $Q(0) \neq 0$. Let $\beta_1, \ldots, \beta_n$ be the roots of Q. Then

$$\begin{aligned} b_k &= e_{n-k}(-\beta_1, \ldots, -\beta_n) \\ &= \sum_{S \subseteq [n], |S| = n-k} (-1)^{n-k} \beta^S \\ &= \prod_{j=1}^{n} \beta_j \sum_{S \subseteq [n], |S| = n-k} (-1)^{n-k} \frac{1}{\beta^{[n] \setminus S}} \\ &= \prod_{j=1}^{n} \beta_j \sum_{S \subseteq [n], |S| = k} (-1)^{n} \frac{1}{(-\beta)^S} \\ &= A e_k(-1/\beta_1, \ldots, -1/\beta_n) \end{aligned}$$

where $A = (-1)^n \prod_{j=1}^n \beta_j$. Note that above computation is well-defined since $Q(0) \neq 0$ so the roots β_j are all non-zero. Let C' be the complement of C. Then P is C'-stable so by Corollary 2.5 it follows that

$$h(z) = A \sum_{k=0}^{n} a_k e_k(-z/\beta_1, \ldots, -z/\beta_n) / \binom{n}{k}.$$

is $(C')^n$-stable. Thus if $(-z/\beta_1, \ldots, -z/\beta_n) \in (C')^n$ then $h(z) \neq 0$. Therefore if ζ is a root of $h(z)$ then there exists some j such that $-\zeta/\beta_j = \alpha$ for some $\alpha \in C$, in other words $\zeta = -\alpha\beta_j$ for some β_j a root of Q. Now suppose $Q(0) = 0$ and consider $h_\epsilon = P(z) * Q(z - \epsilon)$ for sufficiently small $\epsilon > 0$. Then by the previous part the zeros of h_ϵ lie in

$$-(\beta_1 + \epsilon)C \cup \cdots \cup -(\beta_n + \epsilon).$$

Since C is closed the zeros of $h = \lim_{\epsilon \to 0} h_\epsilon$ lie in

$$-\beta_1 C \cup \cdots \cup -\beta_n C$$

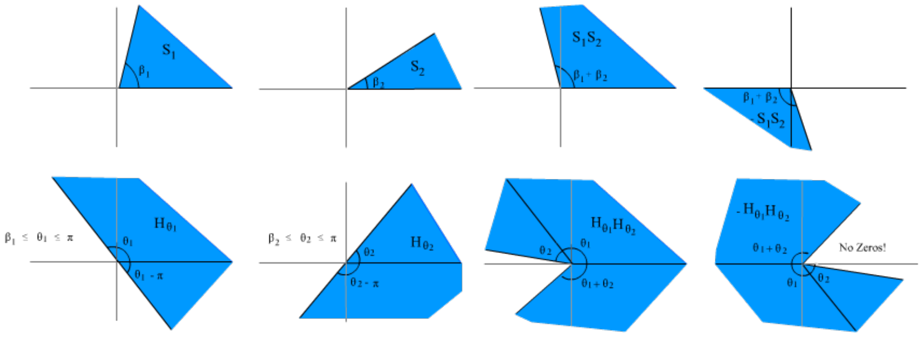
by Hurwitz theorem. This proves the theorem for C closed. Finally suppose that C is open. Then C may be shrunk to an open region D properly contained in C containing the roots of P. Then $\overline{D} \subset C$. Now the theorem applied to the closed region $\overline{D}$ gives the statement for C. □

Exercise 2: Define a sector to be a set of the form $S = \{re^{i\theta} : r \geq 0, \alpha \leq \theta \leq \beta\}$. Prove that if P has all zeros in a sector S_1 and Q in S_2, then $P * Q$ has all its zeros in $-S_1 S_2$.

Proof. Let $S_1 = \{re^{i\theta} : r \geq 0, \alpha_1 \leq \theta \leq \beta_1\}$. and $S_2 = \{re^{i\theta} : r \geq 0, \alpha_2 \leq \theta \leq \beta_2\}$. We first show that we may w.l.o.g assume $\alpha_1, \alpha_2 = 0$. Suppose the theorem holds for $\alpha_1, \alpha_2 = 0$. Given $\alpha_1, \alpha_2 \neq 0$ then $P(e^{\alpha_1 i} z)$ has its zeros in $e^{-\alpha_1 i} S_1$ and $Q(e^{\alpha_2 i} z)$ its zeros in $e^{-\alpha_2 i} S_2$ (both sectors starting at $\theta = 0$). Our assumption therefore applies to $h(e^{(\alpha_1+\alpha_2)i} z) = P(e^{\alpha_1 i} z) * Q(e^{\alpha_2 i} z)$ whereby its zeros lie in $-e^{-(\alpha_1+\alpha_2)i} S_1 S_2$. To state the obvious: if $-e^{-(\alpha_1+\alpha_2)i} \zeta_1 \zeta_2$ is a zero of $h(e^{(\alpha_1+\alpha_2)i} z)$ where $\zeta_1 \in S_1, \zeta_2 \in S_2$, then

$$h(-\zeta_1 \zeta_2) = h(e^{(\alpha_1+\alpha_2)i}(-e^{-(\alpha_1+\alpha_2)i} \zeta_1 \zeta_2)) = 0$$

Hence the zeros of $h(z)$ must lie in $-S_1S_2$ as required and so we may assume $\alpha_1, \alpha_2 = 0$. The only sectors which are circular regions are the ones where $\beta_1 - \alpha_1 = \pi$ (i.e the half-planes) so in this case the Schur-Szegö composition theorem applies and the statement follows. For arbitrary $0 \leq \beta_1, \beta_2 \leq \pi$ we will argue using half-planes containing S_1 and S_2 in order to exclude regions that do not contain any zeros of $h(z)$. Below figure illustrates the idea:



The half-planes containing S_1 and S_2 respectively make angles θ_1 and θ_2 with the positive x-axis (as depicted above) where $\beta_1 \leq \theta_1 \leq \pi$ and $\beta_2 \leq \theta_2 \leq \pi$. The zeros of $h(z)$ are therefore contained in the intersection of the regions $-H_{\theta_1}H_{\theta_2}$ which are the sectors bounded by the rays making angle $\pi - \theta_1 - \theta_2$ to the rays making angle $\pi + \theta_1 + \theta_2$. The remaining part of the plane does not contain any zeros of $h(z)$. Varying θ_1 and θ_2 in respective range we find that $h(z)$ does not contain any zeros in the sector with angle $\pi + \beta_1 + \beta_2$ to π. This leaves the zeros in the desired sector $-S_1S_2$ reaching from π to $\pi + \beta_1 + \beta_2$. $\square$

Exercise 3: Let $U = (U_{ij})_{i,j=1}^n$ be a unitary matrix. Define the linear operator T on $\mathbb{C}_{\mathbf{1}}[z_1, \ldots, z_n]$ by

$$T = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma) \left(\prod_{i=1}^{n} U_{i\sigma(i)} \right) \sigma.$$

Prove that T preserves stability.

Proof. Look over this argument, it is possibly broken

Suppose $P \in \mathbb{C}_{\mathbf{1}}[z_1, \ldots, z_n]$ is stable and let $D \subseteq H^n$ such that $\overline{D} \subseteq H^n$. Then

- P stable $\implies$ $\operatorname{Sym}_n(P(z))$ stable.
- $\inf_{z \in D}\{|P(z)|\} = \inf_{z \in D}\{|\pi(P(z))|\}$ for all $\pi \in \mathfrak{S}_n$.

We have

$$
\begin{aligned}
\inf_{z\in D}\{|T(P(z))|\} &= \frac{1}{n!}\sum_{\pi\in\mathfrak{S}_n}\inf_{z\in D}\{|\pi(T(P(z)))|\}\\
&= \inf_{z\in D}\left\{\frac{1}{n!}\sum_{\pi\in\mathfrak{S}_n}|\pi(T(P(z)))|\right\}\\
&\geq \inf_{z\in D}\left\{\left|\frac{1}{n!}\sum_{\pi\in\mathfrak{S}_n}\pi(T(P(z)))\right|\right\}\\
&= \inf_{z\in D}\left\{\left|\frac{1}{n!}\sum_{\pi\in\mathfrak{S}_n}\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sign}(\sigma)\left(\prod_{i=1}^{n}U_{i\sigma(i)}\right)\pi\sigma(P(z))\right|\right\}\\
&= \inf_{z\in D}\left\{\left|\frac{1}{n!}\sum_{\pi\in\mathfrak{S}_n}\left(\underbrace{\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sign}(\sigma)\left(\prod_{i=1}^{n}U_{i\sigma(i)}\right)}_{\det(U)=1}\right)\pi(P(z))\right|\right\}\\
&= \inf_{z\in D}\{|\mathrm{Sym}_n(P(z))|\}\\
&> 0
\end{aligned}
$$

Thus $|T(P(z))| > 0$ for all $z \in H^n$ so that T preserves stability.

$\square$

3 Polarization Procedures

Definition 3.1. *(Polarization operator)*
Let $(\kappa_1, \ldots, \kappa_n) \in \mathbb{N}^n$ and let $\mathbb{C}^{\kappa}_{MA}$ be the space of multiaffine polynomials in the independent variables $\{z_{ij} : 1 \leq j \leq \kappa_i\}$. Define a (linear) ***polarization operator***

$$\Pi^{\uparrow}_{\kappa} : \mathbb{C}_{\kappa}[z_1, \ldots, z_n] \to \mathbb{C}^{\kappa}_{MA}$$

that associates to each $P \in \mathbb{C}_{\kappa}[z_1, \ldots, z_n]$ the unique polynomial $\Pi^{\uparrow}_{\kappa}(P) \in \mathbb{C}^{\kappa}_{MA}$ such that
(a) For all $1 \leq i \leq n$ the polynomial $\Pi^{\uparrow}_{\kappa}$ is symmetric in $\{z_{ij} : 1 \leq j \leq \kappa_i\}$
(b) Putting $z_{ij} = z_i$ for all $1 \leq i \leq n$ and $1 \leq j \leq \kappa_i$ in $\Pi^{\uparrow}_{\kappa}$ recovers P.

In other words, if $\alpha \leq \kappa$ then

$$\Pi^{\uparrow}_{\kappa}(z^{\alpha}) = \binom{\kappa}{\alpha}^{-1} e_{\alpha_1}(z_{11}, \ldots, z_{1\kappa_1}) \ldots e_{\alpha_n}(z_{n1}, \ldots, z_{n\kappa_n}),$$

where $\binom{\kappa}{\alpha} = \prod_{j=1}^{n}\binom{\kappa_j}{\alpha_j}$.

Definition 3.2. *(Projection operator)*
Define the ***projection operator***

$$\Pi^{\downarrow}_{\kappa} : \mathbb{C}^{\kappa}_{MA} \to \mathbb{C}_{\kappa}[z_1, \ldots, z_n]$$

by letting $z_{ij} \mapsto z_i$ and extending linearly.

Remark 3.3. *Note that by (b) in Definition 3.1 it follows that* $\Pi_\kappa^\downarrow \circ \Pi_\kappa^\uparrow = id_{\mathbb{C}_\kappa[z_1,\dots,z_n]}$ *and* $\Pi_\kappa^\uparrow \circ \Pi_\kappa^\downarrow$ *is the operator that for each* $1 \le i \le n$ *symmetrizes the variables in* $\{z_{ij} : 1 \le i \le \kappa_i\}$.

Definition 3.4. *(Polarization of linear operator)*
Let $T : \mathbb{C}_\kappa[z_1,\dots,z_n] \to \mathbb{C}_\gamma[z_1,\dots,z_n]$ *be a linear operator. The* ***polarization of* T** *is defined as the linear operator* $\Pi(T) : \mathbb{C}^\kappa_{MA} \to \mathbb{C}^\gamma_{MA}$ *given by*

$$\Pi(T) = \Pi_\gamma^\uparrow \circ T \circ \Pi_\kappa^\downarrow.$$

Conversely

$$T = \Pi_\gamma^\downarrow \circ \Pi(T) \circ \Pi_\kappa^\uparrow.$$

Remark 3.5. *It is immediate by specialization that* $\Pi_\kappa^\downarrow$ *preserves stability. More remarkably however is that* $\Pi_\kappa^\uparrow$ *preserves stability as well (See Proposition 3.6 below).*

Proposition 3.6. *Let* $P \in \mathbb{C}_\kappa[z_1,\dots,z_n]$*. Then* P *is stable if and only if* $\Pi_\kappa^\uparrow(P)$ *is stable.*

Proof.
If $\Pi_\kappa^\uparrow(P)$ is stable then note that $\Pi_\kappa^\downarrow(\Pi_\kappa^\uparrow(P)) = P$ by Remark 3.3 and so P is stable since $\Pi_\kappa^\downarrow$ preserves stability by Remark 3.5. Conversely suppose P is stable. Write

$$P(z_1,\dots,z_n) = \sum_{j=0}^{n} Q_j(z_2,\dots,z_n) z_1^j$$

treating $z_2,\dots,z_n$ as constants. Then

$$\begin{aligned}
\Pi_\kappa^\uparrow(P(z_1,\dots,z_n)) &= \Pi_{\kappa_n}^{\uparrow z_n} \circ \cdots \circ \Pi_{\kappa_1}^{\uparrow z_1}(P(z_1,\dots,z_n)) \\
&= \Pi_{\kappa_n}^{\uparrow z_n} \circ \cdots \circ \Pi_{\kappa_2}^{\uparrow z_2}\left(\sum_{j=0}^{n} Q_j(z_2,\dots,z_n)\Pi_{\kappa_1}^{\uparrow z_1}(z_1^j)\right) \\
&= \Pi_{\kappa_n}^{\uparrow z_n} \circ \cdots \circ \Pi_{\kappa_2}^{\uparrow z_2}\left(\underbrace{\sum_{j=0}^{n} Q_j(z_2,\dots,z_n)\binom{\kappa_1}{j}^{-1} e_j(z_{11},\dots,z_{1\kappa_1})}_{\text{stable in } H^{\kappa_1} \text{ by Corollary 2.5, fixing } z_2,\dots,z_n \text{ in } H}\right).
\end{aligned}$$

Arguing inductively for $z_2,\dots,z_n$, polarizing one variable at a time and applying Corollary 2.5 we find that $\Pi_\kappa^\uparrow(P)$ is stable. □

Definition 3.7. *(Symbol of* T*)*
The ***symbol*** *of a linear operator* $T : \mathbb{C}_\kappa[z_1,\dots,z_n] \to \mathbb{C}_\gamma[z_1,\dots,z_n]$ *is the polynomial* $G_T(z,w)$ *in* $2n$ *variables defined by*

$$G_T(z,w) = T((z+w)^\kappa) = \sum_{\alpha \le \kappa} \binom{\kappa}{\alpha} T(z^\alpha) w^{\kappa-\alpha}.$$

Lemma 3.8. *Let $T : \mathbb{C}_\kappa[z_1, dots, z_n] \to \mathbb{C}_\gamma[z_1, \ldots, z_n]$ be a linear operator. The the symbol of the polarization of T is the polarization of the symbol of T, that is,*

$$G_{\Pi(T)} = \Pi^{\uparrow}_{\gamma \oplus \kappa}(G_T)$$

where $\Pi^{\uparrow}_{\gamma\oplus\kappa} : \mathbb{C}_{\gamma\oplus\kappa}[z_1, \ldots, z_n, w_1, \ldots, w_n] \to \mathbb{C}^{\gamma\oplus\kappa}_{MA}$, and $\gamma\oplus\kappa = (\gamma_1, \ldots, \gamma_m, \kappa_1, \ldots, \kappa_n)$.

Proof. We have

$$\begin{aligned}
G_{\Pi(T)} &= (\Pi^{\uparrow z}_{\gamma} \circ T \circ \Pi^{\downarrow z}_{\kappa}) \left[\prod_{i=1}^{n} \prod_{j=1}^{\kappa_i} (z_{ij} + w_{ij}) \right] \\
&= (\Pi^{\uparrow z}_{\gamma} \circ T \circ \Pi^{\downarrow z}_{\kappa}) \left[(\Pi^{\uparrow z}_{\kappa} \circ \Pi^{\uparrow w}_{\kappa}) \left[(z + w)^{\kappa} \right] \right] \\
&= (\Pi^{\uparrow z}_{\gamma} \circ T \circ \Pi^{\uparrow w}_{\kappa}) \left[(z + w)^{\kappa} \right] && [\text{since } \Pi^{\downarrow z}_{\kappa} \circ \Pi^{\uparrow z}_{\kappa} = id] \\
&= (\Pi^{\uparrow z}_{\gamma} \circ \Pi^{\uparrow w}_{\kappa} \circ T) \left[(z + w)^{\kappa} \right] && [\text{since T acts only on z-variables}] \\
&= \Pi^{\uparrow}_{\gamma\oplus\kappa}(G_T).
\end{aligned}$$

□

4 Further Properties of Stable Polynomials

Lemma 4.1. *Let $P = Q + iR \in \mathbb{C}[z_1, \ldots, z_n]$ where P and Q are real polynomials. Then the following are equivalent:*
(1) P is stable;
(2) $W + wR \in \mathbb{R}[z_1, \ldots, z_n, w]$ is stable.

Proof. Clearly (2) implies (1) by specialization $w = i$. Conversely suppose P is stable. Let $\zeta = \alpha + i\beta \in H^n$, where $\alpha, \beta \in \mathbb{R}^n$. We want to show that the univariate polynomial $H(w) := Q(\zeta) + wR(\zeta) \neq 0$ for all $w \in H$. Since $\alpha, \beta \in \mathbb{R}^n$ we also have that the polynomial $p(t) := P(\alpha + t\beta)$ is stable. Write

$$P(\alpha + t\beta) = C \prod_{j=1}^{d} (t - \xi_j).$$

Then by stability $Im(\xi_j) \leq 0$ for all $1 \leq j \leq d$ and so $-\xi_j$ is closer to i than ξ_j is to i. In other words $|i - \xi_j| \geq |i - (-\xi_j)| = |i + \xi_j|$ for all $1 \leq j \leq d$. Hence

$$\begin{aligned}
|Q(\zeta) + iR(\zeta)| = |P(\zeta)| = |C \prod_{j=1}^{d} (i - \xi_j) &\geq |C \prod_{j=1}^{n} (i + \xi_j)| = |C \prod_{j=1}^{n} (-i - \xi_j)| \\
&= |P(\alpha - i\beta)| = |P(\bar{\zeta})| = |\overline{P(\bar{\zeta})}| = |\overline{Q(\bar{\zeta})} - i\overline{R(\bar{\zeta})}| \\
&= |Q(\zeta) - iR(\zeta)| \qquad (*)
\end{aligned}$$

If $R(\zeta) = 0$ then $H(w) = P(\zeta) \neq 0$ by stability of P. We may therefore assume $R(\zeta) \neq 0$. Then dividing $(*)$ by $R(\zeta)$ we have

$$|Q(\zeta)/R(\zeta) + i| \geq |Q(\zeta)/R(\zeta) - i|$$

which implies $Im(Q(\zeta)/R(\zeta)) \geq 0$ and thus $Im(Q(\zeta)/R(\zeta)+w) > 0$ for all $w \in H$. Hence $H(w) \neq 0$ for all $w \in H$ making it stable since ζ was arbitrary in H^n. □

Corollary 4.2. *Let $Q, R \in \mathbb{R}[z_1, \dots, z_n]$. Then $Q+iR$ is stable if and only if $Q-iR$ is $(-H)^n$-stable.*

Proof. We claim that $f(z_1, \dots, z_n)$ real stable implies $f(-z_1, \dots, -z_n)$ real stable. By Proposition 1.2 it follows that the univariate polynomial $f(\alpha+\beta t)$ is real rooted for all $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^n_+$. If $f(-z_1, \dots, -z_n)$ is not real stable then there exists $\zeta_i = a_j + ib_j \in H^n$ where $a_j \in \mathbb{R}$ and $b_j \in \mathbb{R}_+$ for $j = 1, \dots, n$, such that $f(-\zeta_1, \dots, -\zeta_n) = 0$. But then $f(\alpha+\beta t) = 0$ for $\alpha = (-a_1, \dots, -a_n)$, $\beta = (b_1, \dots, b_n)$ and $t = -i$ contradicting stableness of $f(z_1, \dots, z_n)$, so the claim follows. Now by above claim and Lemma 4.1 we have

$$\begin{aligned} Q+iR \text{ stable} \iff Q+wR \text{ real stable} &\iff Q(-z)-wR(-z) \text{ real stable} \\ &\iff Q(-z)-iR(-z) \text{ stable} \\ &\iff Q-iR \text{ is } (-H)^n\text{-stable} \end{aligned}$$

□

Lemma 4.3. *Let $Q, R \in \mathbb{C}[z_1, \dots, z_n]$, $\Omega \subset \mathbb{C}^n$ a connected subset, $C_1, C_2 \subset \mathbb{C}$ two closed sets such that $C_1 \cup C_2 = \mathbb{C}$, and $J = C_1 \cap C_2$ a simple curve separating C_1 and C_2. If $P = Q + z_{n+1}R$ is $\Omega \times J$-stable and R is Ω-stable, then P is either $\Omega \times C_1$-stable or $\Omega \times C_2$-stable.*

Proof. If $P(\zeta_1, \dots, \zeta_{n+1}) = Q(\zeta_1, \dots, \zeta_n) + \zeta_{n+1}R(\zeta_1, \dots, \zeta_n) = 0$ where $(\zeta_1, \dots, \zeta_n) \in \Omega$ then $\zeta_{n+1} \notin J$ by $\Omega \times J$-stability of P. Thus solving for ζ_{n+1} it follows that

$$\zeta \in \Omega \implies -\frac{Q(\zeta)}{R(\zeta)} \notin J.$$

Note also that $R(\zeta) \neq 0$ by Ω-stability of R. From standard topology the continuous image of a connected set is connected, so the image $\{-\frac{Q(\zeta)}{R(\zeta)} : \zeta \in \Omega\}$ is connected. Since J is a simple curve separating C_1 from C_2 and $-\frac{Q(\zeta)}{R(\zeta)}$ never hits the separating boundary J it follows by connectivity that we cannot have $-\frac{Q(\zeta_1)}{R(\zeta_1)} \in C_1$ and $-\frac{Q(\zeta_2)}{R(\zeta_2)} \in C_2$ for some $\zeta_1, \zeta_2 \in \Omega$. Hence $\{-\frac{Q(\zeta)}{R(\zeta)} : \zeta \in \Omega\}$ lies exclusively in the interior of either C_1 or C_2. If w.l.o.g $-\frac{Q(\zeta)}{R(\zeta)} \in C_1$ for all $\zeta \in \Omega$ then P is $\Omega \times C_2$-stable, for if $P(\zeta_1, \dots, \zeta_{n+1}) = 0$ for $(\zeta_1, \dots, \zeta_n) \in \Omega$ and $\zeta_{n+1} \in C_2$ then

$$Q(\zeta_1, \dots, \zeta_n) + \zeta_{n+1}R(\zeta_1, \dots, \zeta_n) = 0 \implies -\frac{Q(\zeta_1, \dots, \zeta_n)}{R(\zeta_1, \dots, \zeta_n)} = \zeta_{n+1} \in C_2$$

which is a contradiction. □

Corollary 4.4. *Let Q and R be real polynomials that are not constant multiples of each other. Then*

$$\alpha Q + \beta R$$

is stable for all $\alpha, \beta \in \mathbb{R}$ for which $\alpha^2 + \beta^2 \neq 0$ if and only if $Q+iR$ or $Q-iR$ is stable.

Proof. By Lemma 4.1, $Q \pm iR$ stable implies $Q \pm wR$ stable. Now by specialization (see Proposition 1.8) $Q + \beta R$ is stable for all $\beta \in \mathbb{R}$ since $Im(\beta) \geq 0$. Therefore $\alpha(Q + \frac{\beta}{\alpha}R) = \alpha Q + \beta R$ is stable for all $\alpha, \beta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 \neq 0$ (i.e $\alpha, \beta \neq 0$) since Q and R are not constant multiples of each other. Conversely suppose $\alpha Q + \beta R$ is stable for all $\alpha, \beta \in \mathbb{R}$ for which $\alpha^2 + \beta^2 \neq 0$. Then $Q + wR$ is $H^n \times \mathbb{R}$-stable. Hence by Lemma 4.3 it follows that $Q + wR$ is stable (i.e $H^n \times H$-stable) or $H^n \times (-H)$-stable. In other words $Q + wR$ is stable or $Q - wR$ is stable. □

Lemma 4.5. *Let $V \subseteq \mathbb{K}[z_1, \ldots, z_n]$ be a $\mathbb{K}$-linear space, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.*
(i) If $\mathbb{K} = \mathbb{R}$ and every non-zero element of V is real stable then $\dim V \leq 2$.
(ii) If $\mathbb{K} = \mathbb{C}$ and every non-zero element of V is stable then $\dim V \leq 1$.

Proof. We first prove (ii). Let P and Q be two linearly independent polynomials in V. Then the linear combination $P + \zeta Q$ is non-zero for all $\zeta \in \mathbb{C}$ and lies in V (since V is a linear space) so it is stable by assumption on V. This however cannot happen since for any $\xi \in H^n$ we have $P(\xi), Q(\xi) \neq 0$ by stability of P and Q and so for $\zeta := -\frac{P(\xi)}{Q(\xi)}$ we get $P(\xi) + \zeta Q(\xi) = 0$ for any $\xi \in H^n$ contradicting stability. Hence the dimension of V can be at most 1. To prove (i) suppose there exists three linearly independent polynomials $P_1, P_2, P_3 \in V$. Then the linear combination $P_1 + vP_2 + wP_3 \in V$ is $H^n \times \mathbb{R}^2$-stable. By multiplying P_2 or P_3 by -1 if necessary we may assume via two applications of Lemma 4.3 with $J = \mathbb{R}, C_1 = H, C_2 = -H$ that $P_1 + vP_2 + wP_3$ is stable. Via scaling (see Poposition 1.8) it follows that $\lambda^{-1}(P_1 + \lambda vP_2 + \lambda wP_3)$ is stable for $\lambda > 0$. Hence by Hurwitz theorem, letting $\lambda \to \infty$ we get that $vP_2 + wP_3$ is stable, so $P_2 + \frac{w}{v}P_3$ is stable for $w, v \in H$. We have that $\{w/v : (v, w) \in H^2\} = \mathbb{C} \setminus \mathbb{R}_{<0}$ and yet $P_2 + \zeta P_3 \in V$ is supposed to be stable for all real ζ, a contradiction. Hence the dimension of V is at most 2. □

Corollary 4.6. *Let $P \in \mathbb{C}[z_1, \ldots, z_n]$. Then the following are equivalent:*
(1) P is stable and $(-H)^n$-stable;
(2) ξP is real stable for some $\xi \in \mathbb{C}$.

Proof. Real stable polynomials are automatically $(-H)^n$-stable since roots in $(-H)^n$ give roots in H^n via complex conjugation (and vice versa). Hence (2) $\implies$ (1). Now assume (1) and write P as $P = Q + iR$. By Lemma 4.1 we have that $Q + wR$ and $Q(-z) + wR(-z)$ are real stable (by $f(z_1, \ldots, z_n)$ real stable $\implies$ $f(-z_1, \ldots, -z_n)$ real stable), which means that the non-zero elements of the complex vector space V spanned by Q and R are stable. Hence V is one-dimensional by Lemma 4.5. Therefore there exists $c \in \mathbb{C}$ such that $Q = cR$ which implies $(i + c)^{-1}P = (i + c)^{-1}(Q + iR) = Q$ is real stable. □

Lemma 4.7. *Let $P \in \mathbb{C}_\kappa[z_1, \ldots, z_n]$ where $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{N}^n$, and $W = (W_1, \ldots, W_n) \in H^n$. Then for all $\epsilon > 0$ sufficiently small, the polynomial*

$$(z + W)^\kappa + \epsilon P(z)$$

is stable where $(z + W)^\kappa = (z_1 + W_1)^{\kappa_1} \ldots (z_n + W_n)^{\kappa_n}$.
Write $(z + W)^\kappa = Q + iR$, where Q and R are real polynomials. Suppose that $P \in \mathbb{R}_\kappa[z_1, \ldots, z_n]$, then

$$Q + \epsilon P$$

is real stable for all sufficiently small $\epsilon > 0$.

Proof. Set $Y = (Im(W_1), \ldots, Im(W_n)) \in \mathbb{R}^n_+$. For $\alpha \leq \kappa$ we have

$$\left|\frac{(z+W)^\alpha}{(z+W)^\kappa}\right| = \frac{1}{\prod_{j=1}^n |(z_j+W_j)|^{\kappa_j - \alpha_j}} \leq Y^{\alpha-\kappa} \text{ for } z \in H^n$$

since $|z_j + W_j| = \sqrt{(Re(z_j+W_j))^2 + \underbrace{(Im(z_j+W_j))^2}_{>Im(W_j) \text{ since } z_j, W_j \in H}} \geq Im(W_j) = Y_j$.

Thus expanding P in powers of $z+W$ we see that there exists $\epsilon_0 > 0$ such that

$$\frac{|P(z)|}{(z+W)^\kappa} < \frac{1}{\epsilon_0} \text{ for } z \in H^n.$$

Hence by (reverse) triangle inequality

$$|(z+W)^\kappa + \epsilon P(z)| \geq |(z+W)^\kappa| - \epsilon |P(z)| > 0 \text{ for all } z \in H^n \text{ and } \epsilon \in (0, \epsilon_0).$$

In particular $(z+W)^\kappa + \epsilon P(z)$ is stable for all such ϵ. If P is a real polynomial, then since $(z+W)^\kappa + \epsilon P(z) = (Q+\epsilon P) + iR$ is stable it follows directly from Corollary 4.4 with $\alpha = 1, \beta = 0$ that $Q + \epsilon P$ is stable. □

5 Algebraic Characterization of Stability Preservers

Definition 5.1. *(Multiplier Sequence)*
A sequence $\Lambda = \{\lambda\}_{k=0}^\infty \subset \mathbb{R}$ is a ***multiplier sequence*** *if the diagonal linear operator, T_Λ, defined by $T_\Lambda(z^k) = \lambda_k z^k$ preserves the property of having only real zeros.*

Theorem 5.2. *(Polya-Schur theorem)*
Let $\Lambda = \{\lambda\}_{k=0}^\infty \subset \mathbb{R}$. Then the following are equivalent:
(1) Λ is a multiplier sequence;
(2) For each $n \in \mathbb{N}$, either $T_\Lambda((1+z)^n) \equiv 0$, or all zeros of $T_\Lambda((1+z)^n)$ are real, and all its nonzero zeros are of the same sign; (Algebraic characterization)
(3) The series

$$\sum_{n=0}^\infty \frac{\lambda_n}{n!} z^n$$

is an entire function which is the limit, uniformly on compact subsets of $\mathbb{C}$ of polynomials with only real zeros which are of the same sign (Transcendental characterization).

Theorem 5.3. *Let $T : \mathbb{C}_\kappa[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ be a linear operator. Then T preserves stability if and only if*
(1) The range of T is at most one-dimensional and T is of the form

$$T(P) = \alpha(P) Q$$

where $\alpha : \mathbb{C}_\kappa[z_1, \dots, z_n] \to \mathbb{C}$ is a linear functional and Q is a stable polynomial, or
(2) The symbol $G_T(z, w) = T((z + w)^\kappa)$ is stable.

Proof. Clearly if T is a linear operator satisfying (1) then T preserves stability (by scaling a stable polynomial). If G_T is stable then $G_{\Pi(T)}$ is stable since $\Pi^\uparrow_{\gamma\oplus\kappa}$ preserves stability by Proposition 3.6 and $G_{\Pi(T)} = \Pi^\uparrow_{\gamma\oplus\kappa}(G_T)$ by Lemma 3.8. Now by Proposition 1.14 stability of $G_{\Pi(T)}$ implies stability of $\Pi(T)$. By definition

$$T = \Pi^\downarrow_\gamma \circ \Pi(T) \circ \Pi^\uparrow_\kappa.$$

and so T is a composition of stability preservers (recall projection is stable), so T itself preserves stability. Conversely suppose $T : \mathbb{C}_\kappa[z_1, \dots, z_n] \to \mathbb{C}[z_1, \dots, z_n]$ preserves stability. Given $W \in H^n$ we have that $(z+W)^\kappa$ is stable or identically zero so $T((z + W)^\kappa)$ is stable or identically zero. Suppose first the former case that $T((z + W)^\kappa) \equiv 0$ and let $P \in \mathbb{C}_\kappa[z_1, \dots, z_n]$. Then by Lemma 4.7 there exists $\epsilon > 0$ such that $(z + W)^\kappa + \epsilon P(z)$ is stable. It follows that

$$\epsilon T(P) = T((z + W)^\kappa + \epsilon P(z)).$$

is stable or identically zero. Hence the image of T is a complex linear space whose non-zero elements are all stable polynomials. By Lemma 4.5 (ii) it follows that the image has dimension 1 so that $T(P) = \alpha(P)Q$ where Q is a stable polynomial and $\alpha(P)$ is a linear functional. Suppose now that $T((z+W)^\kappa) \not\equiv 0$ for all $W \in H^n$. Then $T((z+W)^\kappa)$ is stable for all $W \in H^n$. Hence $G_T(z, w) = T((z+W)^\kappa)$ is stable since T preserves stability. ☐

Theorem 5.4. *Let $T : \mathbb{R}_\kappa[z_1, \dots, z_n] \to \mathbb{R}[z_1, \dots, z_n]$ be a linear operator. Then T preserves stability if and only if*
(1) The range of T is at most two-dimensional and T is of the form

$$T(P) = \alpha(P)Q + \beta(P)R,$$

where $\alpha, \beta : \mathbb{R}_\kappa[z_1, \dots, z_n] \to \mathbb{R}$ are linear functionals and $R + iQ$ is a stable polynomial, or
(2) The symbol $G_T(z, w) = T((z + w)^\kappa)$ is stable, or
(3) $G_T(z, -w) = T((z - w)^\kappa)$ is stable.

Proof. If $T : \mathbb{R}_\kappa[z_1, \dots, z_n] \to \mathbb{R}[z_1, \dots, z_n]$ is a linear operator as in (1) then T is stable by Corollary 4.4. If $T, T' : \mathbb{R}_\kappa[z_1, \dots, z_n] \to \mathbb{R}[z_1, \dots, z_n]$ are linear operators whose symbol satisfy $G_{T'}(z, w) = G_T(z, -w)$ then by definition

$$\begin{aligned}\sum_{\alpha\leq\kappa} \binom{\kappa}{\alpha} T'(z^\alpha) w^{\kappa-\alpha} &= \sum_{\alpha\leq\kappa} \binom{\kappa}{\alpha} T(z^\alpha)(-w)^{\kappa-\alpha}\\ &= \sum_{\alpha\leq\kappa} \binom{\kappa}{\alpha} T(z^\alpha)(-1)^{\kappa-\alpha} w^{\kappa-\alpha}\\ &= \sum_{\alpha\leq\kappa} \binom{\kappa}{\alpha} (-1)^\kappa T((-z)^\alpha) w^{\kappa-\alpha}.\end{aligned}$$

Comparing terms we see that T' and T are related via

$$T'(P)(z) = (-1)^\kappa T(P(-z)).$$

Thus they preserve stability simultaneously (by $f(z_1, \dots, z_n)$ real stable iff $f(-z_1, \dots, -z_n)$ real stable). We may therefore assume that (2) holds i.e that $G_T(z, w)$ is stable. But by Proposition 1.14 G_T stable implies T stable and hence sufficiency is proved. Suppose conversely that T preserves stability. Consider $G_T(z, w) = T((z + w)^\kappa)$ and let $W \in H^n$. Write

$$(z + W)^\kappa = F(z) + iG(z), \qquad \text{where } F, G \in \mathbb{R}_\kappa[z_1, \dots, z_n].$$

Then by Corollary 4.4 we have that $\alpha F + \beta G$ is stable or identically zero for all $\alpha, \beta \in \mathbb{R}$ and hence $T(\alpha F + \beta G) = \alpha T(F) + \beta T(G)$ is stable or identically zero for all $\alpha, \beta \in \mathbb{R}$ since T is a linear operator preserving stability by hypothesis. Thence by Corollary 4.4 again we have that

$$T(F) + iT(G) = G_T(z, W)$$

is stable, $(-H)^n$-stable or identically zero. Suppose there exists $W_1, W_2 \in H^n$ such that $G_T(z, W_1)$ is stable or identically zero and $G_T(z, W_2)$ is $(-H)^n$-stable or identically zero. By a homotopy argument details? we deduce that there exists $t \in [0, 1]$ such that $G_T(z, W')$ is stable and $(-H)^n$-stable, or identically zero where $W' = (1 - t)W_1 + tW_2 \in H^n$. Therefore by Corollary 4.6 there exists $\xi \in \mathbb{C}$ such that $\xi G_T(z, W')$ is real stable i.e there exists a real stable polynomial $P(z)$ such that $G_T(z, W') = \xi^{-1}P(z)$. Write $\xi^{-1} = a + bi$ where $a, b \in \mathbb{R}$ and write $(z + W')^\kappa = Q(z) + iR(z)$ where $Q(z)$ and $R(z)$ have real coefficients. Then $T(Q) = aP$ and $T(R) = bP$ so that

$$T(bQ - aR) == bT(Q) - aT(R) = baP - abP = 0.$$

As noted in the proof of Lemma 4.7 we have that for all $h \in \mathbb{R}_\kappa[z_1, \dots, z_n]$ there exists $\epsilon > 0$ such that

$$\frac{|h|}{(z + W')^\kappa} < \frac{1}{\epsilon}$$

whenever $Im(z_i) \geq 0$ for $1 \leq i \leq n$. Hence if $a^2 + b^2 \neq 0$ we get

$$\begin{aligned} \frac{|h|}{|bQ - aR + i(aQ + bR)|} = \frac{|h|}{|b(Q + iR) + ia(Q + iR)|} &= \frac{|h|}{|(b + ia)(z + W')^\kappa|} \\ &< \frac{1}{\epsilon|b + ia|} = \frac{1}{\epsilon'}. \end{aligned}$$

This implies by (reverse) triangle inequality that

$$0 < |bQ - aR + \epsilon' h + i(aQ + bR)| - \epsilon'|h| \leq |bQ - aR + \epsilon' h + i(aQ + bR)|$$

whenever $z \in H^n$. In particular, $bQ - aR + \epsilon' h + i(aQ + bR)$ is stable so by Corollary 4.4 $bQ - aR + \epsilon' h$ is stable. It follows that

$$T(h) = \frac{1}{\epsilon'} T(bQ - aR + \epsilon' h)$$

is stable or identically zero since T preserves stability. Thus all non-zero polynomials in the image of T are real stable and so we may conclude by Lemma 4.5 (i) that the Image of T has linear space dimension at most two. Hence T is of the form (1). We may therefore assume the symbol $T((z + W)^\kappa)$ is stable for all $W \in H^n$ or $T((z - W)^\kappa)$ is stable for all $W \in H^n$. This amounts to saying that $G_T(z, w)$ or $G_T(z, -w)$ is stable. $\square$

6 Transcendental Characterization of Stability Preservers

Definition 6.1. *(Laguerre-Pólya class)*
We say that an entire function in $f(z)$ in n variables is in the ***complex Laguerre-Pólya class****, $f(z) \in \mathcal{L} - \mathcal{P}_n(\mathbb{C})$, if there is a sequence of stable polynomials $\{P_k(z)\}_k$ such that $f(z)$ is the limit, uniformly on compact subsets of $\mathbb{C}$, of $\{P_k(z)\}$. The* ***real Laguerre-Pólya class****, $\mathcal{L} - \mathcal{P}_n(\mathbb{R})$ consists of those functions in $\mathcal{L} - \mathcal{P}_n(\mathbb{C})$ with real coefficients.*

Theorem 6.2. *A real entire function $f(z)$ is in the Laguerre-Poólya class if and only if it may be written as*

$$f(z) = Cz^n e^{az-bz^2} \prod_{k=1}^{\omega} (1 + x_k z) e^{-x_k z}$$

where $C, a, x_k \in \mathbb{R}$ for all $k, b \geq 0$, $\omega \in \mathbb{N} \cup \{\infty\}$ and $\sum_k x_k^2 < \infty$.

Theorem 6.3. *A real entire function $f(z)$ with nonnegative coefficients is in the Laguerre-Pólya class if and only if it may be written as*

$$f(z) = Cz^n e^{\sigma_z} \prod_{k=1}^{\omega} (1 + x_k z)$$

where $C, \sigma, x_k \geq 0$ for all $k, \omega \in \mathbb{N} \cup \{\infty\}$ and $\sum_k x_k < \infty$.

Definition 6.4.
The ***symbol*** *of a linear operator $T : \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ is the formal power series*

$$G_T(z, w) = T(e^{-z \cdot w}) = \sum_{\alpha \in \mathbb{N}^n} T(z^\alpha)(-1)^\alpha \frac{\omega^\alpha}{\alpha!}$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$ and $z \cdot w 0 z_1 w_1 + \cdots + z_n w_n$

Theorem 6.5. *Let $T : \mathbb{C}[z_1 \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ be a linear operator. Then T preserves stability if and only if*
(1) The range of T is at most one-dimensional and T is of the form

$$T(P) = \alpha(P)Q,$$

where $\alpha_{\mathbb{C}}[z_1, \ldots, z_n] \to \mathbb{C}$ is a linear functional and Q is a stable polynomial, or
(2) $G_T(z, w) \in \mathcal{L} - \mathcal{P}_n(\mathbb{C})$

Theorem 6.6. *Let $T : \mathbb{R}[z_1 \ldots, z_n] \to \mathbb{R}[z_1, \ldots, z_n]$ be a linear operator. Then T preserves real stability if and only if*
(1) The range of T is at most two-dimensional and T is of the form

$$T(P) = \alpha(P)Q + \beta(P)R,$$

where $\alpha, \beta : \mathbb{R}[z_1, \ldots, z_n] \to \mathbb{R}$ are linear functionals and $Q + iR$ is a stable polynomial, or
(2) $G_T(z, w) \in \mathcal{L} - \mathcal{P}_n(\mathbb{R})$, or
(3) $G_T(z, -w) \in \mathcal{L} - \mathcal{P}_n(\mathbb{R})$

Definition 6.7. *(Multivariate Jensen multipliers)*
For $\alpha, \beta \in \mathbb{N}^n$ let $J(\alpha, \beta) = (\beta)_\alpha \beta^{-\alpha}$ (using the convention that $0^{\pm 0} = 1$). For fixed $\beta \in \mathbb{N}^n$ the sequences $\{J(\alpha, \beta)\}_{\alpha \le \beta}$ and $\{(\beta)_\alpha\}_{\alpha \le \beta}$ are called ***multivariate Jensen multipliers****.*

Lemma 6.8. *Fix $1 \le i \le n$ and $\beta \in \mathbb{N}$. The linear operator on $\mathbb{C}[z_1, \ldots, z_n]$ that replaces z_i^k with $(\beta)_k z_i^k$ for all $k \in \mathbb{N}$ preserves stability.*
Let $\beta \in \mathbb{N}^n$. Hence, the linear operators on $\mathbb{C}[z_1, \ldots, z_n]$ defined by

$$\begin{aligned} z^\alpha &\mapsto J(\alpha, \beta) z^\alpha, \quad \alpha \in \mathbb{N}^n \\ z^\alpha &\mapsto (\beta)_\alpha z^\alpha, \qquad \alpha \in \mathbb{N}^n, \end{aligned}$$

preserve stability.

Proof. Fix $\beta \in \mathbb{N}^n$. Since the first operator is a composition of the second operator along with a scaling of variables, it is enough to prove the lemma only for the second operator. Denote by T_κ the restriction of the given operator to $\mathbb{C}_\kappa[z_1, \ldots, z_n]$ where $\kappa \in \mathbb{N}^n$. By Theorem 5.3 it is enough to show that the symbol of T_κ is stable for every $\kappa \in \mathbb{N}^n$. That is,

$$\begin{aligned} G_{T_\kappa}(z, w) = T_\kappa[(z + w)^\kappa] &= \sum_{\alpha \le \kappa} \binom{\kappa}{\alpha} T(z^\alpha) w^{\kappa - \alpha} \\ &= \sum_{\alpha \le \kappa} \binom{\kappa}{\alpha} (\beta)_\alpha z^\alpha w^{\kappa - \alpha} \\ &= \prod_{i=1}^n \left[\sum_{j=0}^{\kappa_i} j! \binom{\kappa_i}{j} \binom{\beta_i}{j} z_i^j w_i^{\kappa_i - j} \right] \end{aligned}$$

is stable. This amounts to showing that for any $m, n \in \mathbb{N}$ the univariate polynomial

$$g(t) = \sum_{j=0}^n j! \binom{n}{j} \binom{m}{j} t^j$$

is real-rooted [why? by Proposition 1.3?] whence its roots are necessarily negative since the polynomial has all positive coefficients. To prove this note that

$$\begin{aligned} \left(1 + \frac{d}{dt}\right)^n \Big|_{t \to t^{-1}} (t^m) &= \sum_{j=0}^n \binom{n}{j} \left(\frac{d}{dt}\right)^j (t^m) \Big|_{t \to t^{-1}} \\ &= \sum_{j=0}^n \binom{n}{j} m(m-1) \cdots (m-j+1) t^{m-j} \Big|_{t \to t^{-1}} \\ &= \sum_{j=0}^n j! \binom{n}{j} \binom{m}{j} t^{m-j} \Big|_{t \to t^{-1}} \\ &= \sum_{j=0}^n j! \binom{n}{j} \binom{m}{j} t^{j-m} \\ &= t^{-m} g(t) \end{aligned}$$

To prove that $\left(1+\frac{d}{dt}\right)^n$ preserves stability it is enough to consider $n=1$, thence the operator is a composition of stability preservers and the statement follows. Indeed the symbol of $\left(1+\frac{d}{dt}\right)$ is given by

$$\left(1+\frac{d}{dt}\right)[(t+w)^m] = (m+(t+w))(t+w)^{m-1}$$

which is clearly a stable polynomial. Hence the symbol $G_{T_\kappa}(z,w)$ is stable for every $\kappa \in \mathbb{N}^n$ and so the lemma follows. $\square$

Lemma 6.9. *(Szász)*
Suppose $f(z) = 1 + \sum_{i=1}^k a_i z^i = \prod_{j=1}^k (1+\xi_j z)$ *is stable. Then*

$$\sum_{j=1}^k |\xi_j|^2 \leq 3|a_1|^2 + 2|a_2|.$$

Proof. By assumption $Im(\xi_j) \leq 0$ for $1, \leq j \leq k$ as otherwise $-\xi_j^{-1} \in H$ is a root contradicting stability. Hence

$$\sum_{j=1}^k Im(\xi_j)^2 \leq \left(\sum_{j=1}^k Im(\xi_j)\right)^2 = Im(a_1)^2.$$

Note that

$$\sum_{j=1}^k \xi_j^2 = \left(-\sum_{j=1}^k \xi_j\right)^2 - 2\left(\sum_{1\leq i<j\leq k} \xi_i\xi_j\right) = a_1^2 - 2a_2.$$

Thus

$$\begin{aligned}
\sum_{j=1}^k |\xi_j|^2 &= \sum_{j=1}^k (Re(\xi_j)^2 + Im(\xi_j)^2) \\
&= \sum_{j=1}^k (Re(\xi_j)^2 - Im(\xi_j)^2) + 2\sum_{j=1}^k Im(\xi_j)^2 \\
&= Re\left(\sum_{j=1}^k (Re(\xi_j) + iIm(\xi_j))^2\right) + 2\sum_{j=1}^k Im(\xi_j)^2 \\
&= Re\left(\sum_{j=1}^k \xi_j^2\right) + 2\sum_{j=1}^k Im(\xi_j)^2 \\
&= Re(a_1^2 - 2a_2) + 2\sum_{j=1}^k Im(\xi_j)^2 \\
&\leq \underbrace{Re(a_1^2)}_{\leq|a_1|^2} \underbrace{-2Re(a_2)}_{\leq 2|a_2|} + \underbrace{2Im(a_1)^2}_{\leq |a_1|^2} \\
&\leq 3|a_1|^2 + 2|a_2|
\end{aligned}$$

as claimed. $\square$

Definition 6.10. *(Phase)*
*The **phase** of a complex number $\zeta = re^{i\theta}$ is given by θ.*

Lemma 6.11. *Let P be a stable homogeneous polynomial. The all nonzero Taylor coefficients of P have the same phase.*

Proof. By passing to the polarization operator if necessary we may assume by Proposition 3.6 that P is multiaffine. The proof is by induction on the number of variables. Write P as $P = Q + z_n R$ where $P, Q \in \mathbb{C}_{\mathbf{1}}[z_1, \ldots, z_{n-1}]$. If either Q or R is zero then we are done by induction. Otherwise assume Q and R are stable homogeneous polynomials of degree d and $d-1$ respectively. By induction R and Q are polynomials with Taylor coefficients of same phase. Multiplying (and scaling suitably) we may assume the Taylor coefficients of R and $e^{-i\theta}Q = \tilde{Q}$ have the same phase and are nonnegative. By Lemma 1.10 it follows that

$$Im\left(\frac{e^{i\theta}\tilde{Q}(z)}{R(z)}\right) \geq 0$$

for all $z \in H^{n-1}$. Let $x \in \mathbb{R}_+^{n-1}$ and $0 \leq \phi \leq \pi$. By homogeneity and the fact that $deg(Q) = d$, $deg(R) = d-1$ we have

$$0 \leq Im\left(\frac{e^{i\theta}\tilde{Q}(e^{i\phi}x)}{R(e^{i\phi}x)}\right) = Im\left(\frac{e^{i(\theta+\phi)}\tilde{Q}(x)}{R(e^{i\phi}x)}\right) = \frac{\tilde{Q}(x)}{R(x)} sin(\theta + \phi).$$

Since $\frac{\tilde{Q}(x)}{R(z)} \geq 0$ for $x \in \mathbb{R}_+^{n-1}$ (by nonnegativity of the Taylor coefficients of R and Q) we must have $\sin(\theta + \phi) \geq 0$ for all $0 \leq \phi \leq \pi$. This forces θ to be a multiple of 2π so the Taylor coefficients of R and Q have the same phase and hence so does P. The proof follows. □

Remark 6.12. *Let $P(z) = \sum_{\alpha \in \mathbb{N}^n} a(\alpha) z^\alpha$ be a stable polynomial. Let*

$$M = \min\{|\alpha| : a(\alpha)z^\alpha\} \text{ and } N = \max\{|\alpha| : a(\alpha) \neq 0\}.$$

Then

$$P_M(z) := \lim_{\lambda \to 0} \lambda^{-M} P(\lambda z_1, \ldots, \lambda z_n) = \sum_{|\alpha| = M} a(\alpha) z^\alpha$$

and

$$P_N(z) := \lim_{\lambda \to \infty} \lambda^{-N} P(\lambda z_1, \ldots, \lambda z_n) = \sum_{|\alpha| = N} a(\alpha) z^\alpha$$

are homogeneous polynomials which are stable by Hurwitz theorem (being a limit of stable polynomials via scaling of P). By Lemma 6.11 all Taylor coefficients of $P_M(z)$ (and $P_N(z)$) have the same phase.

Lemma 6.13. *Let $P(z) = \sum_{\alpha \in \mathbb{N}^n} a(\alpha) z^\alpha$ be a stable polynomial and let M be defined as in Remark 6.12. Let further*

$$A = \min\{\frac{\alpha!}{\alpha^\alpha}|a(\alpha)| : |\alpha| = M \text{ and } a(\alpha) \neq 0\},$$

$$B = \sum_{|\alpha| = M+1} |a(\alpha)|, \qquad C = \sum_{|\alpha| = M+2} |a(\alpha)|,$$

$$D = \left(3\frac{B^2}{A^2} + 2\frac{C}{A}\right)^{1/2}, \text{ and } \qquad E = \sum_{|\alpha| = M} |a(\alpha)|.$$

Then

$$|a(\beta)| \leq E\frac{\beta^\beta}{\beta!}(|\beta|-M)^{-(|\beta|-M)/2}D^{|\beta|-M},$$

for all $\beta \in \mathbb{N}^n$ *with* $|\beta| \geq M$.

Proof. If $a(\beta) = 0$ then there is nothing to prove as the right hand side is always non-negative. Therefore assume $a(\beta) \neq 0$. Then the polynomial

$$g(t) = \sum_{\alpha\leq\beta} J(\alpha,\beta)a(\alpha)t^{|\alpha|} = \sum_{k=M}^{d} A_k(\beta)t^k$$

where $d = |\beta|$, is stable by specialization ($z_i = t$) and Lemma 6.8. By Remark 6.12 we have $A_M(\beta) \neq 0$ since

$$P_M(z) = \sum_{|\alpha|=M} a(\alpha)z^\alpha \text{ stable } \implies \sum_{|\alpha|=M} J(\alpha,\beta)a(\alpha)z^\alpha \text{ stable}$$

$$\implies \left(\underbrace{\sum_{|\alpha|=M} J(\alpha,\beta)a(\alpha)}_{=A_M(\beta)}\right) t^M \text{ stable via specialization } z_i = t \implies A_M(\beta) \neq 0.$$

Hence we may write $g(t)$ as

$$g(t) = A_M(\beta)t^M\prod_{j=1}^{d-M}(1+\xi_j t).$$

Note that $\gamma \mapsto J(\alpha,\gamma)$ is increasing. [Indeed note that $J(\alpha,\gamma) = (\gamma)_\alpha\gamma^{-\alpha} = \frac{\gamma!}{(\gamma-\alpha)!}\frac{1}{\gamma^\alpha} = (1-\frac{1}{\gamma})(1-\frac{2}{\gamma})\cdots(1-\frac{\alpha-1}{\gamma})$ which increases to 1 as $\gamma \to \infty$]. Hence

$$\gamma \mapsto |A_M(\gamma)| = \left|\sum_{|\alpha|=M} a(\alpha)J(\alpha,\gamma)\right|$$

increases as γ increases. Thus it follows that

$$\begin{aligned}
A &= \min\left\{\underbrace{\frac{\alpha!}{\alpha^\alpha}}_{=J(\alpha,\alpha)}|a(\alpha)| : |\alpha| = M, a(\alpha) \neq 0\right\} \\
&\leq |A_M(\alpha)| \\
&\leq |A_M(\beta)| = \left|\sum_{|\alpha|=M} J(\alpha,\beta)a(\alpha)\right| && \text{[since } |A_M(\gamma)| \text{ is increasing]} \\
&\leq \sum_{|\alpha|=M}\underbrace{|J(\alpha,\beta)|}_{\leq 1}|a(\alpha)| && \text{[by triangle inequality]} \\
&\leq \sum_{|\alpha|=M}|a(\alpha)| \\
&= E
\end{aligned}$$

Note that on one hand the coefficient of $t^{|\beta|} = t^d$ in $g(t)$ is given by $J(\beta,\beta)a(\beta) = \frac{\beta! a(\beta)}{\beta^\beta}$ and on the other hand by $A_M(\beta)\prod_{j=1}^{d-M}\xi_j$. Thus

$$\begin{aligned}
\left|\frac{\beta! a(\beta)}{\beta^\beta A_M(\beta)}\right| &= \prod_{j=1}^{d-M}|\xi_j| \\
&\leq \left(\frac{\sum_{j=1}^{d-M}|\xi_j|^2}{d-M}\right)^{(d-M)/2} && \text{[by AM-GM ineq]} \\
&\leq \left(3\frac{|A_{M+1}(\beta)|^2}{|A_M(\beta)|^2} + 2\frac{|A_{M+2}(\beta)|}{|A_M(\beta)|}\right)^{(d-M)/2}(d-M)^{-(d-M)/2} && \text{[by Lemma 6.9]} \\
&\leq D^{d-M}(d-M)^{-(d-M)/2}.
\end{aligned}$$

In combination with $|A_M(\beta)| \leq E$ we get the desired inequality. □

Proposition 6.14. *Let $P(z)$ be a stable polynomial, and keep the notation in Lemma 6.13. Then*

$$\max\{|P(z)| : |z_i| \leq r \textit{ for all } 1 \leq i \leq n\}$$

where $E' = E2^{n+M-1}e^M\dfrac{\sqrt{2e^2-e}}{e-1}$ *and* $D' = 2e^2D^2$.

Proof. From Stirling approximation of $n!$ it follows that

$$e^{-n} \leq \frac{n!}{n^n} \leq (en+1)e^{-n}, \quad n \geq 0 \tag{1}$$

Let $d(n,k) = \sum_{\beta\in\mathbb{N}^n, |\beta|=k}\frac{\beta^\beta}{\beta!}$ and note that $|\{\beta \in \mathbb{N}^n : |\beta| = k\}| = \binom{n+k-1}{k}$ by a standard "stars and bars" argument. By (1) we obtain the estimate

$$d(n,k) = \sum_{\beta\in\mathbb{N}^n,|\beta|=k}\frac{\beta^\beta}{\beta!} \leq \sum_{\beta\in\mathbb{N}^n,|\beta|=k} e^k = \binom{n+k-1}{k}e^k \leq \sum_{j=0}^{n+k-1}\binom{n+k-1}{j}e^k = 2^{n+k-1}e^k \tag{2}$$

for $n, k \in \mathbb{N}$. Thus

$$\begin{aligned}
\max\{|P(z)| : |z_i| \leq r, 1 \leq i \leq n\} &\leq Er^M\sum_{|\beta|\geq M}(|\beta|-M)^{-(|\beta|-M)/2}\frac{\beta^\beta}{\beta!}(Dr)^{|\beta|-M} && \text{[by Lemma 6.13]} \\
&= Er^M\sum_{k=0}^{\infty}d(n,k+M)k^{-k/2}(Dr)^k \\
&\leq E2^{2+M-1}e^Mr^M\sum_{k=0}^{\infty}k^{-k/2}(2eDr)^k && \text{[by (2)]} \\
&\leq E2^{2+M-1}e^Mr^M\sum_{k=0}^{\infty}\sqrt{(ek+1)e^{-k}}\sqrt{\frac{(2eDr)^{2k}}{k!}} && \text{[by (1)]} \\
&\leq E2^{2+M-1}e^Mr^M\sqrt{\left(\sum_{k=0}^{\infty}(ek+1)e^{-k}\right)\cdot\left(\sum_{k=0}^{\infty}\frac{(2eDr)^{2k}}{k!}\right)} && \text{[by C-S ineq]} \\
&= E2^{2+M-1}e^Mr^M\frac{\sqrt{2e^2-e}}{e-1}\cdot e^{(2eDr)^2/2}.
\end{aligned}$$

□

Proposition 6.15. *Let $f(z) = \sum_{\alpha\in\mathbb{N}^n} a(\alpha)z^\alpha$ be a formal power series. Then the following are equivalent*
(1) $f \in \mathcal{L} - \mathcal{P}_n(\mathbb{C})$.
(2) The polynomial

$$\sum_{\alpha\leq\beta} a(\alpha)(\beta)_\alpha z^\alpha$$

is stable or identically zero for each $\beta \in \mathbb{N}^n$.
(3) There is a sequence $\{\beta(k)\}_{k=1}^\infty$, where $\beta(k) = (\beta_1(k), \dots, \beta_n(k)) \in \mathbb{N}^n$ and $\lim_{k\to\infty} \min_{1\leq j\leq n} \beta_j(k) = \infty$, such that

$$\sum_{\alpha\leq\beta(k)} a(\alpha)(\beta(k))_\alpha z^\alpha$$

is stable or identically zero for each $k \in \mathbb{N}$.

Proof. We prove (1) $\implies$ (2) $\implies$ (3) $\implies$ (1). If $f \in \mathcal{L} - \mathcal{P}_n(\mathbb{C})$, then let $P_k(z) = \sum_\alpha a^{(k)}(\alpha)z^\alpha$ be a sequence of stable polynomials with limit f. Then $Q_k(z) = \sum_{\alpha\leq\beta} a^{(k)}(\alpha)(\beta)_\alpha z^\alpha$ is stable or identically zero for all k being a scaling of $P_k(z)$. Since $a^{(k)}(\alpha \to a(\alpha))$ for all α it follows that $\sum_{\alpha\leq\beta} a(\alpha)(\beta)_\alpha z^\alpha$ is the limit, uniformly on compact sets, of $Q_k(z)$, so it is stable or identically zero by Hurwitz theorem. Clearly (2) $\implies$ (3). Now suppose $\sum_{\alpha\leq\beta(k)} a(\alpha)(\beta(k))_\alpha z^\alpha$ is stable or identically zero for all k and let

$$P_k(z) = \sum_{\alpha\leq\beta(k)} a(\alpha)J(\alpha, \beta(k))z^\alpha.$$

Since $\beta \to J(\alpha,\beta)$ is increasing with $\lim_{k\to\infty} J(\alpha, \beta(k)) = 1$ for all α it follows by Proposition 6.14 that there exists constants A, B, M such that

$$\max\{|P_k(z)| : |z_j| \leq r \text{ for all } j\} \leq Ar^M e^{Br^2}$$

for all k and $r > 0$. Hence the sequence $\{P_k(z)\}$ is uniformly bounded on compact sets and so $\{P_k(z)\}$ is a normal family whose convergent subsequences converge to $f(z)$ by Montel's theorem. The proposition now follows from Vitali's theorem [why is this needed?]. □

Proof of Theorem 6.5.
Suppose T preserves stability. If the range of T is at most one-dimensional, then T has the form (1). Suppose therefore T has range of dimension greater than one.. For $\beta \in \mathbb{N}^m$ let Λ_β be the linear operator that sends z^α to $(\beta)_\alpha z^\alpha$. In view of Proposition 6.15 it remains to prove that $\Lambda_{\gamma\oplus\beta}(G_T)$ is stable for γ and β large enough. By Theorem 5.3,

$$\sum_{\alpha\leq\beta} \binom{\beta}{\alpha} T(z^\alpha)w^{\beta-\alpha}$$

is stable for all β large enough. By inversion we thus have that

$$\sum_{\alpha\leq\beta} T(z^\alpha)(-1)^\alpha(\beta)_\alpha \frac{w^\alpha}{\alpha!}.$$

is stable. Then by Lemma 6.8

$$\Lambda_{\gamma\oplus\beta}(G_T)=\sum_{\alpha\leq\beta}\Lambda_\gamma(T(z^\alpha))(-1)^\alpha(\beta)_\alpha\frac{w^\alpha}{\alpha!}$$

is stable or identically zero for β large enough. The converse is immediate from Theorem 5.3 and Proposition 6.15.

Exercise 4: Prove Theorem 5.2 from Theorem 5.4 and Theorem 6.6.

Proof. (**1**) $\implies$ (**2**). Suppose $\Lambda=\{\lambda_k\}_{k=0}^\infty$ is a multiplier sequence and $T_\Lambda((1+z)^n)\not\equiv 0$. Since $(1+z)^n$ has all real roots (-1 with multiplicity n), then so does

$$T_\Lambda((1+z)^n)=\prod_{j=1}^n(z-\xi_j)$$

by assumption where $\xi_j\in\mathbb{R}$ for all $1\leq j\leq n$. Recall that linear transformations preserving real-rootedness for real univariate polynomials is equivalent to being stable. Thus by Theorem 5.4 we have that either the range of T_Λ is at most two-dimensional, or $G_{T_\Lambda}(z,w)=T_\Lambda((z+w)^n)$ is stable, or $G_{T_\Lambda}(z,-w)=T_\Lambda((z-w)^n)$ is stable. If $\dim(Im\,T_\Lambda)=0,1$ then $T_\Lambda((1+z)^n)$ has no non-zero root. Suppose therefore $\dim(Im\,T_\Lambda)=2$. Then

$$T_\Lambda((1+z)^n)=\sum_{j=0}^n\binom{n}{j}\lambda_jz^j=\binom{n}{k}\lambda_kz^k+\binom{n}{l}\lambda_lz^l$$

for some $0\leq k<l\leq n$ with $\lambda_k,\lambda_l\neq 0$. Note that if $l-k>2$ then

$$\binom{n}{k}\lambda_kz^k+\binom{n}{l}\lambda_lz^l=z^k\left(\binom{n}{k}\lambda_k+\binom{n}{l}\lambda_lz^{l-k}\right)$$

has imaginary roots, contrary to assumption. If $l-k=2$ then we have imaginary roots above, provided that λ_k and λ_l have the same sign. If they have different sign then consider the real rooted polynomial $z^k-z^l=z^k(1-z^2)$. By assumption on Λ being a multiplier sequence it follows that $z^k(\lambda_k-\lambda_lz^2)$ has all real roots which is a contradiction since the non-zero roots are given by $z=\pm\sqrt{\frac{\lambda_k}{\lambda_l}}$ which is imaginary as λ_k and λ_l have opposite sign. Hence $l-k=1$ and so there is exactly one non-zero real root which thus satisfies our requirements. We may therefore assume $G_{T_\Lambda}(z,w)=T_\Lambda((z+w)^n)$ is stable or $G_{T_\Lambda}(z,-w)=T_\Lambda((z-w)^n)$ is stable. In the former case we have

$$T_\Lambda((z+w)^n)=\prod_{j=1}^n(z-\xi_jw)$$

stable so

$$Im(z),Im(w)>0\implies\xi_j\leq 0\text{ for all }1\leq j\leq n.$$

Similarly if $T_\Lambda((z-w)^n)$ is stable then

$$Im(z),Im(w)>0\implies\xi_j\geq 0\text{ for all }1\leq j\leq n.$$

Hence the roots of $T_\Lambda((1+z)^n)$ are real with non-zero roots of the same sign.

$(\mathbf{2}) \implies (\mathbf{3})$.
Similarly to above if

$$T_\Lambda((1+z)^n) = \prod_{j=1}^{n}(z-\xi_j)$$

with ξ_j real and of the same sign for $j = 1, \ldots, n$ then

$$G_{T_\Lambda}(z,w) = T_\Lambda((z+w)^n) = \prod_{j=1}^{n}(z-\xi_j w)$$

or

$$G_{T_\Lambda}(z,-w) = T_\Lambda((z-w)^n) = \prod_{j=1}^{n}(z+\xi_j w)$$

is stable which implies T_Λ is stable by Theorem 5.4. Then by Theorem 6.6 it follows that $T_\Lambda(e^{-zw}) \in \mathcal{L}-\mathcal{P}_1(\mathbb{R})$ or $T_\Lambda(e^{zw}) \in \mathcal{L}-\mathcal{P}_1(\mathbb{R})$. Thus we deduce that

$$\sum_{n=0}^{\infty} \frac{\lambda_n}{n!} z^n = T_\Lambda(e^z) \in \mathcal{L}-\mathcal{P}_1(\mathbb{R}).$$

Note further that on one hand

$$T_\Lambda((1+z)^n) = \sum_{j=0}^{n} \binom{n}{j} \lambda_j z^j$$

and on the other hand

$$T_\Lambda((1+z)^n) = \prod_{j=1}^{n}(z-\xi_j^{(n)}) = \sum_{j=0}^{n} \left((-1)^j \sum_{S\subseteq[n],|S|=j} \prod_{i\in S} \xi_i^{(n)} \right) z^{n-j}.$$

By assumption for fixed n, all non-zero roots $\xi_j^{(n)}$ have the same sign. If all non-zero roots $\xi_j^{(n)}$ are positive for some n then by comparing coefficients we see that the λ_j alternate in sign i.e $\lambda_j = (-1)^{n-j}|\lambda_j|$ for $j = 1, \ldots, n$. Thus we cannot have that the non-zero roots $\xi_j^{(n+1)}$ are all positive for then $\lambda_j = (-1)^{n+1-j}|\lambda_j| = -\lambda_j$ for $j = 1, \ldots, n+1$. If $\xi_j^{(n+1)}$ are all instead positive then the λ_j are all positive for $j = 1, \ldots, n+1$ contradicting that $\lambda_j = (-1)^{n-j}|\lambda_j|$ for $j = 1, \ldots, n$. Hence $\xi_j^{(n+1)}$ are always negative so that $\lambda_j \geq 0$ for all $j \in \mathbb{N}$. But then it follows that the non-zero roots of $\displaystyle\sum_{n=0}^{\infty} \frac{\lambda_n}{n!} z^n$ must be all negative since $\lambda_j \geq 0$ for all $j \in \mathbb{N}$. $\square$

7 Hyperbolic Polynomials

Definition 7.1. *(Hyperbolic polynomial)*
*A homogeneous polynomial $h(z) \in \mathbb{R}[z_1, \ldots, z_n]$ is **hyperbolic** with respect to a vector $e \in \mathbb{R}^n$ if $h(e) \neq 0$, and if for all $x \in \mathbb{R}^n$ the univariate polynomial $t \mapsto h(x + et)$ has only real zeros.*

Example 7.2. *(Examples of hyperbolic polynomials).*

(1) Let $h(z) = z_1 \cdots z_n$. Then $h(z)$ is hyperbolic with respect to any vector $e \in \mathbb{R}^n$ that has no coordinate equal to zero

$$h(x + et) = \prod_{j=1}^{n} (x_j + e_j t).$$

(2) Let $Z = (z_{ij})_{i,j=1}^n$ be a matrix of variables where we impose $z_{ij} = z_{ji}$. Then $\det(Z)$ is hyperbolic with respect to $I = \operatorname{diag}(1, \ldots, 1)$. Indeed $h : t \mapsto det(X + tI)$ is the characteristic polynomial of the symmetric matrix X. A symmetric matrix has only real eigenvalues and so $h(X + tI)$ has only real zeros.

(3) Let $h(z) = z_1^2 - z_2^2 - \cdots - z_n^2$. Then h is hyperbolic with respect to $(1, 0, \ldots, 0)^T$.

Remark 7.3.
If h is hyperbolic with respect to e and of degree d, then we may write

$$h(x + et) = h(e) \prod_{j=1}^{d} (t + \lambda_j(x))$$

where $\lambda_1(x) \leq \cdots \leq \lambda_d(x)$. By homogeneity it follows that

$$\lambda_j(sx) = s\lambda_j(x) \text{ and } \lambda_j(x + se) = \lambda_j(x) + s$$

for all $1 \leq j \leq n, x \in \mathbb{R}^n$ and $s \in \mathbb{C}$.

Definition 7.4. *(Hyperbolicity cone)*
*The **hyperbolicity cone** is the set*

$$\Lambda_{++} = \Lambda_{++}(e) = \{x \in \mathbb{R} : \lambda_1(x) > 0\}.$$

Remark 7.5.
Since $h(e + te) = h(e)(1 + t)^d$ we see that $e \in \Lambda_{++}$.

Example 7.6.
The hyperbolicity cones for the hyperbolic polynomials in Example 7.2 are given by:
(1) $\Lambda_{++}(e) = \{x \in \mathbb{R}^n : x_i e_i > 0, i = 1, \ldots, n\}$ since

$$h(x + et) = \prod_{j=1}^{n} (x_j + e_j t) = e_1 \cdots e_n \prod_{j=1}^{n} (e_j^{-1} x_j + t) = h(e) \prod_{j=1}^{n} (e_j^{-1} x_j + t)$$

with $e_j^{-1}x_j > 0 \iff e_jx_j > 0$.
(2) In this case the $\lambda_j(X)$ represent eigenvalues of the matrix X, and so $\lambda_1(X) > 0$ implies all eigenvalues are positive since $0 < \lambda_1(X) \leq \cdots \leq \lambda_n(X)$ which in turn implies that $\Lambda_{++}(I)$ is given by the set of all symmetric positive definite matrices.
(3) Here

$$\begin{aligned} h(x+et) = h((x_1+t, x_2, \ldots, x_n)) &= (x_1+t)^2 - (x_2^2 + \cdots + x_n^2) \\ &= \left((x_1+t) + \sqrt{x_2^2+\cdots+x_n^2}\right)\left((x_1+t) - \sqrt{x_2^2+\cdots+x_n^2}\right). \end{aligned}$$

Hence $\lambda_1(x) = x_1 - \sqrt{x_2^2+\cdots+x_n^2}$ and so $\Lambda_{++}(1,0,\ldots,0)$ is given by the Lorentz cone

$$\{x \in \mathbb{R}^n : x_1 > \sqrt{x_2^2+\cdots+x_n^2}\}.$$

Proposition 7.7. *The hyperbolicity cone is the connected component of*

$$\{x \in \mathbb{R}^n : h(x) \neq 0\}$$

which contains e.

Proof. Let C be the connected component that contains e. Since C is connected, it is also path connected as $C \subset \mathbb{R}^n$. Therefore let $x(s)$ be a continuous path in C from $x(0) = e$ to $x(1) = x$. Note that since e belongs to the hyperbolicity cone by Remark 7.5 it follows that $\lambda_1(x(0)) = \lambda_1(e) > 0$. If there exists $0 \leq t \leq 1$ such that $\lambda_1(x(t)) < 0$ then since $x(s)$ is a continuous function it follows by intermediate value theorem that there exists u with $0 \leq u \leq t \leq 1$ such that $\lambda_1(x(u)) = 0$. Then

$$h(x(u)) = h(x(u) + e.0) = h(e)\prod_{j=1}^{d}(0 + \lambda_j(x(u))) = 0.$$

This contradicts the fact that $x(u) \in C$ and so $\lambda(x(s)) > 0$ for all $0 \leq s \leq 1$. Conversely if $x \in \Lambda_{++}$ then by homogeneity

$$h(tx + (1-t)e) = h(e)\prod_{j=1}^{d}(\lambda_j(tx) + (1-t)) = h(e)\prod_{j=1}^{d}(t\lambda_j(x) + (1-t)).$$

Now $x \in \Lambda_{++} \implies 0 < \lambda_1(x) \leq \lambda_2(x) \leq \cdots \lambda_d(x)$. Hence $h(tx + (1-t)e)$ is never identically zero for $0 \leq t \leq 1$ so $tx + (1-t)e \in C$ for all $0. \leq t \leq 1$ □

Lemma 7.8. *Let $h(z)$ be a homogeneous real polynomial of degree d, and suppose that $a, b \in \mathbb{R}^n$ are such that $h(a)h(b) \neq 0$. Then the following are equivalent:*
(i) h is hyperbolic with respect to a, and $b \in \Lambda_{++}(a)$.
(ii) For all $x \in \mathbb{R}^n$, the polynomial

$$(s,t) \mapsto h(x + sa + tb)$$

is stable.

Proof. Suppose (ii) holds and let $x \in \mathbb{R}^n$. By specialization $t = 0$ we see that the univariate polynomial $s \mapsto h(x + sa)$ is stable. This is equivalent to having only real zeros. Hence h is hyperbolic with respect to a since $h(a)$ is also non-zero by assumption. This proves the first assertion in (ii). By taking $x = 0$ in $(s,t) \mapsto h(x + sa + tb)$ we get that $p(s,t) = h(sa + tb)$ is stable since the former map is stable for all $x \in \mathbb{R}^n$. $p(s,t)$ is moreover homogeneous of degree d by homogeneity of h. By Lemma 6.11 the Taylor coefficients of $q(s) = p(s,1) = h(b + sa)$ must have the same phase being a homogeneous stable polynomial. Since the coefficients of $q(s)$ are real this means they must have the same sign (i.e $\theta = 0$ or π). But then the roots of $q(s)$ cannot be positive. Moreover $q(0) = h(b) \neq 0$. Hence the roots of $q(s)$ must be negative. Hence $b \in \Lambda_{++}(a)$ which proves the second assertion in (ii). Conversely assume (i) holds. Fix $x_0 \in H$ and $x \in \mathbb{R}^n$ and consider the zero set, $Z(x)$, of $t \mapsto h(x + s_0 a + tb)$. We need to prove $Z(x) \subset -\overline{H} = \{z \in \mathbb{C} : Im(z) \leq 0\}$ for all $x \in \mathbb{R}^n$. Consider $Z(0)$. Since $b \in \Lambda_{++}(a)$ and h hyperbolic w.r.t a it follows that all the zeros of $h(b + sa) = h(a)\prod_{j=1}^{d}(s + \lambda_j(b))$ are real and negative since $\lambda_1(b) > 0$ and $0 \leq \lambda_1(b) \leq \lambda_2(b) \leq \cdots \leq \lambda_d(b)$. Hence if $h(s_0 a + tb) = t^d h(b + s_0 t^{-1} a) = 0$ then $s_0/t < 0$. Thus $Z(0) \subseteq -\overline{H}$. Suppose for a contradiction that there exists $x \in \mathbb{R}^n$ such that $Z(x) \nsubseteq -\overline{H}$. By moving from 0 to x along the line segment $\{\theta x : 0 \leq \theta \leq 1\}$ we see that for some $0 \leq \theta \leq 1$ we have $Z(\theta x) \cap \mathbb{R} \neq \emptyset$ (by Hurwitz theorem). Hence there is a number $\alpha \in \mathbb{R}$ such that $h(\theta x + \alpha b + s_0 a) = 0$. By assumption $s_0 \notin \mathbb{R}$ and moreover $\theta x + \alpha b \in \mathbb{R}^n$ so h has a non-real zero contradicting its hyperbolicity with respect to a. Hence $Z(x) \subseteq -\overline{H}$ for all $x \in \mathbb{R}^n$ which implies $Z(x) \subseteq \mathbb{R}$ for all $x \in \mathbb{R}^n$ since h has real coefficients. □

Theorem 7.9. *Suppose that h is hyperbolic with respect to e.*
(i) If $a \in \Lambda_{++}(e)$, then h is hyperbolic with respect to a and $\Lambda_{++}(a) = \Lambda_{++}(e)$.
(ii) $\Lambda_{++}(e)$ is a convex cone.

Proof. If $a \in \Lambda_{++}(e)$ then it follows by Lemma 7.8 that $(s,t) \mapsto h(x + se + ta)$ is stable. Thus switching the roles of e and a it follows by Lemma 7.8 again that h is hyperbolic with respect to a and $e \in \Lambda_{++}(a)$. By Proposition 7.7 the hyperbolicity cones $\Lambda_{++}(a)$ and $\Lambda_{++}(e)$ are the connected components of $\{x \in \mathbb{R}^n : h(x) \neq 0\}$ containing a and e respectively. Since $a \in \Lambda_{++}(e)$ it follows that a and e belong to the same connected component of $\{x \in \mathbb{R}^n : h(x) \neq 0\}$. Thus any $x \in \Lambda_{++}(e)$ can be connected to a via a path through e and so it follows that $x \in \Lambda_{++}(a)$. Therefore $\Lambda_{++}(e) \subseteq \Lambda_{++}(a)$. Similarly since $e \in \Lambda_{++}(a)$ we have that $\Lambda_{++}(a) \subseteq \Lambda_{++}(e)$. This proves (i). For (ii) let $a, b \in \Lambda_{++}(e)$. Then since $\Lambda_{++}(a) = \Lambda_{++}(e)$ it follows by the argument in Proposition 7.7 that $ta + (1-t)b \in \Lambda_{++}(e)$, for all $0 \leq t \leq 1$. This proves convexity. That $\Lambda_{++}(e)$ is a cone follows from the fact that $\lambda_1(kx) = k\lambda_1(x)$ for all $x \in \Lambda_{++}(e)$ so $\Lambda_{++}(e)$ is closed under multiplication by positive scalars. □

Corollary 7.10. *Suppose h is hyperbolic of degree d with respect to e. If $a \in \Lambda_{++}(e)$ then for any $x \in \mathbb{R}^d$ the polynomial $g(t) = h(a + tx)$ only has real roots.*

Proof. Indeed since $a \in \Lambda_{++}(e)$ we have by Theorem 7.9 (i) that h is hyperbolic with respect to a. Therefore for every $x \in \mathbb{R}^d$ the polynomial $t \mapsto h(x + ta)$ has only real roots. By homogeneity $h(x+ta) = t^d h(t^{-1}x+a)$ and so $t \mapsto h(t^{-1}x+a)$ has only real roots so $g(t) := h(a + tx)$ has only real roots. □

Theorem 7.11. *Let $\lambda_1(x) : \mathbb{R}^n \to \mathbb{R}$ be given as in 7.3. Then $\lambda_1(x)$ is concave.*

Proof. Let $x, y \in \mathbb{R}$. Note that $x - \lambda_1(x)e, y - \lambda_1(y)e \in \overline{\Lambda_{++}(e)}$ since by Remark 7.3 we have $\lambda_1(x - \lambda_1(x)e) = \lambda_1(x) - \lambda_1(x) = 0 \geq 0$ (and likewise for y). By Theorem 7.9 we have that $\Lambda_{++}(e)$ is convex. Thus $\frac{1}{2}(x - \lambda_1(x)e) + (1 - \frac{1}{2})(y - \lambda_1(y)e) \in \overline{\Lambda_{++}(e)}$ and so $(x+y) - (\lambda_1(x) + \lambda_1(y))e \in \overline{\Lambda_{++}(e)}$ since $\overline{\Lambda_{++}(e)}$ is closed under multiplication by positive scalars. But the smallest t such that $(x+y) + te \in \overline{\Lambda_{++}(e)}$ is given by $-\lambda_1(x+y)$ (again using Remark 7.3). Thus by minimality $-\lambda_1(x+y) \leq -\lambda_1(x) + \lambda_1(y)$ so that $\lambda_1(x) + \lambda_1(y) \leq \lambda_1(x+y)$. Therefore by the established inequality and Remark 7.3 we have

$$\lambda_1((1-t)x + ty) \geq \lambda_1((1-t)x) + \lambda_1(ty) = (1-t)\lambda_1(x) + t\lambda_1(y)$$

for all $0 \leq t \leq 1$. Hence $\lambda_1(x)$ is concave. □

Theorem 7.12. *Suppose that h is hyperbolic of degree d with respect to e. Then $h^{1/d}$ is concave on $\overline{\Lambda_{++}(e)}$.*

Proof. Let $a, b \in \overline{\Lambda_{++}(e)}$ and consider $f(t) := h^{1/d}(ta + (1-t)b) = h^{1/d}(b + t(a-b))$. It suffices to prove that f is concave on $[0,1]$, that is, $f''(t) \leq 0$. Since $b \in \overline{\Lambda_{++}(e)}$ the zeros of $g(t) := h(b + t(a-b))$ are real by Corollary 7.10. Since

$$f'' = -\frac{1}{d}\frac{1}{d-1}g^{1/d-2}\left(g'^2 - \frac{d}{d-1}gg''\right).$$

The theorem now follows from Newton's inequalities [How?]. □

Proposition 7.13. *Let $P(z_1, \dots, z_n) = \sum_{\alpha \in \mathbb{N}^n} a(\alpha) z^\alpha$ be a real polynomial of degree d, and let $h(z_1, \dots, z_{n+1}) = z_{n+1}^d P(z_1/z_{n+1}, \dots, z_n/z_{n+1})$. The following are equivalent*
(1) P stable;
(2) h is hyperbolic with respect to $e = (1, \dots, 1, 0)^T$ and its hyperbolicity cone contains $\mathbb{R}_+^n \times \{0\}$.

Proof. Suppose P is stable, and let $P_d(z) := \lim_{\lambda \to \infty} \lambda^d P(\lambda z_1, \dots, \lambda z_n) = \sum_{|\alpha|=d} a(\alpha) z^\alpha$. Then by Remark 6.12 the Taylor coefficients of P_d have the same phase so $P_d(1, \dots, 1) \neq 0$. Let $x \in \mathbb{R}^{n+1}$. We must prove all zeros of $h(x + et)$ are real. Suppose first that $x_{n+1} \neq 0$. If $t = a + ib$ where $b \neq 0$ then $x_j/x_{n+1} + t/x_{n+1} \in H$ or $x_j/x_{n+1} + t/x_{n+1} \in -H$ for all j. Thus since P is real stable by assumption we have that

$$h(x + et) = x_{n+1}^d P(x_1/x_{n+1} + t/x_{n+1}, \dots, x_n/x_{n+1} + t/x_{n+1}) \neq 0.$$

If it was zero then by conjugating if necessary there is a root in H^n contradicting stability of P. If on the other hand $x_{n+1} = 0$ then $h(x + et) = \lim_{\lambda \to 0} \lambda^{-M} P(\lambda z_1, \dots, \lambda z_n) = \sum_{|\alpha|=M} a(\alpha) z^\alpha$ where $M = \min\{|\alpha| : a(\alpha) \neq 0\}$. This polynomial only has real zeros by the same argument as above. Hence h is hyperbolic with respect to e. If $x \in \mathbb{R}_+^n \times \{0\}$, then $h(x) = P_d(x) \neq 0$ since by Remark 6.12 all coefficients have the same phase. By Proposition 7.7 the hyperbolicity cone $\Lambda_{++}(e)$ is given by the connected component of $\{x \in \mathbb{R}^n : h(x) \neq 0\}$ which contains e. The continuous path $\gamma(s) = se + (1-s)x$, $0 \leq s \leq 1$ between e and x is contained in $\mathbb{R}_+^n \times \{0\}$ so $h(\gamma(s)) \neq 0$ for all $0 \leq s \leq 1$. Thus e and x belong to the same connected component and hence $x \in \Lambda_{++}(e)$ by Proposition 7.7. Hence $\mathbb{R}_+^n \times \{0\} \subseteq \Lambda_{++}(e)$. Now assume (2) and let $x + iy \in H^n$. Let further

$x' = (x_1, \dots, x_n, 1)$ and $y = (y_1, \dots, y_n, 0)$. Then $P(x + iy) = h(x' + iy') \neq 0$ by Theorem 7.9 (i) since $y \in \Lambda_{++}(e)$. $\square$

Corollary 7.14. *Let $P(z_1, \dots, z_n)$ be a stable polynomial of degree d that has only nonnegative Taylor coefficients. Then the polynomial*

$$h = z_{n+1}^d P(z_1/z_{n+1}, \dots, z_1/z_{n+1})$$

is stable.

Proof. By Proposition 7.13 we have that h is hyperbolic with hyperbolicity cone containing $\mathbb{R}_+^n \times \{0\}$. By Proposition 7.7 the hyperbolicity cone is given by the connected component of $\{x \in \mathbb{R}^{n+1} : h(x) \neq 0\}$ containing e. Since all Taylor coefficients are nonnegative we have that h is positive on $\mathbb{R}_+^{n+1}$ and so we can find a continuous path between any $x \in \mathbb{R}_+^{n+1}$ and e contained in $\mathbb{R}_+^{n+1}$ on which h is non-zero (positive). Hence the hyperbolicity cone contains $\mathbb{R}_+^{n+1}$. Given $z = (z_1, \dots, z_{n+1}) \in H^{n+1}$, write $z_j = a_j + ib_j$ where $b_j \in \mathbb{R}_+$. Then by Lemma 7.8 the polynomial $(s,t) \mapsto h(x + sa + tb)$ is stable for all $x \in \mathbb{R}^{n+1}$, so in particular $h(z) = h((a_1, \dots, a_{n+1}) + i(b_1, \dots, b_{n+1})) \neq 0$. Hence h is stable. $\square$

Proposition 7.15. *Let $h \in \mathbb{R}[z_1, \dots, z_n]$ be hyperbolic and let $a_1, \dots, a_m \in \overline{\Lambda_{++}(e)}$ and $a_0 \in \mathbb{R}^n$. Then the polynomial*

$$P(z_1, \dots, z_m) = h(a_0 + z_1 a_1 + \dots + a_m z_m)$$

is stable or identically zero.

Proof. By Hurwitz theorem we may assume that $a_1, \dots, a_m \in \Lambda_{++}(e)$. let $z = x + iy \in H^m$ where $x \in \mathbb{R}^m, y \in \mathbb{R}_+^m$. Then

$$P(z) = h\left(a_0 + \sum_{j=1}^m x_j a_j + i \sum_{j=1}^m y_j a_j\right).$$

Note that $a := \sum_{j=1}^m y_j a_j \in \Lambda_{++}(e)$ by definition since $\Lambda_{++}(e)$ is a convex cone by Theorem 7.9 (ii). By Theorem 7.9 (i) it therefore follows that h is hyperbolic with respect to a. Hence by Lemma 7.8 we have that the polynomial

$$(s,t) \mapsto h\left(\left(a_0 + \sum_{j=1}^m x_j a_j\right) + se + ta\right)$$

is stable so letting $s \to 0$ we get by Hurwitz theorem that

$$t \mapsto h\left(\left(a_0 + \sum_{j=1}^m x_j a_j\right) + ta\right)$$

is stable. Hence

$$h(z) = h\left(\left(a_0 + \sum_{j=1}^m x_j a_j\right) + ia\right) \neq 0$$

proving h is stable since z is arbitrary in H^m. $\square$

Definition 7.16. (Lineality space)
The **lineality space**, L, of a convex cone C is the largest linear subspace contained in C, that is, $L = C \cap (-C)$. We denote by $L(h)$ the lineality space of the closure of hyperbolicity cone of h.

Proposition 7.17. *Let h be a hyperbolic polynomial of degree d with respect to e. Then*

$$L(h) = \{y \in \mathbb{R}^n : h(y + x) = h(x) \text{ for all } x \in \mathbb{R}^n\}.$$

Proof. If $y \in L(h)$, then $y \in \overline{\Lambda_{++}(e)} \cap -\overline{\Lambda_{++}(e)}$. Recall that

$$h(y + et) = \prod_{j=1}^{d} (t + \lambda_j(y))$$

with $0 \leq \lambda_1(y) \leq \lambda_2(y) \leq \cdots \leq \lambda_d(y)$ since h is hyperbolic w.r.t e. Since $y \in -\overline{\Lambda_{++}(e)}$ we also have that $0 \leq \lambda_1(-y) \leq \lambda_2(-y) \leq \cdots \leq \lambda_d(-y)$ i.e that $0 \leq -\lambda_1(y) \leq -\lambda_2(y) \leq \cdots \leq -\lambda_d(y)$ (by homogeneity) and so $0 = \lambda_1(y) = \cdots = \lambda_d(y)$. Hence $h(y + et) = h(e)t^d$. Therefore $h(y + xt) = h(x)t^d$ for all $x \in \Lambda_{++}(e)$ since $x \in \Lambda_{++}(e)$ implies h is hyperbolic w.r.t x by Theorem 7.9 (ii). Thus $h(y + x) = h(x)$ for all $x \in \Lambda_{++}(e)$. Therefore the polynomial $p(x) = h(y + x) - h(x)$ is identically zero on $\Lambda_{++}(e)$ which is an open set, so $p(x) \equiv 0$ on whole of $\mathbb{R}^n$ being an entire function. Hence $h(y+x) = h(x)$ for all $x \in \mathbb{R}^n$. Conversely let $y \in \mathbb{R}^n$ such that $h(y + x) = h(x)$ for all $x \in \mathbb{R}^n$. Then

$$\begin{aligned} h((1-t)y + te) &= h(y + t(e - y)) = h(t(e - y)) = t^d h(e - y) \\ &= t^d(-1)^d h((-e) + y) = t^d(-1)^d h(-e) = t^d h(e) \neq 0. \end{aligned}$$

for all $0 < t \leq 1$. Thus by Proposition 7.7 it follows that $(1-t)y + te \in \Lambda_{++}(e)$ for all $0 < t \leq 1$. Thus $y \in \overline{\Lambda_{++}(e)}$. Similarly $-y \in \overline{\Lambda_{++}(e)}$ and hence the proposition follows. □

Suppose $A_1, \ldots, A_n$ are symmetric $d \times d$ matrices and $e = (e_1, \ldots, e_n)^T \in \mathbb{R}^n$. Suppose further that $\sum_{i=1}^n e_i A_i = I$, where I is the identity matrix. Then

$$\det(e_1 A_1 + \cdots + e_n A_n) = \det(I) \neq 0$$

and for every $x \in \mathbb{R}^n$ we have that the polynomial

$$t \mapsto \det(x_1 e_1 t A_1 + \cdots + x_n e_n t A_n) = t^n \det(x_1 e_1 A_1 + \cdots + x_n e_n A_n)$$

has only real roots. Hence the polynomial $h(x) = \det(x_1 A_1 + \cdots + x_n A_n)$ is hyperbolic with respect to e. Its hyperbolicity cone is given by those $x \in \mathbb{R}^n$ such that $x_1 A_1 + \cdots + x_n A_n$ has only positive eigenvalues, that is, by

$$\Lambda_{++}(e) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i A_i \text{ is positive definite} \right\}.$$

Hence Λ_{++} is an intersection of the cone of positive definite matrices with a hyperplane. The Generalized Lax Conjecture asks if this is always the case?

Conjecture 7.18. *(Generalized Lax conjecture)*
Suppose that $\Lambda_{++} \subseteq \mathbb{R}^n$ is a hyperbolicity cone. Are there symmetric $d \times d$ matrices $A_1, \ldots, A_n$ such that

$$\Lambda_{++} = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i A_i \textit{ is positive definite} \right\}?$$

Remark 7.19. *The conjecture has been shown to be true for $n = 3$.*

8 The Lee-Yang Theorem

Lemma 8.1. *(Newton's inequalities)*
Let $P(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial with only real zeros and of degree at most n. Then

$$\frac{a_k^2}{\binom{n}{k}^2} \geq \frac{a_{k-1}}{\binom{n}{k-1}} \frac{a_{k+1}}{\binom{n}{k+1}} \quad \textit{for all } 1 \leq k \leq n-1.$$

Proof. Let $P(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$ be a real-rooted polynomial. We want to prove that $b_k^2 \geq b_{k-1} b_{k+1}$ for all $1 \leq k \leq n-1$ since $a_k = \binom{n}{k} b_k$. If $n < 2$ then there is nothing to prove and if $n = 2$ then the inequality amounts to

$$\frac{a_1^2}{\binom{2}{1}^2} \geq \frac{a_0}{\binom{2}{0}} \frac{a_2}{\binom{2}{2}}$$

which is equivalent to the statement that the discriminant of the polynomial $a_2 z^2 + a_1 z + a_0$ is positive which we know holds since the the polynomial is assumed to be real-rooted. Therefore assume $n > 2$. Note that

$$\begin{aligned}
\frac{1}{n} P'(z) &= \frac{1}{n} \sum_{k=1}^{n} k \binom{n}{k} b_k z^{k-1} \\
&= \frac{1}{n} \sum_{k=0}^{n-1} (k+1) \binom{n}{k+1} b_{k+1} z^k \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} b_{k+1} z^k
\end{aligned}$$

and that the roots of $P'(z)$ interlace those of $P(z)$ by Rolle's theorem so $P(z)$ is real-rooted if and only if $\frac{1}{n} P'(z)$ is real-rooted. Note moreover that $\sum_{k=0}^{n} \binom{n}{k} b_k z^k$ is real-rooted if and only if $\sum_{k=0}^{n} \binom{n}{k} b_k z^{n-k}$ is real-rooted, for if r is a non-zero real root of $\sum_{k=0}^{n} \binom{n}{k} b_k z^k$ then $\sum_{k=0}^{n} \binom{n}{k} b_k \left(\frac{1}{r}\right)^{n-k} = \left(\frac{1}{r}\right)^n \sum_{k=0}^{n} \binom{n}{k} b_k r^k = 0$ and vice versa. Given $s \in \{1, \ldots, n-1\}$ we can now differentiate as above repeatedly $s-1$ times to get the real rooted polynomial $\sum_{k=0}^{n-s+1} \binom{n-s-1}{k} b_{k+s-1} z^k$ and then use our second observation to conclude that $\sum_{k=0}^{n-s+1} \binom{n-s-1}{k} b_{k+s-1} z^{(n-s+1)-k}$ is real-rooted. We finally differentiate again $n-s-1$ times to get that

$$\binom{2}{0} b_{s-1} + \binom{2}{1} b_s z + \binom{2}{2} b_{s+1} z^2$$

is real-rooted. Hence the lemma follows from the $n = 2$ case. □

Theorem 8.2. *Let $P \in \mathbb{R}_{\mathbf{1}}[z_1, \ldots, z_n] \setminus \{0\}$. Then P is stable if and only if*

$$\frac{\partial^2 P}{\partial z_i \partial z_j}(x)P(x) \leq \frac{\partial P}{\partial z_i}(x)\frac{\partial P}{\partial z_j}(x), \tag{3}$$

for all $x \in \mathbb{R}^n$ and all $1 \leq i < j \leq n$.

Proof. Suppose P is stable. Write P as $P = Q + z_j R \in \mathbb{R}[z_1, \ldots, z_n]$ where Q and R do not depend on z_j. If $R \equiv 0$ then (3) clearly holds for all i since this implies P does not depend on z_j. Therefore let $x \in \mathbb{R}^{n-1}$ such that $R(x) \neq 0$ and let $q(t) := Q(x + e_i t)/R(x + e_i t)$ where e_i is the i^{th} standard basis vector. Note that

$$\begin{aligned} q'(0) &= \frac{\frac{\partial Q}{\partial z_i}(x)R(x) - Q(x)\frac{\partial R}{\partial z_i}(x)}{R(x)^2} \\ &= \frac{\left(\frac{\partial Q}{\partial z_i}(x) + z_j\frac{\partial R}{\partial z_i}(x)\right)R(x) - (Q(x) + z_j R(x))\frac{\partial R}{\partial z_i}(x)}{R(x)^2} \\ &= \frac{\frac{\partial P}{\partial z_i}(x)\frac{\partial P}{\partial z_j}(x) - \frac{\partial^2 P}{\partial z_i \partial z_j}(x)P(x)}{R(x)^2}. \end{aligned}$$

Taylor expansion therefore gives

$$q(t) = \frac{Q(x)}{R(x)} + \frac{\frac{\partial P}{\partial z_i}(x)\frac{\partial P}{\partial z_j}(x) - \frac{\partial^2 P}{\partial z_i \partial z_j}(x)P(x)}{R(x)^2}t + O(t^2). \tag{4}$$

By Lemma 1.10 it follows that $Im(Q(z)/R(z)) \geq 0$ whenever $z \in \overline{H}^n$. Hence $Im(q(i\lambda)) = Im\left(Q(x + e_i i\lambda)/R(x + e_i i\lambda)\right) \geq 0$ for all $\lambda > 0$. Thus evaluating imaginary parts in (4) we have

$$\frac{\partial P}{\partial z_i}(x)\frac{\partial P}{\partial z_j}(x) - \frac{\partial^2 P}{\partial z_i \partial z_j}(x)P(x) \geq R(x)^2 - O(\lambda)$$

Hence for $\lambda > 0$ sufficiently small the inequality in (3) follows for all $x \in \mathbb{R}^{n-1}$ such that $R(x) \neq 0$. However because the non-roots of R are dense in $\mathbb{R}^{n-1}$ the inequality (3) follows for all $x \in \mathbb{R}^{n-1}$. For the converse we argue by induction on n. The case $n = 1$ is trivial since one variable affine polynomials always have a single real root so P is certainly stable. Suppose $P(z_1, \ldots, z_{n+1}) = Q + z_{n+1}R$ satisfies (3). If $Q \equiv 0$ or $R \equiv 0$ we are done by induction, so assume this is not the case. Clearly the specialization $P_\alpha(z_1, \ldots, z_n) = P(z_1, \ldots, z_n, \alpha)$ still satisfies the inequalities in (3) for all $\alpha \in \mathbb{R}$. Hence by inductive hypothesis $P_\alpha = Q + \alpha R$, Q and R are all stable or identically zero for all $\alpha \in \mathbb{R}$. If $Q + \alpha R \equiv 0$ for some α then $P = Q + z_{n+1}R = -\alpha R + z_{n+1}R = (z_{n+1} - \alpha)R$ and we are done since R is stable. Therefore we may assume this is not the case. Then P is $H^n \times \mathbb{R}$-stable which by Lemma 4.3 with $\Omega = H^n, C_1 = \overline{H}, C_2 = -\overline{H}$ it follows that P is stable or $H^n \times -H$-stable, that is P is stable or $P(z_1, \ldots, z_n, -z_{n+1})$ is stable. If the latter occurs then on one hand we have the inequality in (3) by assumption and on the other hand $P(z_1,\dots,z_n,-z_{n+1})$ stable implies

$$\frac{\partial^2 P(z_1,\dots,z_n,-z_{n+1})}{\partial z_i \partial z_{n+1}}(x)P(x) \le \frac{\partial P(z_1,\dots,z_n,-z_{n+1})}{\partial z_i}(x)\frac{\partial P(z_1,\dots,z_n,-z_{n+1})}{\partial z_{n+1}}(x) \implies$$
$$-\frac{\partial^2 P(z_1,\dots,z_n,z_{n+1})}{\partial z_i \partial z_{n+1}}(x)P(x) \le -\frac{\partial P(z_1,\dots,z_n,z_{n+1})}{\partial z_i}(x)\frac{\partial P(z_1,\dots,z_n,z_{n+1})}{\partial z_{n+1}}(x) \implies$$
$$\frac{\partial^2 P}{\partial z_i \partial z_{n+1}}(x)P(x) \ge \frac{\partial P}{\partial z_i}(x)\frac{\partial P}{\partial z_{n+1}}(x).$$

Hence we have equality in (3) for $j = n+1$, $1 \le i \le n$ and for all $x \in \mathbb{R}^n$. Thus

$$\frac{\partial P}{\partial z_i}(x)\frac{\partial P}{\partial z_{n+1}}(x) = \frac{\partial^2 P}{\partial z_i \partial z_{n+1}}(x)P(x) \implies \left(\frac{\partial Q}{\partial z_i}(x) + z_{n+1}\frac{\partial R}{\partial z_i}(x)\right)R(x) = \frac{\partial R}{\partial z_i}(x)(Q(x)+z_{n+1}R(x))$$
$$\implies \frac{\partial Q}{\partial z_i}(x)R(x) = \frac{\partial R}{\partial z_i}(x)Q(x) \implies \frac{\partial Q}{\partial z_i}(x)\frac{1}{Q(x)} = \frac{\partial R}{\partial z_i}(x)\frac{1}{R(x)}$$
$$\implies \left(\frac{\partial}{\partial z_i}\ln(Q)\right)(x) = \left(\frac{\partial}{\partial z_i}\ln(R)\right)(x) \implies \left(\frac{\partial}{\partial z_i}\ln\left(\frac{Q}{R}\right)\right)(x) = 0$$

for all $1 \le i \le n$ and $x \in \mathbb{R}^n$. Therefore

$$\ln\left(\frac{Q}{R}\right) = C$$

for some constant $C \in \mathbb{R}$ which implies

$$Q = e^C R$$

so that

$$P(z) = (e^C + z_{n+1})R.$$

Finally since R is stable so is P. □

Definition 8.3. *(Permanent)*
The ***permanent*** *of a square matrix $A = (a_{ij})_{i,j=1}^n$ is the unsigned determinant*

$$per(A) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n a_{i\sigma(i)}.$$

Definition 8.4. *(Doubly stochastic matrix)*
An $n \times n$ matrix is ***doubly stochastic*** *if all entries are nonnegative and each row and column sums to one.*

Remark 8.5. In 1926 Van der Waerden conjectured that if A is a doubly stochastic $n \times n$ matrix then

$$per(A) \ge \frac{n!}{n^n}.$$

with equality if and only if all entries of A are equal to $1/n$. In 1981 the inequality was proved by Falikman and the characterization of the equality proved by Egorychev. Recently Leonid Gurvits came up with a proof for a vast generalization of the Van der Waerden conjecture. His methods uses the theory of stable and hyperbolic polynomials.

Definition 8.6. *(Capacity)*
*The **capacity** of a polynomial $P \in \mathbb{R}[z_1, \ldots, z_n]$ is defined by*

$$Cap(P) = \inf_{x_1,\ldots,x_n>0} \frac{P(x_1, \ldots, x_n)}{x_1 \cdots x_n}$$

For convenience we will also define a function $G : \mathbb{N} \to \mathbb{Q}$ given by $G(0) = G(1) = 1$ and $G(k) = (1 - 1/k)^{k-1}$ for $k > 2$.

Lemma 8.7. *Let P be a stable univariate polynomial with only nonnegative coefficients and of degree $d \geq 0$. Then*

$$P'(0) \geq G(d)\, Cap(P),$$

with equality of an only if P has just one zero.

Proof. If P has just one zero then $P(z) = (z+a)^d$ for some $a \in \mathbb{R}_{\geq 0}, d \geq 1$ since P is univariate real stable making it real-rooted and has nonnegative coefficients making its roots all non-positive. We first seek to evaluate Cap(P). To this end

$$\frac{d}{dx}\left(\frac{(x+a)^d}{x}\right) = 0 \implies \frac{(x+a)^{d-1}((d-1)x-a)}{x^2} = 0 \implies x = -a, \frac{a}{d-1}.$$

We are only looking at $x \in (0, \infty)$ so we can discard $x = -a$. We check that the critical point $x = \frac{a}{d-1}$ indeed gives a minimum.

$$\frac{d^2}{dx^2}\left(\frac{(x+a)^d}{x}\right)\Bigg|_{x=\frac{a}{d-1}} =$$

$$\frac{x^2\left((d-1)(x+a)^{d-2}((d-1)z-a) + (x+a)^{d-1}(d-1)\right) + 2x\left((x+a)^{d-1}((d-1)x-a)\right)}{x^4}\Bigg|_{x=\frac{a}{d-1}} =$$

$$\frac{(d-1)\left(\frac{a}{d-1}\right)^2 (\frac{a}{d-1}+a)^{d-1}}{\left(\frac{a}{d-1}\right)^4} > 0.$$

Hence $x = \frac{a}{d-1}$ gives a global minimum for $\frac{(x+a)^d}{x}$ in $(0, \infty)$ since the expression tends to ∞ as $x \to 0$ and to ∞ as $x \to \infty$ so

$$\text{Cap}(P) = \inf_{x>0} \frac{(x+a)^d}{x} = \frac{\left(\frac{a}{d-1}+a\right)^d}{\frac{a}{d-1}} = da^{d-1}\left(\frac{d}{d-1}\right)^{d-1} = da^{d-1}G(d)^{-1}.$$

Hence

$$G(d)\text{Cap}(P) = da^{d-1} = P'(0).$$

Now consider the general case where we may assume by continuity that all of the non-positive roots of P are strictly negative. Then $P(z) = \prod_{j=1}^{d}(1+\theta_j z)$ for some $\theta_j > 0$, $j = 1, \ldots, d$. By AM-GM inequality we thus have

$$\text{Cap}(P) = \inf_{x>0} \frac{\prod_{j=1}^{d}(1+\theta_j x)}{x} \leq \inf_{x>0} \frac{\left(\sum_{j=1}^{d}(1+\theta_j x)\right)^d}{x} = \inf_{x>0} \frac{\left(1+\frac{\theta_1+\cdots+\theta_d}{d}x\right)^d}{x} = a^{-d} \inf_{x>0} \frac{(x+a)^d}{x}.$$

where $a = \dfrac{d}{\theta_1 + \cdots + \theta_d}$. The right hand side is minimized according to our previous calculation, giving

$$\mathrm{Cap}(P) \leq a^{-d} \inf_{x>0} \frac{(x+a)^d}{x} = a^{-d} d a^{d-1} G(d)^{-1} = \frac{\theta_1 + \cdots + \theta_d}{G(d)}.$$

Finally note that

$$P'(z) = \sum_{i=1}^{d} \theta_i \prod_{j=1, j\neq i}^{d} (1+\theta_j z)$$

so $P'(0) = \theta_1 + \cdots + \theta_d$. Hence

$$P'(0) \geq G(d)\mathrm{Cap}(P)$$

as required. Equality occurs whenever the AM-GM inequality gives equality. This precisely happens when $\theta_1 = \cdots = \theta_d$, that is, P has only one root. □

Lemma 8.8. *Let $P \in \mathbb{R}[z_1, \ldots, z_n]$ be a stable polynomial with nonnegative coefficients, and let $Q = \left.\frac{\partial P}{\partial z_j}\right|_{z_j=0} \in \mathbb{R}[z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n]$.*
Then

$$\mathit{Cap}(Q) \geq G(k)\,\mathit{Cap}(P),$$

where k is the degree in z_j of P.

Proof. Without loss of generality assume $j = n$ and let $x_1, \ldots, x_{n-1} > 0$. Then via specialization it follows that $p(z) = P(x_1, \ldots, x_{n-1}, z)$ is stable. Hence by Lemma 8.7 we have

$$\frac{Q(x_1, \ldots, x_{n-1})}{x_1 \cdots x_{n-1}} = \frac{p'(0)}{x_1 \cdots x_{n-1}} \geq \frac{1}{x_1 \cdots x_{n-1}} G(k)\mathrm{Cap}(p) = G(k) \inf_{x_n>0} \frac{P(x_1, \ldots, x_n)}{x_1 \cdots x_n} \geq G(k)\mathrm{Cap}(P)$$

and the lemma follows. □

Theorem 8.9. *Let $P \in \mathbb{R}[z_1, \ldots, z_n]$ be a stable polynomial with nonnegative coefficients, of total degree n, and of degree d_i in z_i for each $i \in [n]$. Let $e_i = \min\{i, d_i\}$. Then*

$$\mathit{Cap}(P) \geq \frac{\partial^n}{\partial z_1 \cdots \partial z_n} P(0) \geq \mathit{Cap}(P) \prod_{j=1}^{n} G(e_j) \geq \mathit{Cap}(P) \frac{n!}{n^n}.$$

Proof.
Define the sequence of polynomials $\{Q_j\}_{j=1}^n$ by $Q_n = P$ and $Q_{j-1} = \left.\frac{\partial Q_j}{\partial z_j}\right|_{z_j=0}$.
By Lemma 8.8 we have $\mathrm{Cap}(Q_{j-1}) \geq \mathrm{Cap}(Q_j) G(\deg_j(Q_j))$ for each $1 \leq j \leq n$.
Note that $\mathrm{Cap}(Q_0) = Q_0 = \partial^n / \partial z_1 \cdots \partial z_n P(0)$. Note also that $\frac{d}{dx}\left((x-1)\ln(1-1/x)\right) = \ln(1-1/x) - \frac{(x-1)}{x^2-x} = \ln(x-1) - \ln(x) - \frac{1}{x} < 0$ for all $x > 1$. Thus it follows that $\ln(G(k))$ is decreasing and so $G(k)$ is decreasing. Moreover $\deg_j(Q_j) \leq e_j \leq j$ and hence

$$\prod_{j=1}^{n} G(\deg_j(Q_j)) \geq \prod_{j=1}^{n} G(e_j) \geq \prod_{j=1}^{n} G(j) = \prod_{j=2}^{n} \left(\frac{j-1}{j}\right)^{j-1} = \frac{(n-1)!}{n^{n-1}} = \frac{n!}{n^n}.$$

□

Definition 8.10. *($\mathcal{H}_n^+$)*
Let $\mathcal{H}_n^+$ denote the set of all homogeneous polynomials of degree n in $\mathbb{R}[z_1, \dots, z_n]$ that have nonnegative coefficients.

Definition 8.11. *(Doubly stochastic polynomial)*
*A polynomial $P \in \mathcal{H}_n^+$ is **doubly stochastic** if*

$$\frac{\partial P}{\partial z_j}(\mathbf{1}) = 1, \textit{ for all } 1 \leq j \leq n,$$

where $\mathbf{1} = (1, \dots, 1)$.

Remark 8.12. *Let $P \in \mathcal{H}_n^+$. Since P is homogeneous we have the identity*

$$t^n P(z_1, \dots, z_n) = P(tz_1, \dots, tz_n)$$

Differentiating both sides with respect to t we have

$$nt^{n-1} P(z_1, \dots, z_n) = \frac{d}{dt} P(tz_1, \dots, tz_n) = \sum_{j=1}^{n} z_j \frac{\partial P}{\partial z_j}(tz_1, \dots, tz_n).$$

By setting $t = 1$ we get that

$$P = \frac{1}{n} \sum_{j=1}^{n} z_j \frac{\partial P}{\partial z_j}.$$

Hence if P is doubly stochastic then

$$P(\mathbf{1}) = \frac{1}{n} \sum_{j=1}^{n} \frac{\partial P}{\partial z_j}(\mathbf{1}) = \frac{1}{n} \sum_{j=1}^{n} 1 = 1.$$

Lemma 8.13. *Let $\theta_1, \dots, \theta_n$ be positive numbers which sum to 1, and let $x_1, \dots, x_n$ be positive. Then*

$$\log \left(\sum_{i=1}^{n} \theta_i x_i \right) \geq \sum_{i=1}^{n} \theta_i \log(x_i)$$

with equality if and only if $x_1 = \cdots = x_n$.

Proof. Follows straight from Jensen's inequality since $\log(x)$ is a convex function. ☐

Lemma 8.14. *Suppose that $P \in \mathcal{H}_+^n$ and $P(\mathbf{1}) = 1$. Then P is doubly stochastic if and only if Cap$(P) = P(\mathbf{1}) = 1$.*

Proof. Suppose that $P = \sum_\alpha a(\alpha) z^\alpha$ is doubly stochastic. Then

$$1 = \frac{\partial P}{\partial z_j}(\mathbf{1}) = \frac{\partial}{\partial z_j} \left(\sum_\alpha a(\alpha) z^\alpha \right) \Bigg|_{z=\mathbf{1}} = \sum_\alpha \alpha_j a(\alpha).$$

for every $1 \leq j \leq n$. By Lemma 8.13 we have

$$\log(P(x)) \geq \sum_\alpha a(\alpha) \log(x^\alpha) = \sum_\alpha a(\alpha) \sum_{j=1}^{n} \alpha_j \log(x_j) = \sum_{j=1}^{n} \log(x_j) \underbrace{\left(\sum_\alpha \alpha_j a(\alpha) \right)}_{=1} = \log(x_1 \cdots x_n).$$

for all $x \in \mathbb{R}_+^n$. Thus

$$0 \leq \log(P(x)) - \log(x_1 \cdots x_n) = \log\left(\frac{P(x)}{x_1 \cdots x_n}\right) \implies 1 \leq \frac{P(x)}{x_1 \cdots x_n}.$$

Therefore $1 \leq \inf_{x_1,\ldots,x_n>0} \dfrac{P(x)}{x_1 \cdots x_n} = \mathrm{Cap}(P)$. On the other hand the lower bound is obtained by setting $x = \mathbf{1}$ since $P(\mathbf{1}) = 1$ by Remark 8.12. Hence $\mathrm{Cap}(P) = P(\mathbf{1}) = 1$. Conversely if $\mathrm{Cap}(P) = P(\mathbf{1}) = 1$, then consider the function $f : (-1,1) \to \mathbb{R}$ defined by $f(t) = P(1-t, 1+t, 1, \ldots, 1)/(1-t^2)$. Then $f(t) \geq \mathrm{Cap}(P) = 1$ and $f(0) = 1$ so $t = 0$ gives a global minimum for $f(t)$. Thus

$$0 = f'(0) = -\frac{\partial P}{\partial z_1}(\mathbf{1}) + \frac{\partial P}{\partial z_2}(\mathbf{1}).$$

Similarly we get $-\partial P/\partial z_i(\mathbf{1}) + \partial P/\partial z_j(\mathbf{1}) = 0$ for all $1 \leq i,j \leq n$. Hence by Remark 8.12 we have for each $i = 1, \ldots, n$ that

$$1 = P(\mathbf{1}) = \frac{1}{n}\sum_{j=1}^{n} \frac{\partial P}{\partial z_j}(\mathbf{1}) = \frac{1}{n}\frac{\partial P}{\partial z_i}(\mathbf{1}) \sum_{j=1}^{n} 1 = \frac{\partial P}{\partial z_i}(\mathbf{1}).$$

□

Lemma 8.15. *Suppose that all coefficients of $P \in \mathcal{H}_+^n$ are positive. Then there is a unique vector $x \in \mathbb{R}^n$, with all entries positive, such that*

$$\frac{P(x_1, \ldots, x_n)}{x_1 \cdots x_n} = \mathit{Cap}(P).$$

Proof. We first prove existence. By homogeneity it follows that

$$\frac{P(x_1, \ldots, x_n)}{x_1 \cdots x_n} = \frac{P\left((x_1 \cdots x_n)^{1/n} x_1', \ldots, (x_1 \cdots x_n)^{1/n} x_n'\right)}{x_1 \cdots x_n} = P(x_1', \ldots, x_n').$$

Therefore it suffices to consider $x \in \mathbb{R}^n$ such that $x_1 \cdots x_n = 1$. If $P(x) = \sum_\alpha a(\alpha) x^\alpha$, then $P(x) \geq a(n, 0, \ldots, 0) x_1^n$ for all $x \in \mathbb{R}_+^n$. Hence when computing the infimum in the capacity it suffices to consider $x_1 < C_1$ for some constant C_1. Similarly if $x_1 \leq \epsilon \leq 1$ we have for all $x \in \mathbb{R}_+^n$ that

$$\begin{aligned}
P(x) &\geq \sum_{j=2}^{n} a(\mathbf{1} - e_1 + e_j) x_2 \cdots x_{j-1} x_j^2 x_{j+1} \cdots x_n && [\text{subset of terms of } P] \\
&= \frac{1}{x_1} \sum_{j=2}^{n} a(\mathbf{1} - e_1 + e_j) x_j && [\text{since } x_1 \cdots x_n = 1] \\
&\geq \frac{1}{\epsilon} \min_{2 \leq j \leq n} a(\mathbf{1} - e_1 + e_j) && [x_1 \cdots x_n = 1 \text{ and } x_1 \leq 1 \implies x_j \geq 1 \text{ for some } j].
\end{aligned}$$

Thus since $P(x)$ is bounded below by a positive number there must be a positive constant B_1 such that $x_1 \geq B_1$. Similarly we find constants C_j, B_j such that $B_j \leq x_j \leq C_j$ for $j = 2, \ldots, n$. Hence $P(x)/x_1 \cdots x_n$ attains its infimum for some $x \in \mathbb{R}_+^n$ being a continuous function on a compact domain. For uniqueness we may assume via rescaling and normalization that $P(\mathbf{1}) = \mathrm{Cap}(P) = 1$. Then by Lemma 8.14 it follows that P is doubly stochastic. Suppose x is another vector which realizes the capacity of P so that

$$\frac{P(x_1,\ldots,x_n)}{x_1\cdots x_n} = \mathrm{Cap}(P) = 1$$

then

$$\log(P(x_1,\ldots,x_n)) = \log(x_1\cdots x_n).$$

But since P is stochastic we have as in Lemma 8.14 that

$$\log(x_1\cdots x_n) = \sum_{j=1}^{n}\log(x_j)\underbrace{\left(\sum_\alpha \alpha_j a(\alpha)\right)}_{=1} = \sum_\alpha a(\alpha)\sum_{j=1}^{n}\alpha_j\log(x_j) = \sum_\alpha a(\alpha)\log(x^\alpha).$$

Thus

$$\log\left(\sum_\alpha a(\alpha)x^\alpha\right) = \log(P(x_1,\ldots,x_n)) = \sum_\alpha a(\alpha)\log(x^\alpha)$$

where $\sum_\alpha a(\alpha) = 1$ since $P(\mathbf{1}) = 1$. Therefore by Lemma 8.13 we have that $x^\alpha = x^\beta$ for all $\alpha,\beta\in\mathbb{N}^n$ such that $|\alpha| = |\beta| = n$. In particular $x_i^n = x_i^{n-1}x_j$ so that $x_i = x_j$ for all $1\le i,j\le n$. Since P is doubly stochastic we have $x = \mathbf{1}$ as desired [Explain the punchline]. □

Theorem 8.16. *Let $P\in\mathcal{H}_n^+$ be stable. Then*

$$\frac{\partial^n P}{\partial z_1\cdots\partial x_n}(0) = \mathit{Cap}(P)\frac{n!}{n^n}$$

if and only if $\mathit{Cap}(P) = 0$ or

$$P = (a_1z_1+\cdots+a_nx_n)^n,$$

where $a_j > 0$ for each j.

Proof. Recall that by Theorem 8.9 we have

$$\mathrm{Cap}(P) \ge \frac{\partial^n}{\partial z_1\cdots\partial z_n}P(0) \ge \mathrm{Cap}(P)\frac{n!}{n^n}. \tag{5}$$

Thus if $\mathrm{Cap}(P) = 0$ then we clearly have equality. If $P = (a_1z_1+\cdots+a_nz_n)^n$ then

$$\frac{n^n}{n!}\frac{\partial^n P}{\partial z_1\cdots\partial z_n}(0) = \frac{n^n}{n!}\frac{\partial^n(a_1z_1+\cdots+a_nz_n)^n}{\partial z_1\cdots\partial z_n}(0) = \frac{n^n}{n!}n!\prod_{i=1}^{n}a_i = n^n\prod_{i=1}^{n}a_i.$$

Moreover by AM-GM inequality we have

$$
\begin{aligned}
\mathrm{Cap}(P) &= \inf_{x_1,\dots,x_n>0} \frac{(a_1x_1+\cdots a_nx_n)^n}{x_1\cdots x_n} \\
&= \inf_{\substack{x_1,\dots,x_n>1 \\ x_1\cdots x_n=1}} (a_1x_1+\cdots+a_nx_n)^n \\
&= \inf_{\substack{x_1,\dots,x_n>1 \\ x_1\cdots x_n=1}} n^n\left(\frac{a_1x_1+\cdots+a_nx_n}{n}\right)^n \\
&\overset{\text{AM-GM}}{\geq} n^n\prod_{i=1}^n a_ix_i \\
&= n^n\prod_{i=1}^n a_i \\
&= \frac{n^n}{n!}\frac{\partial^n P}{\partial z_1\cdots\partial z_n}(0)
\end{aligned}
$$

Therefore

$$\frac{n!}{n^n}\mathrm{Cap}(P) \geq \frac{\partial^n P}{\partial z_1\cdots\partial z_n}(0)$$

and hence we have equality. Suppose conversely that

$$\frac{n!}{n^n}\mathrm{Cap}(P) = \frac{\partial^n P}{\partial z_1\cdots\partial z_n}(0) \tag{6}$$

and $\mathrm{Cap}(P)>0$. Then by Theorem 8.9 it follows that

$$\prod_{j=1}^n G(e_j) = \prod_{j=1}^n G(j) = \frac{n!}{n^n}.$$

If the coefficient in front of z_n^n is zero then $e_n = \min\{n, \deg_n(P)\} \leq \min\{n, n-1\}$. But then since G is strictly decreasing and $e_j \leq j$ for all $j=1,\dots,n$ we have

$$\prod_{j=1}^n G(e_j) > \prod_{j=1}^n G(j)$$

which is a contradiction. Thus the coefficient in front of z_n^n must be non-zero. But the inequality in Theorem 8.9 remains invariant under permutation of the variables since mixed partials commute and the infimum in Cap(P) remains the same under permutation. Hence we conclude that the coefficient of z_j^n must be non-zero for all $j=1,\dots,n$. Thus by an induction argument on the degree of z_i for each $i=1,\dots,n$ it follows via differentiation that all coefficients of P are strictly positive. Moreover $P(e_i)>0$ (where e_i is the standard basis vector in $\mathbb{R}^n$) since the only term that survives evaluation in e_i is z_n^i (because P is homogeneous of degree n) and all coefficients are strictly positive. Since P is real stable by hypothesis it follows by Proposition 1.3 that $t \mapsto P(x+e_it)$ is real rooted for every $x\in\mathbb{R}^n$. Hence P is hyperbolic with respect to e_i for every

$i = 1, \dots, n$. Now let $c_1, \dots, c_n > 0$ and $R(z_1, \dots, z_n) := P(c_1 x_1, \dots, c_n x_n)$. Then

$$\begin{aligned}\mathrm{Cap}(R) &= \inf_{x_1,\dots,x_n>0} \frac{R(x_1,\dots,x_n)}{x_1 \cdots x_n} \\ &= \inf_{x_1,\dots,x_n>0} \frac{P(c_1x_1,\dots,c_nx_n)}{x_1 \cdots x_n} \\ &= c_1 \cdots c_n \inf_{x_1,\dots,x_n>0} \frac{P(c_1x_1,\dots,c_nx_n)}{(c_1x_1) \cdots (c_nx_n)} \\ &= c_1 \cdots c_n \mathrm{Cap}(P).\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial^n R(z_1,\dots,z_n)}{\partial z_1 \cdots \partial z_n}(0) &= \frac{\partial^n P(c_1z_1,\dots,c_nz_n)}{\partial z_1 \cdots \partial z_n}(0) \\ &= c_1 \cdots c_n \frac{\partial^n P(z_1,\dots,z_n)}{\partial z_1 \cdots \partial z_n}(0) \\ &= c_1 \cdots c_n \frac{n!}{n^n}\mathrm{Cap}(P) \\ &= \frac{n!}{n^n}\mathrm{Cap}(R)\end{aligned}$$

Thus the equality in (6) remains true under scaling of the variables by positive real numbers. Therefore by Lemma 8.15 we may assume that P is doubly stochastic and that the vector $\mathbf{1}$ uniquely realizes the capacity. Suppose $Q = \partial P/\partial z_i \Big|_{z_i=0}$ for some $1 \le i \le n$ and $\mathrm{Cap}(Q) > G(n)\mathrm{Cap}(P)$. Recall that by defining $Q_n := P$, $Q_{j-1} := \partial Q_j/\partial z_j \Big|_{z_j=0}$ we have $Q_0 = \partial^n/\partial z_1 \cdots \partial z_n P(0)$ and by Lemma 8.8 that $\mathrm{Cap}(Q_{j-1}) \ge \mathrm{Cap}(Q_j) G(\deg_j(Q_j))$. Thus by permuting variables if necessary we have strict inequality in (5) contradicting our assumption in (6). Hence by the equality case in Lemma 8.7 it follows that the stable (by specialization) univariate polynomial $t \mapsto P(\mathbf{1} - e_j + te_j)$ has only one zero. Thus since $P(\mathbf{1}) = 1$ we may write

$$P(\mathbf{1} + te_j) = (1 + \lambda_j t)^n$$

for some $\lambda_j \in \mathbb{R}$. Since P is doubly stochastic we have

$$n\lambda_j = \frac{d}{dt}(1+\lambda_j t)^n \Big|_{t=0} = \frac{d}{dt}P(\mathbf{1} + te_j)\Big|_{t=0} = \frac{\partial P}{\partial z_j}(\mathbf{1}) = 1.$$

Hence

$$P(\mathbf{1} - ne_j + nte_j) = P(\mathbf{1} + n(t-1)e_j))) = (1 + \lambda_j n(t-1))^n = (1 + \frac{1}{n}n(t-1))^n = t^n$$

for all $1 \le j \le n$. Thus $\mathbf{1} - ne_j$ belongs to the closure of the hyperbolicity cone of P with respect to e_j. Similarly by homogeneity of P we have

$$P(-(\mathbf{1} - ne_j) - nte_j) = P(-(\mathbf{1} + n(t-1)e_j)) = (-1)^n P(\mathbf{1} + n(t-1)e_j) = (-1)^n t^n$$

so that $-(\mathbf{1} - ne_j)$ lies in the closure of the hyperbolicity cone of P and thereby $\mathbf{1} - ne_j \in L(P)$ for $1 \le j \le n$. Thus since $L(P)$ is a linear subspace we have

$$\sum_{j=1}^{n} x_j(e_j - \frac{1}{n}\mathbf{1}) = -\frac{x_j}{n}(\mathbf{1} - ne_j) \in L(P).$$

As such it follows by Proposition 7.17 that

$$P(x) = P\left(\sum_{j=1}^{n} x_j e_j\right) = P\left(\sum_{j=1}^{n} x_j(e_j - \frac{1}{n}\mathbf{1}) + \frac{1}{n}\left(\sum_{j=1}^{n} x_j\right)\mathbf{1}\right) = P\left(\frac{1}{n}\left(\sum_{j=1}^{n} x_j\right)\mathbf{1}\right) = \left(\frac{\sum_{j=1}^{n} x_j}{n}\right)^n$$

which is of the desired form.

□

Proof. (Van der Waerden Conjecture)
If $A = (a_{ij})_{i,j=1}^n$ is a matrix with nonnegative entries then the polynomial

$$P_A(z) = \prod_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ji} z_j\right)$$

is identically zero, or homogeneous stable of degree n since $Im\left(\sum_{j=1}^{n} a_{ji} z_j\right) > 0$ for all $z \in H^n$ (so in particular the factors are non-zero). Note that

$$\frac{\partial^n P}{\partial z_1 \cdots \partial z_n}(0) = \mathrm{per}(A)$$

since only the terms $\left(\prod_{j=1}^{n} a_{j\sigma(j)}\right) z_1 \cdots z_n$, where $\sigma \in \mathfrak{S}_n$, survive the derivation. If A is doubly stochastic then for every $k = 1, \ldots, n$ we have

$$\frac{\partial P_A}{\partial z_k}(\mathbf{1}) = \underbrace{\sum_{r=1}^{n} a_{rk}}_{=1} \prod_{\substack{i=1 \\ i \neq k}}^{n}\left(\sum_{j=1}^{n} a_{ij} z_j\right)\Bigg|_{z=\mathbf{1}} = \prod_{\substack{i=1 \\ i \neq k}}^{n}\underbrace{\left(\sum_{j=1}^{n} a_{ij}\right)}_{=1} = 1.$$

Hence P_A is doubly stochastic. Note that $P_A \in \mathcal{H}_+^n$ and $P_A(\mathbf{1}) = 1$. By Lemma 8.14 we therefore have $\mathrm{Cap}(P_A) = 1$ and by Theorem 8.9 that

$$\mathrm{per}(A) = \frac{\partial^n}{\partial z_1 \cdots \partial z_n} P_A(0) \geq \mathrm{Cap}(P_A)\frac{n!}{n^n} = \frac{n!}{n^n}.$$

By Theorem 8.16 it follows that we have equality if and only if $P_A = (a_1 z_1 + \cdots + a_n z_n)^n$ where $a_j > 0$ for each j. Thus $a_{ji} = a_j$ for all $1 \leq i, j \leq n$. Since $1 = \sum_{i=1}^{n} a_{ji} = na_j$ it follows that $a_j = 1/n$ and so $a_{ji} = a_j = 1/n$ for $1 \leq i, j \leq n$ as required. □

Definition 8.17. *(Complete polarized form)*
Let $h(z_1, \ldots, z_n)$ be a homogeneous polynomial of degree d. Let $v_j = (v_{1j}, \ldots, v_{nj})^T$ for $1 \leq j \leq d$. The **complete polarized form** of h is defined as the form $H : (\mathbb{R}^n)^d \to \mathbb{R}$ defined by

$$H(v_1, \ldots, v_d) = \frac{1}{d!}\prod_{j=1}^{d}\left(\sum_{i=1}^{n} v_{ij}\frac{\partial}{\partial z_i}\right) h(z).$$

Remark 8.18.
Note that
(1) H is clearly symmetric.
(2) H is multilinear: Indeed by symmetry it is enough to show linearity in the first entry.

$$\begin{aligned}
H(\lambda v_1 + \mu v_1', v_2, \ldots, v_n) &= \frac{1}{d!}\left(\sum_{i=1}(\lambda v_{i1} + \mu v_{i1}')\frac{\partial}{\partial z_i}\right)\prod_{j=2}^{n}\left(\sum_{i=1}^{n} v_{ij}\frac{\partial}{\partial z_i}\right)h(z) \\
&= \frac{1}{d!}\left(\lambda\sum_{i=1} v_{i1}\frac{\partial}{\partial z_i} + \mu\sum_{i=1} v_{i1}'\frac{\partial}{\partial z_i}\right)\prod_{j=2}^{n}\left(\sum_{i=1}^{n} v_{ij}\frac{\partial}{\partial z_i}\right)h(z) \\
&= \lambda\frac{1}{d!}\prod_{j=1}^{n}\left(\sum_{i=1}^{n} v_{ij}\frac{\partial}{\partial z_i}\right)h(z) + \mu\frac{1}{d!}\left(\sum_{i=1}^{n} v_{i1}'\frac{\partial}{\partial z_i}\right)\prod_{j=2}^{n}\left(\sum_{i=1}^{n} v_{ij}\frac{\partial}{\partial z_i}\right)h(z) \\
&= \lambda H(v_1, v_2, \ldots, v_n) + \mu H(v_1', v_2, \ldots, v_n)
\end{aligned}$$

(3) $H(v, \ldots, v) = h(v)$: If we write $h(z) = \sum\limits_{\substack{\alpha \\ |\alpha|=d}} a(\alpha)z^\alpha$ then

$$H(v, \ldots, v) = \frac{1}{d!}\left(\sum_{i=1}^{n} v_i\frac{\partial}{\partial z_i}\right)^d h(z) = \frac{1}{d!}\left(\sum_{\substack{\alpha \\ |\alpha|=d}}^{n}\frac{d!}{\alpha!}v^\alpha\frac{\partial^d}{\partial z_1^{\alpha_1}\cdots\partial z_d^{\alpha_d}}\right)\left(\sum_{\substack{\alpha \\ |\alpha|=d}} a(\alpha)z^\alpha\right) = \sum_{\substack{\alpha \\ |\alpha|=d}} a(\alpha)v^\alpha = h(v)$$

Exercise 5: Prove that

$$H(v_1, \ldots, v_d) = \frac{1}{d!}\frac{\partial^d}{\partial z_1\cdots\partial z_d}h\left(\sum_{j=1}^{d} z_j v_j\right)$$

Proof. We argue by induction on d.

$$\begin{aligned}
H(v_1, \ldots, v_d) &= \frac{1}{d!}\prod_{j=1}^{d}\left(\sum_{i=1}^{n} v_{ij}\frac{\partial}{\partial z_i}\right)h(z) \\
&= \frac{1}{d}\left(\sum_{i=1}^{n} v_{id}\frac{\partial}{\partial z_i}\right)\left(\frac{1}{(d-1)!}\prod_{j=1}^{d-1}\left(\sum_{i=1}^{n} v_{ij}\frac{\partial}{\partial z_i}\right)h(z)\right) \\
&= \frac{1}{d}\left(\sum_{i=1}^{n} v_{id}\frac{\partial}{\partial z_i}\right)\left(\frac{1}{(d-1)!}\frac{\partial^{d-1}}{\partial z_1\cdots\partial z_{d-1}}h\left(\sum_{j=1}^{d-1} z_j v_j\right)\right) \\
&= \frac{1}{d!}\frac{\partial^{d-1}}{\partial z_1\cdots z_{d-1}}\left(\sum_{i=1}^{n} v_{id}\frac{\partial h}{\partial z_i}\left(\sum_{j=1}^{d-1} z_j v_j\right)\right) \\
&= \frac{1}{d!}\frac{\partial^{d-1}}{\partial z_1\cdots z_{d-1}}\left(\frac{\partial}{\partial z_d}h\left(\sum_{j=1}^{d} z_j v_j\right)\right) \\
&= \frac{1}{d!}\frac{\partial^d}{\partial z_1\cdots\partial z_d}h\left(\sum_{j=1}^{d} z_j v_j\right).
\end{aligned}$$

□

Exercise 6: Gårding's inequality reads as follows. Suppose that h is hyperbolic of degree n, and that $a_1, \ldots, a_n \in \Lambda_{++}$, then

$$h(a_1)^{1/n} \cdots h(a_n)^{1/n} \leq H(a_1, \ldots, a_n).$$

Prove that Gårding's inequality follows from Gurvits' inequality (Theorem 8.9).

Proof. First note that

$$\begin{aligned}
h^{1/n}(a) + h^{1/n}(b) &= \left(h\left((a+b)\frac{a}{a+b}\right)\right)^{1/n} + \left(h\left((a+b)\frac{b}{a+b}\right)\right)^{1/n} \\
&= \left(\left(\frac{a}{a+b}\right)^n h(a+b)\right)^{1/n} + \left(\left(\frac{b}{a+b}\right)^n h(a+b)\right)^{1/n} \\
&= \frac{a}{a+b} h^{1/n}(a+b) + \frac{b}{a+b} h^{1/n}(a+b) \\
&= h^{1/n}(a+b).
\end{aligned}$$

Set $P(z_1, \ldots, z_n) := h\left(\sum_{j=1}^n z_j a_j\right)$. Then since $a_1, \ldots, a_n \in \Lambda_{++}$ we have by Proposition 7.15 that P is stable. By Lemma 6.11 P stable and homogeneous implies all coefficients of P have the same phase. Thus by hyperbolicity, all coefficients of P are nonnegative [might be talking BS in this sentence, fix it!]. Hence

$$\begin{aligned}
H(a_1, \ldots, a_n) &= \frac{1}{n!} \frac{\partial^n}{\partial z_1 \cdots \partial z_n} h\left(\sum_{j=1}^n z_j a_j\right) \\
&= \frac{1}{n!} \frac{\partial^n}{\partial z_1 \cdots \partial z_n} P(z_1, \ldots, z_n) \\
&= \frac{1}{n!} \frac{\partial^n}{\partial z_1 \cdots \partial z_n} P(0) \\
&\geq \frac{\mathrm{Cap}(P)}{n^n} && \text{[Gurvits ineq]} \\
&= \inf_{x_1, \ldots, x_n > 0} \frac{\left(\frac{h^{1/n}\left(\sum_{j=1}^n x_j a_j\right)}{n}\right)^n}{x_1 \cdots x_n} \\
&= \inf_{x_1, \ldots, x_n > 0} \frac{\left(\frac{\sum_{j=1}^n h^{1/n}(x_j a_j)}{n}\right)^n}{x_1 \cdots x_n} \\
&\geq \inf_{x_1, \ldots, x_n > 0} \frac{\left(\sqrt[n]{\prod_{j=1}^n h^{1/n}(x_j a_j)}\right)^n}{x_1 \cdots x_n} && \text{[AM-GM ineq]} \\
&= \inf_{x_1, \ldots x_n > 0} \frac{x_1 \cdots x_n \prod_{j=1}^n h^{1/n}(a_j)}{x_1 \cdots x_n} \\
&= h^{1/n}(a_1) \cdots h^{1/n}(a_n).
\end{aligned}$$

□

9 Negative Dependence and The Geometry of Polynomials

In this section we are concerned with discrete probability measures on $\{0,1\}^S$ where S is a finite set, by which we mean functions $\mu : \{0,1\}^S \to \mathbb{R}_{\geq 0}$ such that $\sum_{\eta \in \{0,1\}^S} \mu(\eta) = 1$. Hence if $A \subseteq \{0,1\}^S$, we set $\mu(A) := \sum_{\eta \in A} \mu(\eta)$.

Definition 9.1. *(PNC)*
*A probability measure μ is **pairwise negatively correlated**, PNC, if*

$$\mu(\{\eta : \eta(i) = \eta(j) = 1\}) \leq \mu(\{\eta : \eta(i) = 1\})\mu(\{\eta : \eta(j) = 1\})$$

for all $i \neq j$ in S.

Definition 9.2. *(NLC)*
*A probability measure μ is said to satisfy the **negative lattice condition**, NLC, if*

$$\mu(\eta \vee \xi)\mu(\eta \wedge \xi) \leq \mu(\eta)\mu(\xi)$$

for all $\nu, \xi \in \{0,1\}$.

Definition 9.3. *(NA)*
*A probability measure μ is said to be **negatively associated**, NA, if*

$$\int fg d\mu \leq \int f d\mu \int g d\mu$$

for all increasing functions $f, g : \{0,1\}^S \to \mathbb{R}$ that depend on disjoint sets of variables, that is, there exists a subset $A \subseteq S$ such that $f(\nu_1, \ldots, \nu_n)$ only depends on $\{\eta_j : j \in A\}$ and g only depends on $\{\eta_j : j \in S \setminus A\}$.

Definition 9.4. *An **external field** is a vector $x \in \mathbb{R}_+^S$ giving rise to a measure μ_x given by*

$$\mu_x(\eta) := \frac{x^\eta \mu(\eta)}{Z_\mu(x)} \quad \textit{for all } \eta \in \{0,1\}^S$$

where

$$Z_\mu(x) = \sum_{\eta \in \{0,1\}^S} \mu(\eta) x^\eta, \quad x^\eta = \prod_{j \in S} x_j^{\eta(j)}$$

*is the **partition function** of μ.*

Definition 9.5. *If P is a property of measures, we say that μ satisfies P_+ if μ_x satisfies P for all $x \in \mathbb{R}_+$.*

Remark 9.6. *Note that*

$$\frac{\partial Z_\mu(x)}{\partial z_j} = x_j^{-1} \sum_{\eta : \eta(j) = 1} \mu(\eta) x^\eta = x_j^{-1} Z_\mu(x) \frac{\sum_{\eta : \eta(j)=1} \mu(\eta) x^\eta}{Z_\mu(x)} = x_j^{-1} Z_\mu(x) \mu_x(\{\eta : \eta(j) = 1\}).$$

Thus μ_x is negatively correlated if and only if

$$
\begin{aligned}
Z_\mu(x)\frac{\partial^2 Z_\mu(x)}{\partial x_i \partial x_j} &= Z_\mu(x)\frac{\partial}{\partial x_i}\left(x_j^{-1}\sum_{\eta:\eta(j)=1}\mu(\eta)x^\eta\right)\\
&= Z_\mu(x)x_j^{-1}x_i^{-1}\sum_{\eta:\eta(j)=1,\eta(i)=1}\mu(\eta)x^\eta\\
&= Z_\mu^2(x)x_j^{-1}x_i^{-1}\mu_x(\{\eta:\eta(i)=\eta(j)=1\})\\
&\le Z_\mu^2(x)x_j^{-1}x_i^{-1}\mu_x(\{\eta:\eta(i)=1\})\mu_x(\{\eta:\eta(j)=1\})\\
&= (x_i^{-1}Z_\mu(x)\mu_x(\{\eta:\eta(i)=1\}))(x_j^{-1}Z_\mu(x)\mu_x(\{\eta:\eta(j)=1\}))\\
&= \frac{\partial Z_\mu(x)}{\partial x_i}\frac{\partial Z_\mu(x)}{\partial x_j}
\end{aligned}
$$

Definition 9.7. *(Rayleigh)*
*A probability measure μ is **Rayleigh** if*

$$Z_\mu(x)\frac{\partial^2 Z_\mu(x)}{\partial x_i \partial x_j} \le \frac{\partial Z_\mu(x)}{\partial x_i}\frac{\partial Z_\mu(x)}{\partial x_j} \qquad \text{for all } x\in\mathbb{R}_+^S,\ i,j\in S$$

Remark 9.8. *By Theorem 3 it follows immediately that if Z_μ is a stable polynomial, then μ is Rayleigh. In fact, more is true. If Z_μ is stable then μ is said to be **strongly Rayleigh**, or SR for short. Strongly Rayleigh measures are NA_+.*

Definition 9.9. *(Constant sum)*
*A measure μ has **constant sum** if $\mu(\xi)\mu(\eta)\neq 0$ implies $|\xi|=|\eta|$, that is, if Z_μ is homogeneous.*

Theorem 9.10. *(Feder and Mihail)*
If μ is a constant sum Rayleigh measure then mu is NA_+.

Definition 9.11. *(Projection of measure)*
*Let μ be a probability measure on $\{0,1\}^S$ and let $R\subseteq S$. Then the **projection of** μ **onto** $\{0,1\}^R$ is defined as the measure whose partition function is obtained from Z_μ by setting $z_j=1$ for all $j\in S$.*

Remark 9.12. *Since stability is a closed property under specialization by Proposition 1.8 it follows that the class of strongly Rayleigh measures are closed under projection, and so is the class of negatively associated measures. Hence in view of Theorem 9.10, to prove that SR measures are NA it suffices to show that each SR measure is the projection of a constant sum SR measure.*

Definition 9.13. *(Homogeneous symmetrization)*
*Let μ be a probability measure on $\{0,1\}^S$ and let R be a set such that $R\cap S=\emptyset$ and $|R|=|S|$. The **homogeneous symmetrization** of μ is defined as the unique measure, μ, on $\{0,1\}^{S\cup R}$ satisfying:*
(1) μ_h has contant sum $|S|$;
(2) The sites in R are indistinguishable, that is, $\mu_h(\sigma(\eta))=\mu_h(\eta)$ whenever $\sigma\in\mathfrak{S}_{|R|}$;
(3) The projection of μ_h onto $\{0,1\}^S$ is μ.

In terms of partition functions, μ_h is the measure with partition function

$$Z_{\mu h} = \sum_{\eta \in \{0,1\}^S} \mu(\eta) z^\eta \frac{e_{n-|\eta|}(z_R)}{\binom{n}{k}},$$

where $n = |S|$ and $e_k(z_R)$ is the k^{th} elementary symmetric polynomial in the variables $z_R = (z_j)_{j \in R}$.

Corollary 9.14. *μ is strong Rayleigh if and only if μ_h is strong Rayleigh.*

Proof. If μ_h is strong Rayleigh, then so is μ since the strong Rayleigh property is closed under projections. Conversely suppose that μ is strong Rayleigh. Then Z_μ is stable by definition and has nonnegative coefficients, so by Corollary 7.14 we have that

$$P = \sum_{\eta \in \{0,1\}^S} \mu(\eta) z^\eta w^{n-|\eta|}$$

is stable in z and w. By Proposition 3.6 the polarization $\Pi^\uparrow(P) = Z_{\mu h}$ is also stable showing μ_h is strong Rayleigh. □

Theorem 9.15. *Strong Rayleigh measures are negatively associated*

Proof. If μ is strong Rayleigh then by Corollary 9.14 so is μ_h. Since μ_h is constant sum we have by Theorem 9.10 that μ_h is negatively associated. But μ_h projects onto μ and the class of negatively associated measures is closed under projections. Hence μ is negatively associated. □

Exercise 7: Prove that μ is negatively associated whenever Z_μ is stable.

Proof. If Z_μ is stable then μ is strong Rayleigh by definition. The statement now follows by Theorem 9.15. □

Definition 9.16. *(The exclusion process)*
The **exclusion process** is one of the main models considered in the area of Interacting Particle Systems. The idea is that particles move in continuous on a countable set S of **sites** in such a way that there is always at most one particle per site.

Definition 9.17. *(The symmetric exclusion process)*
The **symmetric exclusion process** is a continuous time Markov chain (SEP) on a state space $\{0,1\}^S$ where S is a countable set of sites. To avoid technicalities we will only consider the case where $|S| < \infty$. If $\eta \in \{0,1\}^S$ we think of the indices j such that $\eta(j) = 1$ as **occupied sites** and those with $\eta(j) = 0$ as **vacant sites**. The transitions of the Markov chain are: For each $i, j \in S, \eta \mapsto \tau_{ij}(\eta)$ at rate q_{ij}, where τ_{ij} is the transposition that exchanges i and j. Ligett and Pemantle conjectured independently that if the initial distribution of SEP is a product measure or deterministic, then the distribution at all positive times is negatively associated. It turns out much more is true, namely SEP preserves the strong Rayleigh property. This was proved by Borcea, Bränden and Ligett. It is convenient to view the Markov chain as acting on the partition functions of the measures. We may then view a Markov chain as acting on $\mathbb{C}_{\mathbf{1}}[z_1, \ldots, z_n]$

as a family of linear operators (or matrices) $\{T_t\}_{t\geq 0}$ indexed by the continuous variable t. It follows that T_t satisfies

$$\frac{d}{dt}T_t = \mathcal{L}T_t, \qquad \text{for all } t \geq 0, \tag{7}$$

where $\mathcal{L} : \mathbb{C}_{\mathbf{1}}[z_1, \ldots, z_n] \to \mathbb{C}_{\mathbf{1}}[z_1, \ldots, z_n]$ is the (linear) generator. In the case of SEP

$$\mathcal{L} = \sum_{i<j} q_{ij}(\tau_{ij} - id)$$

where id is the identity. Clearly (7) is equivalent to

$$T_t = e^{t\mathcal{L}} = \sum_{n=0}^{\infty} t^n \mathcal{L}^n/n! = \lim_{n\to\infty}\left(1 + \frac{t\mathcal{L}}{n}\right)^n.$$

Thus if the distribution at $t = 0$ has partition function Z, then the partition function at time $t > 0$ is $T_t(Z)$.

Theorem 9.18. *(Lie-Trotter product formula)*
Let A and B be complex square matrices, or bounded operators on a Banach space. Then

$$\lim_{n\to\infty}\left(e^{A/n}e^{B/n}\right)^n = e^{A+B}$$

in the operator norm.

Proof. We have

$$e^{A/n}e^{B/n} = \left(\sum_{k=0}^{\infty}\frac{A^k}{n^k k!}\right)\left(\sum_{l=0}^{\infty}\frac{B^l}{n^l l!}\right) = \sum_{k,l=0}\frac{A^k B^l}{n^{k+l}k!l!} = I + (A+B)/n + n^{-2}C(n)$$

where $||C(n)|| \leq K$ for all n and some $K > 0$. By triangle inequality we get that

$$||(M+N)^n - M^n|| \leq (||M|| + ||N||)^n - ||M||^n.$$

Thus

$$\begin{aligned}
\left|\left|\left(e^{A/n}e^{B/n}\right)^n - \left(I + \frac{A+B}{n}\right)^n\right|\right| &= \left|\left|\left(\left(I + \frac{A+B}{n}\right) + n^{-2}C(n)\right)^n - \left(I + \frac{A+B}{n}\right)^n\right|\right| \\
&\leq \left(\left|\left|\left(I + \frac{A+B}{n}\right)\right|\right| + ||n^{-2}C(n)||\right)^n - \left|\left|\left(I + \frac{A+B}{n}\right)\right|\right|^n \\
&\leq \left(\left|\left|\left(I + \frac{A+B}{n}\right)\right|\right| + \frac{K}{n^2}\right)^n - \left|\left|\left(I + \frac{A+B}{n}\right)\right|\right|^n \\
&\to (||I|| + 0)^n - ||I||^n = 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

Finally by the standard realization of e^x as the limit $\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$ we get

$$\left|\left|e^{A+B} - \left(I + \frac{A+B}{n}\right)^n\right|\right| \to 0 \qquad \text{as } n \to \infty$$

and hence the theorem follows. $\square$

Theorem 9.19. *If the initial distribution of the symmetric exclusion process is strong Rayleigh, then the distribution is strongly Rayleigh, and thus negatively associated for all $t \geq 0$.*

Proof. We show that $T_t : \mathbb{C}_{\mathbf{1}}[z_1, \dots, z_n] \to \mathbb{C}_{\mathbf{1}}[z_1, \dots, z_n]$ preserves stability for all $t \geq 0$ i.e if Z is the partition function at $t = 0$ then $T_t(Z)$ is stable for all $t \geq 0$. In the case of symmetric exclusion processes the generator of T_t is given by

$$\mathcal{L} = \sum_{i<j} q_{ij}(\tau_{ij} - id)$$

To show that T_t preserves stability we wish to reduce the problem to the case of a single transposition with $\mathcal{L} = q(\tau - id)$. To this end if $T_t^{(ij)}$ and $T_t^{(kl)}$ preserve stability for all $t \geq 0$ and are generated by $\mathcal{L}^{(ij)} = q_{ij}(\tau_{ij} - id)$, $\mathcal{L}^{(kl)} = q_{kl}(\tau_{kl} - id)$ respectively, then by Lie-Trotter product formula the linear operator generated by $\mathcal{L}^{(ij)} + \mathcal{L}^{(kl)}$ is given by

$$e^{t(\mathcal{L}^{(ij)}+\mathcal{L}^{(kl)})} = \lim_{n\to\infty} \left(e^{t\mathcal{L}^{(ij)}} e^{t\mathcal{L}^{(kl)}}\right)^n = \lim_{n\to\infty} \left(T_t^{(ij)} T_t^{(kl)}\right)^n.$$

Given that $\left(T_t^{(ij)} T_t^{(kl)}\right)^n$ preserves stability for all $t \geq 0$ it follows by Hurwitz' theorem that so does $e^{t(\mathcal{L}^{(ij)}+\mathcal{L}^{(kl)})}$. Since $\mathcal{L}$ is a sum of generators of above form we conclude by repeating the argument a finite number of times that T_t preserves stability. It thus remains to prove the result for $\mathcal{L} = q(\tau - id)$ where τ is a transposition. Given that $\tau^2 = id$ we have

$$\begin{aligned}
T_t &= e^{tq(\tau - id)} = e^{tq\tau} e^{-tq} = \left(\sum_{n=0}^{\infty} \tau^n \frac{(tq)^n}{n!}\right) e^{-tq} = \left(\sum_{n=0}^{\infty} \frac{(tq)^{2n+1}}{(2n+1)!}\tau + \sum_{n=0}^{\infty} \frac{(tq)^{2n}}{(2n)!}\right) e^{-tq} \\
&= \left(\frac{e^{tq} - e^{-tq}}{2}\tau + \frac{e^{tq} + e^{-tq}}{2} id\right) e^{-tq} = \frac{1 - e^{-2qt}}{2}\tau + \frac{1 + e^{-2qt}}{2} id \\
&= (1 - p(t))\tau + p(t) id.
\end{aligned}$$

where $p(t) := \dfrac{1 - e^{-2qt}}{2}$. Hence by Theorem 2.1 it follows that T_t preserves stability for every $t \geq 0$. $\square$

10 The Matrix Tree Theorem

Let $G = (V, E)$ be a graph without loops and multiple edges, and define a $V \times E$ matrix U by $e_n = \begin{cases} \pm 1, & \text{if } i \in e \\ 0, & \text{if } i \notin e \end{cases}$
where we require that $U_{ie}U_{je} = -1$ for each edge $e = \{i, j\} \in E$.

Definition 10.1. *(Rooted forest)*
A ***rooted forest*** *in G is a pair $\mathcal{F} = [F, R]$, where $F \subseteq E$ contains no cycle, $R \subseteq V$, and if $C_1, \dots, C_k \subseteq V$ are the connected components of the graph (F, V), then there is a bijection $\phi : [k] \to R$ such that $\phi(j) \in C_j$ for every $1 \leq j \leq k$. If $\mathcal{F}$ is a rooted forest let roots$(\mathcal{F}) = R$ and edges$(\mathcal{F}) = F$.*

Definition 10.2. *(Rooted tree)*
*A **rooted tree** is a rooted forest where* $|R| = 1$.

Definition 10.3. *(Total unimodularity)*
An $m \times n$ *matrix is **totally unimodular** if the determinant of each square submatrix of* A *is either* $0, -1$ *or* 1.

Lemma 10.4. *The matrix* U *is totally unimodular. Moreover if* $F \subseteq E$ *and* $W \subseteq V$ *with* $|F| = |W|$*, then* $\det(U(W, F)) = \pm 1$ *if and only if* $[FV \setminus W]$ *is a rooted forest.*

Proof. We argue by induction on the size of the submatrix. By definition of U the result is clearly true for square submatrices of size 1. Let M be a square submatrix of size ≥ 1. If M has a zero column then $\det(M) = 0$. Likewise if M has a column with only one non-zero entry then expanding the determinant along this column gives us a square submatrix of size one less and so $\det(M) \in \{0, \pm 1\}$ by induction. Notice that U has exactly two non-zero entries per column as given by the fact that each edge is incident to exactly two vertices in G. Moreover by the condition that $U_{ie}U_{je} = -1$ if $(i, j) = e$ it follows that the two non-zero entries are necessarily of opposite sign. In particular each column sum to zero. Thus if M has exactly two non-zero entries per column it follows that $M^T \mathbf{1} = 0$ where $\mathbf{1} = (1, \ldots, 1)$ and so M has non-zero null-space which implies $\det M = 0$. Hence every submatrix of U has determinant $0, 1$ or -1 by induction and so U is totally unimodular. From the reasoning above that no column can be zero nor all columns contain exactly two non-zero entries, it follows that $\det(U(W, F)) = \pm 1$ if and only if we may reorder the columns and rows, multiplying certain columns by -1 if necessary (operations which only changes the sign of the determinant) so that $U(W, F)$ has the form

$$\begin{pmatrix}
* & * & * & \ldots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
* & * & * & \ldots & * \\
1 & * & * & \ldots & * \\
0 & 1 & * & \ldots & * \\
0 & 0 & 1 & \ldots & * \\
\vdots & \vdots & \vdots & \ddots & * \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}$$

where the first rows correspond to the vertices in $V \setminus W$. This matrix represents a bipartite graph with independent sets W and $V \setminus W$. In particular $[F, V \setminus W]$ has no cycles and hence is a rooted forest.

□

If A is an $m \times n$ matrix and $S \subseteq [m], T \subseteq [n]$ with $|S| = |T|$, let $A(S, T)$ denote the determinant of the submatrix of A that has rows and columns indexed by S and T, respectively.

Theorem 10.5. *(Binet-Cauchy)*
Let A *be an* $m \times n$ *matrix,* B *an* $n \times q$ *matrix and suppose that* $S \subseteq [m]$ *and* $T \subseteq [q]$ *satisfy* $|S| = |T| = k$*. Then*

$$(AB)(S,T) = \sum_{R,|R|=k} A(S,R)B(R,T).$$

Theorem 10.6. *(Principal Minors Matrix-Tree Theorem)*
Let $G = (V, E)$ be a graph without loops and multiple edges, and Z and W diagonal matrices with variables $z_i, (i \in V)$ and $w_e, (e \in E)$, where the variables are ordered in the same order as in the matrix U. Then

$$\det(Z + UWU^T) = \sum_{\mathcal{F}} z^{roots(\mathcal{F})} w^{edges(\mathcal{F})}$$

where the sum is over all rooted forests in G.

Proof. If A is an $n \times n$ matrix and $Z = \operatorname{diag}(z_1, \ldots, z_n)$, then

$$\det(A + Z) = \sum_{S \subseteq [n]} A(S,S) z^{[n]\setminus S}$$

Thus by above observation and the Binet-Cauchy theorem we have

$$\begin{aligned}
\det(Z + UWU^T) &= \sum_{S \subseteq V} z^{V\setminus S} (UWU^T)(S,S) \\
&= \sum_{S \subseteq V} z^{V\setminus S} \sum_{R,|R|=|S|} w^R U(S,R) U^T(R,S) \\
&= \sum_{S \subseteq V} \sum_{R \subseteq E, |R|=|S|} z^{V\setminus S} w^R U(S,R)^2.
\end{aligned}$$

Finally, since by Lemma 10.4 (the determinant) $U(S,R) = \pm 1$ if and only if $\mathcal{F} = [R, V \setminus S]$ is a rooted forest (and zero otherwise) we have that

$$\det(Z + UWU^T) = \sum_{S \subseteq V} \sum_{R \subseteq E, |R|=|S|} z^{V\setminus S} w^R U(S,R)^2 = \sum_{\mathcal{F}} z^{\text{roots}(\mathcal{F})} w^{\text{edges}(\mathcal{F})}.$$

Hence the proof follows. □

Corollary 10.7. *(Matrix-Tree theorem)*
Let $G = (V, E)$ be a connected graph and $i \in V$. Then

$$(UWU^T)(V \setminus \{i\}, V \setminus \{i\}) = \sum_{T} w^{edges(T)},$$

where the sum is over all spanning trees of G.

Definition 10.8. *(Uniform spanning tree measure)*
The ***uniform spanning tree measure*** *is the measure μ on $\{0,1\}^E$ such that*
$$\mu(F) = \frac{1}{t} \begin{cases} 1, & \text{if } F \text{ is a spanning tree,} \\ 0, & \text{otherwise} \end{cases}$$
where t is the number of spanning trees.

Corollary 10.9. *The polynomial $\det(Z + UWU^T)$ is stable, and the uniform spanning tree measure is strong Rayleigh.*

Proof. Suppose that $V = [n]$ and let $\{e_i\}_{i \in [n]}$ be the standard basis of $\mathbb{R}^n$. If u_e, $(e \in E)$ are the columns of U, then

$$Z + UWU^T = \sum_{i\in[n]} z_u e_i e_i^T + \sum_{e\in E} w_e u_e u_e^T.$$

Note that $e_i e_i^T = E_{ii}$ where E_{ii} is the standard basis vector of $M_n(\mathbb{R})$. The matrix E_{ii} has eigenvalue spectrum $\sigma(E_{ii}) = \{1, 0, \dots, 0\}$ so is therefore positive semidefinite and moreover Hermitian being real symmetric. Note also that $u_e u_e^T = (e_i - e_j)(e_i - e_j)^T = e_i e_i^T - e_i e_j^T - e_i e_j^T + e_j e_j^T = E_{ii} - E_{ij} - E_{ji} + E_{jj}$ which is clearly real symmetric and therefore Hermitian. Moreover expanding $\det(\lambda I - u_e u_e^T)$ along column i we get

$$\begin{aligned}
\det(\lambda I - u_e u_e^T) &= (-1)^{(i-1)+(i-1)}(\lambda - 1)^2\lambda^{n-2} + (-1)^{(i-1)+(j-1)}\left((-1)^{(j-2)+(i-1)}\lambda^{n-2}\right) \\
&= (\lambda - 1)^2\lambda^{n-2} - \lambda^{n-2} \\
&= \lambda^{n-2}((\lambda - 1)^2 - 1) \\
&= \lambda^{n-1}(\lambda - 2).
\end{aligned}$$

Thus the matrix $u_e u_e^T$ has eigenvalue spectrum $\sigma(u_e u_e^T) = \{2, 0, \dots, 0\}$ so it is positive semidefinite for every $e \in E$. Hence by Proposition 1.9 it follows that $\det(Z + UWU^T)$ is stable. The same proof works for the uniform spanning tree measure by the Matrix-Tree theorem. $\square$

11 Stable Polynomials and Matroid Theory

Matroid theory tries to capture the essence of independence, as linear independence in linear algebra, algebraic independence, or the notion of cycles in graphs.

Definition 11.1. *(Matroid)*
*A **matroid** is a pair $(\mathcal{M}, E)$, where $\mathcal{M}$ is a collection of subsets of a finite set E satisfying:*
(1) $\mathcal{M}$ is hereditary, i.e if $B \in \mathcal{M}$ and $A \subseteq B$, then $A \in \mathcal{M}$.
(2) The collection $\mathcal{B}$ consisting of maximal elements with respect to inclusion of $\mathcal{M}$ respects the basis change axiom:

$$A, B \in \mathcal{B} \text{ and } x \in A \setminus B \implies \exists y \in B \setminus A \text{ such that } A \setminus \{x\} \cup \{y\} \in \mathcal{B}.$$

*The elements of $\mathcal{M}$ are called **independent sets** and the set $\mathcal{B}$ is called the **set of bases** of **M**.*

Example 11.2. The fundamental motivating example of a matroid arises from a list of vectors $v_1, \dots, v_n$ in a k-linear space V. If $E = [n]$ then $A \subseteq [n]$ is an independent set of the matroid $\mathcal{M}$ over E if and only if the vectors v_i, $(i \in A)$ are linearly independent. The basis change axiom follows from Steinitz exchange lemma in linear algebra. Such a matroid is said to be **representable** over k. A subclass of the representable matroids are the **graphic matroids**. If $G = (V, E)$ is a graph, we define a matroid on E by declaring $S \subseteq E$ to be independent if S contains no cycle. Hence the set of bases are the spanning trees if G is connected and the maximal spanning forests otherwise. By the Matrix-Tree theorem it follows that the graphic matroids are representable over $\mathbb{R}$. However the arguments in Lemma 10.4 hold over any field and so the graphic matroids are representable over any field.

Definition 11.3. *(Jump system)*
Let $\alpha, \beta \in \mathbb{Z}^n$ and define $|\alpha| = \sum_{i=1}^{n} |\alpha_i|$. The set of ***steps from*** *α* ***to*** *β is defined by*

$$St(\alpha, \beta) = \{\sigma \in \mathbb{Z}^n : |\sigma| = 1, |\alpha + \sigma - \beta| = |\alpha - \beta| - 1\}$$

A collection $\mathcal{F}$ of vectors in $\mathbb{Z}^n$ is called a ***jump system*** *if it respects the* ***Two-step Axiom****:*

If $\alpha, \beta \in \mathcal{F}$, $\sigma \in St(\alpha, \beta)$ and $\alpha + \sigma \notin \mathcal{F}$,
then there exists $\tau \in St(\alpha + \sigma, \beta)$ such that $\alpha + \sigma + \tau \in \mathcal{F}$.

Remark 11.4. Jump systems generalizes matroids to arbitrary collections of finite subsets of $\mathbb{Z}^n$. Suppose $\mathcal{J} \subseteq \{0,1\}^n$ has constant sum, i.e $|\xi| = |\eta|$ for all $\xi, \eta \in \mathcal{J}$. If we identify $\{0,1\}^n$ with $\{S : S \subseteq [n]\}$ we see that $\mathcal{J}$ is a jump system if and only if $\mathcal{J}$ is the set of bases of a matroid.

Definition 11.5. *(Support of a polynomial)*
The ***support****, supp(P), of a polynomial $P(z) = \sum_{\alpha \in \mathbb{N}^n} a(\alpha) z^\alpha \in \mathbb{C}[z_1, \dots, z_n]$ is defined by*

$$supp(P) = \{\alpha \in \mathbb{N}^n : a(\alpha) \neq 0\}.$$

Suppose that $P(z) = \sum_{0 \le \gamma \le \kappa} a(\gamma) z^\gamma \in \mathbb{C}[z_1, \dots, z_n]$ is a stable polynomial of degree κ_j in z_j for each j and suppose that $\alpha, \beta \in \operatorname{supp}(P)$ with $\alpha \le \beta$. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, let

$$\partial^\alpha P = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}} P.$$

Let

$$G(z) = \partial^{\kappa - \beta}(z^\kappa P(-1/z)), \qquad -1/z = (-1/z_1, \dots, -1/z_n).$$

Then $G(z)$ is stable since stability is preserved under inversion and differentiation. Let further

$$P_{\alpha,\beta}(z) = \partial^\alpha (z^\beta G(-1/z)).$$

For $\alpha, \beta \in \mathbb{Z}^n$, let $[\alpha, \beta] = \{\gamma \in \mathbb{Z}^n : \alpha \le \gamma \le \beta\}$ and $(\alpha, \beta) = \{\gamma \in \mathbb{Z}^n : \alpha < \gamma < \beta\}$. Again $P_{\alpha,\beta}$ is stable and

$$\operatorname{supp}(P_{\alpha,\beta}) = \{\gamma - \alpha : \gamma \in \operatorname{supp}(P) \cap [\alpha, \beta]\}.$$

Theorem 11.6. *Suppose that P is stable. Then the support of P is a jump system.*

Proof. Let $\alpha, \beta \in \operatorname{supp}(P)$ and let $\mu(P)$ be the change of variables

$$z_i \mapsto \begin{cases} -z_i^{-1} & \text{if } \alpha_i > \beta_i, \\ z_i & \text{otherwise} \end{cases}$$

and let $\gamma \in \mathbb{N}^n$ be sufficiently large so that $G(z) := z^\gamma P(\mu(z))$ is a polynomial. By inversion $P(z)$ is stable if and only if $G(z)$ is. Under this transformation $\alpha, \beta \in \text{supp}(P)$ are translated into $\alpha', \beta' \in \text{supp}(G)$ where $\alpha' \leq \beta'$. Thus we may assume $\alpha \leq \beta$ when checking the two-step axiom. Suppose there is a stable polynomial P and $\alpha, \beta \in \text{supp}(P)$ with $\alpha \leq \beta$ and α, β minimal with respect to $|\alpha - \beta|$ for which the two-step axiom is violated. Note that if P, α, β constitutes a counterexample then so does $P_{\alpha,\beta}, 0,\ \beta - \alpha \in \text{supp}(P_{\alpha,\beta})$. Hence we may assume that our minimal counterexample is of the form $\sum_\gamma a(\gamma) z^\gamma \in \mathbb{C}[z_1, \ldots, z_n]$ with $a(0), a(\beta) \neq 0$ where $\beta_i > 0$ for all $1 \leq i \leq n$ and $\text{supp}(P) \subseteq [0, \beta]$. Let $e_1, \ldots, e_n$ be the standard orthonormal basis of $\mathbb{R}^n$. Note that $e_1, \ldots, e_n \in \text{St}(0, \beta)$ since $\beta_i > 0$ for all $1 \leq i \leq n$. By symmetry we may assume $\sigma = e_1$ in the two-step axiom. Then by failure of the two-step axiom for this counterexample we have $e_1, 0 + \sigma + e_1 = 2e_1, 0 + \sigma + e_2 = e_1 + e_2, \ldots, 0 + \sigma + e_n = e_1 + e_n \notin \text{supp}(P)$. If there was $\xi \in (e_1, \beta) \cap \text{supp}(P)$ then there would be a smaller counterexample given by $P_{0,\xi}$. Hence if $\gamma \in \mathbb{N}^n$ with $\gamma_1 > 0$ then $a(\gamma) = 0$ unless $\gamma = \beta$. Let $\lambda > 0$ and $r = \beta_1^{-1} \sum_{i=2}^n \beta_i$. Then by scaling the univariate polynomial $P(\lambda^{-r} z, \lambda z, \ldots, \lambda z)$ is stable. Letting $\lambda \to 0$ we end up with the polynomial

$$a(0) + a(\beta) z^{|\beta|},$$

which is stable by Hurwitz's theorem. We cannot have $|\beta| \leq 2$, since then the two-step axiom would be valid, so $|\beta| \geq 3$. But this is a contradiction, since when $|\beta| \geq 3$ the equation

$$a(0) + a(\beta) z^{|\beta|} = 0 \implies z^{|\beta|} = -\frac{a(0)}{a(\beta)}$$

necessarily has non-real solutions contradicting stability of the univariate polynomial. $\square$

Corollary 11.7. *The support of a stable, multiaffine and homogenous polynomial is the set of bases of a matroid.*

Example 11.8. A finite subset $\mathcal{F}$ of $\mathbb{N}$ is a jump system if and only if it has holes of size at most 1 i.e,

$$i, k \in \mathcal{F}, i < k \text{ and } j \notin \mathcal{F} \text{ for all } i < j < k \implies k - i \leq 2.$$

One may ask whether all finite jump systems in $\mathbb{N}$ are supports of polynomials with the half-plane property? The answer is in fact Yes. If we assume that $0 \in \mathcal{F}$ then we claim there is a real-rooted polynomial P with simple zeros such that $\mathcal{F} = \text{supp}(P)$. The proof of this is by induction over the maximal element of $\mathcal{F}$. If $1 \in \mathcal{F}$ then

$$\mathcal{F}_1 = \{i - 1 : i \geq 1, i \in \mathcal{F}\}.$$

is a jump system with $0 \in \mathcal{F}_1$. Hence by induction, there is a real- and simple-rooted polynomial Q such that $\text{supp}(Q) = \mathcal{F}_1$. If $\epsilon > 0$ is small enough then $\epsilon + zQ$ will be real- and simple-rooted (amounts to vertically perturbing the real- and simple-rooted polynomial zQ by a very small positive amount). Moreover $\text{supp}(\epsilon + zQ) = \mathcal{F}$ since the factor z shifts back the support up one step and ϵ makes sure 0 belongs to the support, ensuring we get back $\mathcal{F}$. If $1 \notin \mathcal{F}$ then $\mathcal{F} = \{0\}$ or $2 \in \mathcal{F}$ since holes can be of size at most 1. In the latter case we have that

$$\mathcal{F}_2 = \{i - 2 : i \geq 2, i \in \mathcal{F}\}$$

is a jump system with $0 \in \mathcal{F}_2$. Hence by induction there is a real- and simple-rooted polynomial Q such that $\text{supp}(Q) = \mathcal{F}_2$. For small $\epsilon > 0$ the polynomial $-\epsilon Q(0) + z^2 Q$ will be real- and simple-rooted and $\text{supp}(-\epsilon Q(0) + z^2 Q) = \mathcal{F}$.

A well known property of real-rooted polynomials with non-negative coefficients is that the coefficients have no internal zeros, i.e, if $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is real-rooted and $a_i \geq 0$ for $0 \leq i \leq n$, then

$$i < j < k \text{ and } a_i a_k \neq 0 \implies a_j \neq 0.$$

(Ref: M Aissen, A Edrei, I.J Schoenberg, A Whitney, On the Generating Functions of Totally Positive Sequences, Proc. Nat. Acad. Sci. U.S.A., 37 (1951), pp. 303307). This extends to several variables.

Corollary 11.9. *Let P be a real stable polynomial with nonnegative coefficients. If $\alpha \leq \gamma \leq \beta$ and $\alpha, \beta \in supp(P)$ then $\gamma \in supp(P)$.*

Proof. If the corollary is false then there is a real stable polynomial P with nonnegative coefficients, and points $\alpha, \beta \in \mathbb{N}^n$ with $\alpha < \beta$, $\alpha, \beta \in \text{supp}(P)$ but $\alpha + e_i \notin \text{supp}(P)$ for some $1 \leq i \leq n$ with $\alpha + e_i < \beta$. By the two-step axiom there exists a $1 \leq j \leq n$ such that $\xi = \alpha + e_i + e_j \in \text{supp}(P)$. Now $P_{\alpha,\xi} = a + bz_j^2 + cz_i z_j$ with $a, b, c > 0$, $ac > 0$ is real stable. If $i = j$ then $P_{\alpha,\xi} = a + cz_i^2$ is not real stable, so we must have $i \neq j$. By letting $z_i = \lambda z$ and $z_j = \lambda^{-1} z$ and letting $\lambda \to \infty$ we have by Hurwitz's theorem that the univariate polynomial $a + cz^2$ is real stable which is a contradiction. □

Lemma 11.10. *If $\mathcal{J} \subset \mathbb{Z}^n$ is a finite jump system and $\alpha, \beta \in \mathcal{J}$ are maximal (or minimal) with respect to $\leq$, then $|\alpha| = |\beta|$.*

Proof. We argue by contradiction. Let M be the set of maximal elements β of $\mathcal{J}$ with $|\beta| = d$ maximal. Suppose further that $\beta \in M$ is of minimal L^1-distance to the set of all maximal (w.r.t $\leq$) elements α with $|\alpha| < d$. Let α be a maximal element that realizes the above distance to β. Clearly $\alpha_j > \beta_j$ for some j as otherwise α would not be a maximal element. Thus e_j is a step from β to α and $\beta + e_j \notin \mathcal{J}$ by maximality of β. Thus by the two-step axiom, $\beta' = \beta + e_j + s \in \mathcal{J}$ for some step s from $\beta + e_j$ to α. Since β is maximal, the non-zero coordinate in s is negative. Now $|\beta'| = |\beta|$, so β' is maximal (w.r.t $\leq$). However $|\beta' - \alpha| < |\beta - \alpha|$ since $\beta' \in M$ is a step closer to α than β, but this contradicts the minimality of $\beta \in M$. □

Definition 11.11. *(HPP and WHPP)*
*A matroid has the **half-plane property** (HPP) if the **bases generating polynomial***

$$P_{\mathcal{B}}(z) = \sum_{B \in \mathcal{B}} \prod_{j \in B} z_j$$

*is stable. A matroid has the **weak half-plane property** (WHPP) if there is a function $a : \mathcal{B} \to \mathbb{C} \setminus \{0\}$ such that*

$$\sum_{B \in \mathcal{B}} a(B) \prod_{j \in B} z_j$$

is stable.

By Lemma 6.11 there is no restriction in assuming that $a : \mathcal{B} \to \mathbb{R}_+$ [Why?].

Proposition 11.12. *All matroids representable over* $\mathbb{C}$ *have the weak half-plane property.*

Proof. Suppose that $v_1, \ldots, v_n \in \mathbb{C}^m$ realizes the matroid $\mathcal{M}$. We may assume that $m = r$, where r is the rank of $\mathcal{M}$. Let U be the matrix with $v_1, \ldots, v_n$ as columns. Then

$$\det(z_1 v_1 v_1^* + \cdots + z_1 v_n v_n^*) = \det(UZU^*) = \sum_{S, |S|=r} |U([r], S)|^2 z^S,$$

by the Binet-Cauchy theorem. Since $u_i u_i^*$ is positive semidefinite, the above polynomial is stable by Proposition 1.9. The support of the polynomal is the set of bases of $\mathcal{M}$. □

Corollary 11.13. *Graphic matroids have the half-plane property.*

Proof. By definition the maximal elements of graphic matroids are the maximal edge subsets which contain no cycles, that is, the spanning trees if G is connected. If G is not connected then the base generating polynomial naturally becomes the product of the base generating polynomial of each connected component with each component having a disjoint set of variables. We may therefore assume G is connected, in which case stability simply follows from Corollary 10.9. □