ON A CLASS OF EXTREMAL POLYMATROIDS

Abstract. We study a new family of extremal problems over a class $\mathcal{C}$ of polymatroids. The extremal numbers we define are partially motivated by [2], where they were used to establish bounds in counterexamples to stronger algebraic versions of the Generalized Lax Conjecture. We prove a global bound for all instances of our problem. In the rank two case we note a couple of interesting combinatorial consequences and compute exact values in some special cases.

1. Introduction

Matroids serve as a common generalization of objects from many different areas including graph theory, linear algebra and finite geometry. From a broad perspective one could therefore also view the extremal theory of matroids as a generalization of the extremal theories of either areas. Extremal problems on matroids have been studied by a number of authors, see surveys by Kung [5], Bonin [3] and references therein. In the words of Kung [5] it is reasonable to say that extremal matroid theory, and by extension extremal polymatroid theory, is predominantly concerned with the following set of questions:

Let $\mathcal{C}$ be a class of polymatroids satisfying given properties. Determine the size function

$$h(\mathcal{C}; n) = \max\{|P| : P \in \mathcal{C} \text{ and } \operatorname{rank}(P) = r\}.$$

and characterise the polymatroids of maximum size for each rank.

In this paper we are concerned with polymatroids up to some fixed maximum rank with prescribed spanning conditions that have the property of not containing a given number of spanning sets (of size equal to the maximal rank) confined to a subset of fixed size. The extremal numbers we define are interesting in their own generality, but they also form upper bounds to some intriguing families of extremal problems that arise by restricting the extremum over specific subclasses of polymatroids.

2. Preliminaries

Let E be a finite set. A *polymatroid* $P = (E, r)$ is given by a function $r : 2^E \to \mathbb{N}$ satisfying

(i) $r(\emptyset) = 0$,
(ii) $r(S) \leq r(T)$ whenever $S \subseteq T \subseteq E$,
(iii) r is *submodular*, i.e.,

$$r(S) + r(T) \geq r(S \cap T) + r(S \cup T),$$

for all $S, T \subseteq E$.

A *matroid* $M = (E, r)$ is a polymatroid for which $r(\{x\}) \leq 1$ for all $x \in E$. The *rank* of P is defined as $r(E)$. With abuse of notation we will also denote the rank of P simply by r. A subset $S \subseteq E$ is said to be *spanning* if $r(S) = r$. A *hyperplane*

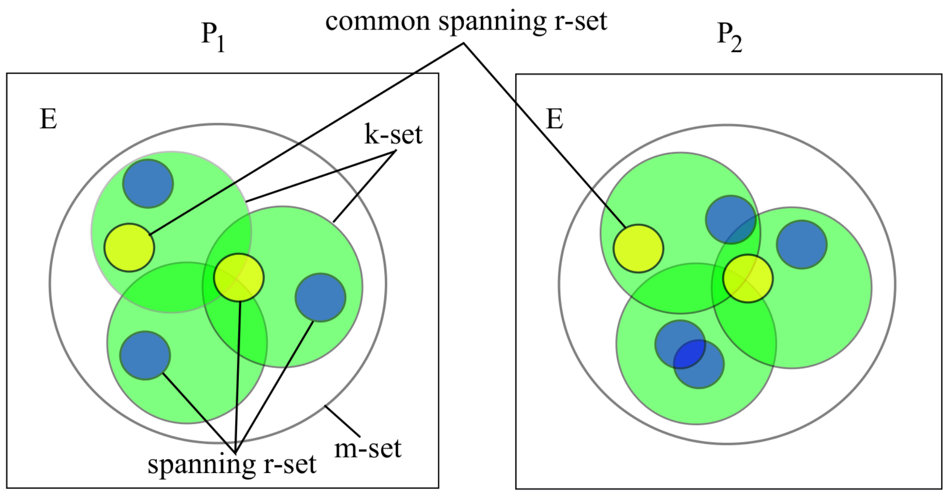

FIGURE 1. The sets involved in Definition 3.1

of P is a maximal subset of E that is not spanning. An element $x \in E$ is called a *loop* if $r(\{x\}) = 0$.

Below we list some examples of polymatroids that we shall mention again later in the context of our extremal number.

Example 2.1.

(i) Let V be a vector space of rank r and let $V_1, \dots, V_n$ be a collection of subspaces of V. Define $r : 2^{[n]} \to \mathbb{N}$ by $r(S) = \dim\left(\sum_{i \in S} V_i\right)$ for all $S \subseteq [n]$. Then $r : 2^{[n]} \to \mathbb{N}$ is a polymatroid on $[n]$ of rank r. In particular if $V_1, \dots, V_n$ are subspaces of dimension at most one, then we obtain a matroid of rank r on $[n]$.

(ii) Let P be a set of points and $L \subseteq 2^P$ a set of lines. A point-line configuration (P, L) defines a rank 3 matroid on P if and only if all lines in L pairwise meet in at most one point. Three points define a base in such a configuration if and only if the points are non-collinear (i.e not all contained in a single line).

(iii) Let G be a bipartite graph with vertex classes A and B of size n and r respectively. For a subset $S \subseteq A$ let $\Gamma(S)$ denote the set of neighbours of S in G. Then $r : 2^A \to \mathbb{N}$ given by $r(S) = |\Gamma(S)|$ defines a rank r polymatroid. In particular if $f : A \to B$ is a function, then $r(S) = |f(S)|$ defines a rank r matroid where $f(S)$ denotes the image of S.

Recall that a *hypergraph* H consists of a set $V(H)$ of *vertices* together with a set $E(H) \subseteq 2^{V(H)}$ of *hyperedges*. We say that a hypergraph H is r*-uniform* if all hyperedges have size r. The *complete* r*-uniform hypergraph* on n vertices $K_n^{(r)}$, is the r-uniform hypergraph on $[n]$ with all possible hyperedges.

3. Definition and basic properties

Definition 3.1. Let $\mathcal{C}$ be a class of polymatroids and let r, s, k, ℓ, m be nonnegative integers such that $k \geq r+1, m \geq r$ and $\ell \leq \binom{m}{r}$. The number $A^{\mathcal{C}}_{r,s}(k, \ell, m)$ is the least number n such that if $P_1, \dots, P_s \in \mathcal{C}$ are polymatroids of rank at most r on an n-element set E where all subsets $S \subseteq E$ of size k are spanning (for each P_i, $i = 1, \dots, s$), then there exists a subset $T \subseteq E$ of size m containing at least ℓ distinct r-subsets of T which are spanning in P_i for all $i = 1, \dots, s$.

We may think of r and s as fixed ambient parameters. If $\mathcal{C} = \mathcal{P}$, the set of all polymatroids, then we will abbreviate and write $A_{r,s}(k, \ell, m)$, instead of $A^{\mathcal{P}}_{r,s}(k, \ell, m)$. Clearly for any class $\mathcal{C} \subseteq \mathcal{P}$ we have $A^{\mathcal{C}}_{r,s}(k, \ell, m) \leq A_{r,s}(k, \ell, m)$. In particular this enables us to prove global estimates on problems over any given $\mathcal{C}$ by finding upper bounds on $A_{r,s}(k, \ell, m)$. We will therefore mainly be concerned with the numbers $A_{r,s}(k, \ell, m)$.

It is in place to justify the shape of our definition, especially given the introduction of so many parameters. The original source of motivation for studying the numbers $A_{r,s}(k, \ell, m)$ comes from the following lemma figuring in [2], used to establish bounds in counterexamples to stronger algebraic versions of the Generalized Lax Conjecture.

Lemma 3.1. *Let P_i, $i = 1, \dots, s$, be polymatroids on $[n]$ of rank at most r such that no hyperplane has more than $r+1$ elements. If $n \geq (2s+1)(r+1) - 1$, then there is a set T of size $r+1$ such that there are at least two r-subsets of T that are spanning in all P_i, $i = 1, \dots, s$.*

Remark 3.2. In our notation, Lemma 3.1 gives an upper bound for the number $A_{r,s}(r+2, 2, r+1)$.

The parameters s and ℓ are introduced in Definition 3.1 to strictly generalize the situation in Lemma 3.1. We also have a general bound for the numbers $A_{r,s}(k, \ell, m)$ via Theorem 4.2. However for many purposes it is more natural to focus on the special case $s = 1$ and $\ell = \binom{m}{r}$. As may be seen in Example 3.1 there are also many interesting extremal problems that arise by varying the polymatroid class $\mathcal{C}$.

It is not a priori clear that the numbers $A_{r,s}(k, \ell, m)$ exist. To merely show existence and to better illustrate the definition we will give a preliminary short proof using Ramsey theory. The Ramsey bound however will be improved significantly in Theorems 4.2 and 4.3 by better taking into account the polymatroid structure.

Recall that the *multicolour hypergraph Ramsey number* $R_r(t_1, \dots, t_s)$ is the least number n such that any s-colouring of the edges of the complete r-uniform hypergraph $K^{(r)}_n$ on $[n]$ contains a monochromatic i-coloured copy of $K^{(r)}_{t_i}$ for some $i \in [s]$. The well-definedness of this number follows from Ramseys theorem [6]. The numbers $R_r(t_1, \dots, t_s)$ grow rapidly and have known superexponential lower bound for $r \geq 3$ (see [4]). We may view a rank r polymatroid P as a complete r-uniform hypergraph on E where a hyperedge is coloured blue if it is spanning in P and coloured red otherwise. Thus rank r polymatroids on $R_r(k, m)$ elements suffices to guarantee a monochromatically blue m-subset $T \subseteq E$, since by definition we are prohibiting the existence of monochromatically red (non-spanning) k-subsets. The following argument generalizes this observation.

Proposition 3.3. *The number $A_{r,s}(k, \ell, m)$ is well-defined.*

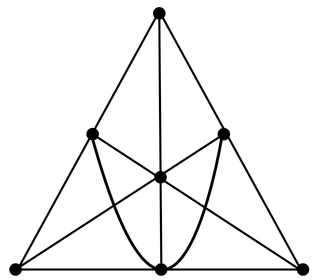

FIGURE 2. The Fano configuration. Every line contains 3 points and no 5 points lie in general position. Thus $A^{\mathcal{G}}_{3,1}\left(4, \binom{5}{3}, 5\right) > 7$.

Proof. Colour an r-uniform hypergraph on E using 2^s colours indexed by $\mathbf{c} = (c_1, \dots, c_s) \in \{0,1\}^s$. We label a hyperedge e by 1 with respect to P_i if e is spanning in P_i, and label e by 0 otherwise. We colour a hyperedge e with colour $\mathbf{c}$ if e is labelled by c_i with respect to P_i for all $i = 1, \dots, s$. We require an m-subset $T \subseteq E$ containing at least ℓ common r-subsets in colour $\mathbf{c} = \mathbf{1}$. It follows that

$$A_{r,s}(k, \ell, m) \leq A_{r,s}\left(k, \binom{m}{r}, m\right) \leq R_r(\underbrace{k, \cdots, k}_{2^s-1}, m)$$

so the number $A_{r,s}(k, \ell, m)$ is well-defined by the existence of Ramsey numbers. $\square$

Below we give examples of interpretations of our extremal number for some interesting polymatroid classes $\mathcal{C}$. Clearly since the examples below concern extremal problems on a more restricted class, we may only get a crude upper bound to the problems using the numbers $A_{r,s}(k, \ell, m)$. Nevertheless the same setup could be used to study these problems in isolation using the additional structure contained in the class. The numbers $A^{\mathcal{C}}_{r,s}(k, \ell, m)$ (as far as we know) have not been studied for any polymatroid class $\mathcal{C}$, not even for the natural class of representable matroids (see Example 3.1 (i)).

Example 3.1.

(i) Let $\mathcal{R}$ denote the class of representable matroids in Example 2.1 (i). Then any $r \times n$ matrix, where $n \geq A^{\mathcal{R}}_{r,1}\left(k, \binom{m}{r}, m\right)$, in which any k columns have full rank must contain a subset of m columns in which any r columns have full rank.

(ii) Let $\mathcal{G}$ denote the class of matroids in Example 2.1 (ii). Any point-line configuration in $\mathcal{G}$ of order at least $A^{\mathcal{G}}_{3,1}\left(k, \binom{m}{3}, m\right)$ having at most $k-1$ collinear points, contains m points in general position (i.e no three of which are collinear).

(iii) Let $\mathcal{F}$ denote the class of matroids in Example 2.1 (iii). For any set A with at least $N^{\mathcal{F}}_{r,1}\left(k, \binom{m}{r}, m\right)$ elements and any set B with r elements we have that any function $f : A \to B$ which is surjective when restricted to any k-subset of A, must contain a subset of size m on which f restricts to a bijection on any r-subset.

(iv) Let $\mathcal{M}$ denote the class of matroids. Then any rank r matroid with at least $A^{\mathcal{M}}_{r,1}\left(k, \binom{m}{r}, m\right)$ elements in which the maximum size of a hyperplane is $k-1$, must contain a copy of the uniform matroid $U_{r,m}$ of rank r with m elements.

Clearly the number $A_{r,s}(k,\ell,m)$ is monotone non-decreasing in the parameters r,s,k,ℓ and monotone non-increasing in the parameter m, keeping remaining parameters fixed (and valid). Below we show that the numbers must at least be strictly increasing in the parameter k.

Lemma 3.4. *If $k' < k$, then $A_{r,s}(k',\ell,m) < A_{r,s}(k,\ell,m)$.*

Proof. It suffices to consider $k' = k-1$. Let $P_1,\dots,P_s$ be polymatroids on E of size $A_{r,s}(k-1,\ell,m)-1$ with rank at most r for which any $(k-1)$-subset of E is spanning in P_i (for each $i = 1,\dots,s$) and contains no m-subset of E in which there exists at least ℓ distinct r-subsets that are spanning in P_i for all $i = 1,\dots,s$. Extend the polymatroids P_i to polymatroids P'_i of the same rank by adding a new element $x \notin E$ which is a loop in P'_i for all $i = 1,\dots,s$. Then $P'_1,\dots,P'_s$ are rank at most r-polymatroids on $A_{r,s}(k-1,\ell,m)$ elements with every k-element subset of $E \sqcup x$ spanning. For this set of polymatroids there is no m-subset of $E \sqcup x$ in which there exists at least ℓ distinct r-subsets which are spanning in P'_i for each $i = 1,\dots,s$. Hence $A_{r,s}(k',\ell,m) < A_{r,s}(k,\ell,m)$.

□

Lemma 3.5. *Let $\mathcal{LF}$ denote the class of all loop-free polymatroids, then*

$$A^{\mathcal{LF}}_{r,1}(k,\ell,m) = A_{r,1}(k,\ell,m).$$

Proof. Consider polymatroids P of rank at most r with non-empty set of loops L ($|L| \le k$), for which any k-subset is spanning in P. This implies that every $(k-|L|)$-subset of $P \setminus L$ must be spanning. Let $\mathcal{L}_i$ denote the class of all polymatroids with i loops. Then it follows that

$$A^{\mathcal{L}_i}_{r,1}(k,\ell,m) = i + A_{r,1}(k-i,\ell,m).$$

By Lemma 3.4 we have that

$$i + A_{r,1}(k-i,\ell,m) \le A_{r,1}(k,\ell,m).$$

Since $\mathcal{P} = \bigcup_{i\ge 1} \mathcal{L}_i \cup \mathcal{LF}$, the statement follows.

□

Lemma 3.6. *Let $\mathcal{P}_r$ denote the class of all rank r polymatroids, then*

$$A^{\mathcal{P}_r}_{r,s}(k,\ell,m) = A_{r,s}(k,\ell,m).$$

Proof. Let $P_i = (r_i, E)$ be polymatroids of rank at most r on E where $|E| = A_{r,s}(k,\ell,m)-1$ such that any k-subset of E is spanning in P_i (for each $i = 1,\dots,s$) but for which there is no m-subset of E in which at least l number of r-subsets are spanning in P_i for all $i = 1,\dots,s$. Define $P'_i = (r'_i, E)$ where

$$r'_i(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ r_i(S) - r_i(E) + r & \text{otherwise} \end{cases}$$

for all $S \subseteq 2^E$ and $i = 1,\dots,s$. It is easy to check that $P'_1,\dots,P'_s$ are indeed rank r-polymatroids. Moreover it is clear from the definition that we have a one-to-one correspondence between the spanning sets of P_i and P_i' for all $i = 1, \dots, s$. Hence the statement follows.

□

4. Bounds

In this section we establish bounds for various instances of the numbers $A_{r,s}(k, \ell, m)$. The following lemma can be seen as the bottleneck in our upper bound estimate of $A_{r,s}(k, \ell, m)$ in Theorem 4.2. The lemma appeared in [2], but we include it here in its rightful context for completeness.

Lemma 4.1. *For $n, r, c \geq 1$, let $\mathcal{P}(n, r, c)$ be the family of all rank at most r polymatroids on n elements such that each hyperplane has at most $r-1+c$ elements. If $\alpha(n, r, c)$ denotes the maximal number of non-spanning sets of size r taken over all polymatroids in $\mathcal{P}(n, r, c)$, then*

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$$\alpha(n, r, c) \leq c\binom{n}{r-1}. \tag{4.1}$$

Proof. If $r = 1$, then each hyperplane has at most c elements, i.e., there are at most c loops so that $\alpha(n, r, c) = c$ as desired. The proof is by induction over $n \geq 1$ where $r \geq 1$. The lemma is trivially true for $n = 1$.

Let $P \in \mathcal{P}(n, r, c)$, where $n, r \geq 2$. If $n \leq r$, then (4.1) is trivially true. Assume $n > r$. Let i be a non-loop of P. If $r(E \setminus i) < r(E)$, then $E \setminus i$ is a hyperplane and hence $n - 1 \leq r - 1 + c$, so that $\binom{n}{r} \leq c\binom{n}{r-1}$. Hence we may assume $r(E \setminus i) = r(E) > 0$.

If S is a non-spanning r-set of P, then either S is a non-spanning r-set of $P \setminus i$, or $S \setminus i$ is a non-spanning $(r-1)$-set of P/i. Hence $P \setminus i \in \mathcal{P}(n-1, r, c)$ and $P/i \in \mathcal{P}(n-1, r-1, c)$, and thus

$$\begin{aligned}\alpha(n, r, c) &\leq \alpha(n-1, r, c) + \alpha(n-1, r-1, c)\\ &\leq c\binom{n-1}{r-1} + c\binom{n-1}{r-2} = c\binom{n}{r-1},\end{aligned}$$

by induction. □

We may now improve upon the Ramsey bound from Proposition 3.3.

Theorem 4.2. *If r, s, k, ℓ, m are nonnegative integers such that $k \geq r+1, r \leq m$ and $\ell \leq \binom{m}{r}$, then*

$$A_{r,s}(k, \ell, m) \leq r - 1 + sr(k-r)\frac{\binom{m}{r}}{\binom{m}{r} - \ell + 1}.$$

Proof. Let $P_1, \dots, P_s$ be polymatroids on a set $E = [n]$ of rank at most r with hyperplanes of size at most $k - 1$. Suppose

$$n > r - 1 + sr(k-r)\frac{\binom{m}{r}}{\binom{m}{r} - \ell + 1}$$

and that there exists no m-subset of E containing at least l common r-subsets which are spanning in P_i for all $i = 1, \dots, s$. Let

$$A = \left\{(S, T) : \binom{[n]}{r} \ni S \subset T \in \binom{[n]}{m}, S \text{ is not spanning in } P_i \text{ for some } i \in [s]\right\}.$$

Then

$$|A| \geq \left(\binom{m}{r} - \ell + 1\right)\binom{n}{m}$$

since each $T \in \binom{[n]}{m}$ contains most $\ell - 1$ distinct r-subsets which are spanning in P_i for all $i = 1, \ldots, s$. Furthermore by Lemma 4.1 we have

$$\begin{aligned}
|A| &= \#\left\{S \subseteq \binom{[n]}{r} : S \text{ is not spanning in } P_i \text{ for some } i \in [s]\right\} \cdot \binom{n-r}{m-r} \\
&\leq s\alpha(n, r, k-r)\binom{n-r}{m-r} \\
&\leq s(k-r)\binom{n}{r-1}\binom{n-r}{m-r}.
\end{aligned}$$

Hence

$$\left(\binom{m}{r} - \ell + 1\right)\binom{n}{m} \leq s(k-r)\binom{n}{r-1}\binom{n-r}{m-r}.$$

Solving for n gives $n \leq r - 1 + sr(k-r)\binom{m}{r} / \left(\binom{m}{r} - \ell + 1\right)$, a contradiction.

□

When requiring m-subsets with very densely populated spanning r-subsets, such as the case $\ell = \binom{m}{r}$, the following theorem essentially gives an $O(rm)$ improvement over the bound in Theorem 4.2.

Theorem 4.3. *If r, s, k, ℓ, m are nonnegative integers such that $k \geq r + 1, r \leq m$ and $r_* = \min\{r - 1, \lfloor (m-1)/2 \rfloor\}$, then*

$$A_{r,s}\left(k, \binom{m}{r}, m\right) \leq m + s(k - r_* - 1)\binom{m-1}{r_*}.$$

Proof. Let $P_1, \ldots, P_s$ be polymatroids on a set $E = [n]$ of rank at most r with hyperplanes of size at most $k - 1$. Suppose a maximal (with respect to inclusion) subset $S \subseteq E$ of size $N < m$ have been chosen such that any r-subset of S has full rank with respect to $P_1, \ldots, P_s$. By maximality of S the elements in $E \setminus S$ must each belong to a hyperplane of P_i containing at most $r - 1$ elements of S for some $i = 1, \ldots, s$. The number of such hyperplanes is at most $s\binom{N}{r_*(N)}$ where $r_*(N) = \min(r - 1, \lfloor N/2 \rfloor)$. Moreover each such hyperplane may contain at most $k - r_*(N) - 1$ elements in E that do not belong to S given that the size of each hyperplane is bounded by $k - 1$. Therefore

$$n \leq N + s(k - r_*(N) - 1)\binom{N}{r_*(N)}.$$

Hence taking $n \geq m + s(k - r_*(m-1) - 1)\binom{m-1}{r_*(m-1)}$ ensures the existence of an m-set with the required properties.

□

Below we give a constructive lower bound for the numbers $A_{r,s}\left(k, \binom{m}{r}, m\right)$.

Theorem 4.4. *If r, s, k, m are nonnegative integers such that $k, m \geq r + 1$, then*

$$\max\left\{m, 1 + \left\lfloor \frac{k-1}{2(r-1)} \right\rfloor (s+1)(m-1)\right\} \leq A_{r,s}\left(k, \binom{m}{r}, m\right).$$

Proof. Note that m is the trivial lower bound. Suppose $k-1 \geq 2(r-1)$. Let $N = \frac{1}{2}(s+1)(m-1)$ and let $E_1, \ldots, E_N$ be disjoint element classes of size $2\left\lfloor \frac{k-1}{2(r-1)} \right\rfloor$ (the smallest even integer less than $(k-1)/(r-1)$). Let M be the rank r matroid on $E = E_1 \sqcup \cdots \sqcup E_N$ with base set $\mathcal{B}$ given by

$$\{\{x_1, \ldots, x_r\} \subset E : x_i \in E_{\sigma(i)} \text{ for all } i = 1, \ldots, r \text{ for some injection } \sigma : [r] \to [N]\}.$$

It is not difficult to verify that M is a matroid by checking the base exchange axiom. Such a matroid is called a partial transversal matroid. By construction any k-subset of E contains an element of $\mathcal{B}$. Hence every k-subset is spanning in M.

We now define matroids $M_1, \ldots, M_s$ on E using the construction above to realize the lower bound in the theorem. Since $|E_i|$ is even we may partition $E_i = E_i' \sqcup E_i''$ where E_i' and E_i'' are disjoint subsets of equal size. Now let $E_1^{(j)}, \ldots, E_N^{(j)}$ denote the disjoint element classes (to be defined) of the matroid M_j on E for $j = 1, \ldots, s$. We partition the classes $E_1, \ldots, E_N$ consecutively into $m-1$ blocks of $t = \frac{1}{2}(s+1)$ classes each. In each block we create obstructions by ensuring that every pair of elements appear together in at least one element class $E_i^{(j)}$ for some $1 \leq j \leq s$. This implies that one may pick at most one point out of the elements in each block to form a subset of E in which all r-subsets simultaneously belong to the base sets $\mathcal{B}_i$ of the matroids M_i for all $i = 1, \ldots, s$. However such a set can have size at most $m-1$ since there are $m-1$ blocks and hence the theorem follows.

We define the classes $E_i^{(j)}$ $(j = 1, \ldots, s)$ in the first block $(i = 1, \ldots, t)$ by considering the set A of all $\binom{s+1}{2}$ pairs of subsets in $\{E_1', E_1'', \ldots, E_t', E_t''\}$ (note that $st = \binom{s+1}{2}$). The set A may be decomposed into s disjoint families of disjoint pairs. This follows from a well-known fact in graph theory that the complete graph K_{2t} admits a decomposition into 1-factors (i.e. perfect matchings). Each of these s families of t disjoint pairs make up the definition of the element classes $E_i^{(j)}$ for the first block in M_j for $i = 1, \ldots, t$ and $j = 1, \ldots, s$. We construct the element classes for the remaining blocks similarly. This definition ensures that all elements within each block appears together at least once in the element class $E_i^{(j)}$ of some matroid M_j and can therefore not be selected together. Hence the lower bound follows. □

5. Special cases

The upper bound in Theorem 4.2 is not sharp, however for $r = 2$ and $\ell = \binom{m}{2}$ Theorems 4.3 and 4.4 together imply an essentially tight bound. Through a graph theoretic characterization of rank 2 polymatroids we find interesting consequences for the existence of common partial transversals of partitioning families with bounded block size, or equivalently, the clique number of an intersection of complete multipartite graphs with bounded class size.

There exists a bijection between loop-free rank 2 matroids and complete multipartite graphs, originally due to Acketa [1]. Given a rank 2 matroid M we may associate a graph G where $V(G) = E(M)$ and $E(G) = \mathcal{B}(M)$, in other words $\{x, y\} \in \binom{E(M)}{2}$ forms an edge in G if and only if $\{x, y\}$ is spanning in M. A graph which corresponds to a rank 2 matroid is said to be *matroidic*. Denote by G' the graph which remains after deleting all degree zero vertices.

Theorem 5.1 (Acketa [1])**.** *A graph G is matroidic if and only if G' is a complete multipartite graph.*

Remark 5.2. Note that the degree zero vertices in a matroidic graph G correspond to the loops of the matroid. Note that Theorem 5.1 also implies a characterization of polymatroidic graphs, since a rank 2 element is simply a vertex class of size one in G'.

Let $\mathcal{A} = \{\mathcal{A}_1, \dots, \mathcal{A}_s\}$ be a family of s partitions of a set E where $\mathcal{A}_i$ has blocks $A_1^{(i)}, \dots, A_{k_i}^{(i)}$ for $i = 1, \dots, s$. A set $X = \{x_1, \dots, x_m\} \subseteq E$ is a *common partial transversal of size* m if there exists injections $\sigma_i : [m] \to [k_i]$ for $i = 1, \dots, s$ such that

$$x_j \in \bigcap_{i=1}^{s} A_{\sigma_i(j)}^{(i)}$$

for all $j = 1, \dots, m$.

Example 5.1. Let $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ be the family of partitions of $[16]$ given by

$$\begin{aligned}
\mathcal{A}_1 &= \{\{1,2,3,4\}, \quad \{5,6,7,8\}, \quad \{9,10,11,12\}, \quad \{13,14,15,16\}\} \\
\mathcal{A}_2 &= \{\{1,2,5,6\}, \quad \{3,4,7,8\}, \quad \{9,10,13,14\}, \quad \{11,12,15,16\}\} \\
\mathcal{A}_3 &= \{\{1,2,7,8\}, \quad \{3,4,5,6\}, \quad \{9,10,15,16\}, \quad \{11,12,13,14\}\}.
\end{aligned}$$

The family $\mathcal{A}$ has a (maximal) common partial transversal of size 2, e.g $\{1, 9\}$ (cf. Corollary 5.3).

Corollary 5.3. *A family of partitions $\mathcal{A} = \{\mathcal{A}_1, \dots, \mathcal{A}_s\}$ of $[n]$ with maximum block size k contains a common partial transversal of size at least $\lfloor \frac{n-1}{s(k-1)+1} \rfloor + 1$.*

Proof. Construct a family of complete multipartite graphs $G_1, \dots, G_s$ from the partitions $\mathcal{A}_1, \dots, \mathcal{A}_s$ with vertex classes given by the partition blocks. By Theorem 5.1 the graphs G_i are matroidic and every $(k+1)$-subset is spanning since each vertex class has size at most k. Thus Theorem 4.3 implies that any family of partitions of a set E of size at least $n = m + s(k-1)(m-1)$ and with blocks of size at most k, contains a common partial transversal of size m. Solving for m proves the corollary. □

In similar vein we have the following interpretation.

Corollary 5.4. *If $G_1, \dots, G_s$ be complete multipartite graphs on $[n]$ with vertex classes of size at most k, then*

$$\omega\left(\bigcap_{i=1}^{s} G_i\right) \geq \left\lfloor \frac{n-1}{s(k-1)+1} \right\rfloor + 1,$$

where $\omega(G)$ denotes the clique number of a graph G.

Corollary 5.5.

$$N_{2,1}\left(k, \binom{m}{2}, m\right) = (k-1)(m-1) + 1.$$

Proof. Follows directly from Theorem 4.3 and Theorem 4.4. Alternatively note that by Lemma 3.5 we may restrict ourselves to loop-free polymatroids. Now use the characterization of rank 2 matroids in Theorem 5.1 (and by extension, rank 2 polymatroids) to see that the extremal polymatroids are constructed via complete multipartite graphs with $m-1$ independent classes of size $k-1$. □

For polymatroids of rank greater than 2 there is no simple characterization similar to Theorem 5.1. Nevertheless we may attempt to find exact values for other notable specializations. Closest to hand are the 'diagonal' numbers $A^{\mathcal{C}}_{r,1}(r+1,r+1,r+1)$. Below we find an exact answer to $A^{\mathcal{M}}_{3,1}(4,4,4)$.

Theorem 5.6. *We have*

$$A^{\mathcal{M}}_{3,1}(4,4,4)=6,$$

where $\mathcal{M}$ denotes the class of all matroids.

Proof. Let $e_1,\dots,e_6$ denote the standard basis of $\mathbb{R}^6$. By considering the vectorial matroid M on $\{x_1,\dots,x_5\}$ where $x_i=e_i$ for $i=1,\dots,3$, $x_4=e_3$ and $x_5=e_1+e_2$, we see that $A^{\mathcal{M}}_{3,1}(4,4,4)>5$.

Suppose for a contradiction that M is a matroid of rank at most 3 on 6 elements in which any 4 elements are spanning, but for which there is no subset of 4 elements in which all 3-subsets are spanning. Clearly since

$$A^{\mathcal{M}}_{r(M),1}(4,4,4)\le A^{\mathcal{M}}_{3,1}(4,4,4),$$

we may assume $r(M)=3$. Thus M contains a base $B=\{x_1,\dots,x_3\}$. It follows by assumption that the remaining three elements $y_1,y_2,y_3\in E(M)\setminus B$ must belong to distinct hyperplanes containing 2 of the elements in B. For elements $z_1,\dots,z_k\in E(M)$ we adopt the shorthand $r(z_1,\dots,z_k)=r(\{z_1,\dots,z_k\})$. Thus without loss of generality assume

$$r(x_2,x_3,y_1)=2,\ r(x_1,x_3,y_2)=2,\ r(x_1,x_2,y_3)=2.$$

If $r(x_2,x_3,y_i)=2$ for some $i\in\{2,3\}$, then by submodularity we have

$$r\left(\{x_2,x_3,y_i\}\cup\{x_2,x_3,y_1\}\right)+r\left(\{x_2,x_3,y_i\}\cap\{x_2,x_3,y_1\}\right)\le r\left(x_2,x_3,y_i\right)+r\left(x_2,x_3,y_1\right),$$

where the left hand side equals $3+2$ and the right hand side equals $2+2$, a contradiction. Thus $r(x_2,x_3,y_i)=3$ for all $i\in\{2,3\}$. Hence without loss of generality

$$r(x_3,y_2,y_3)=2,$$

since not all 3-subsets of the 4-set $\{x_2,x_3,y_2,y_3\}$ can be spanning by assumption. Using submodularity on the sets

$$\{x_3,y_2,y_3\}\text{ and }\{x_1,x_3,y_2\}$$

we deduce that the rank of their intersection satisfies

$$r(x_3,y_2)\le 1\text{ and hence }r(x_3,y_2)=1$$

By an argument similar to above we have (without loss of generality) that

$$r(x_2,y_1)=1.$$

Using submodularity a final time we get

$$r\left(\{x_2,y_1\}\cup\{x_3,y_3\}\right)+r\left(\{x_2,y_1\}\cap\{x_3,y_3\}\right)\le r\left(x_2,y_1\right)+r\left(x_3,y_2\right),$$

where the left hand side equals $3+0$ and the right hand side is equals $1+1$, a contradiction. Hence $A^{\mathcal{M}}_{3,1}(4,4,4)=6$.

$\square$

6. Remarks and open questions

There exist plenty of further questions concerning the numbers $A^{\mathcal{C}}_{r,s}(k,\ell,m)$, especially since they to our knowledge have not been studied elsewhere in the literature (even in special form). To begin with it would be good to understand if working on the level of polymatroids is unnecessary in the sense that there always exist extremal matroids.

Question 1. *Is it true that $A^{\mathcal{M}}_{r,s}(k,\ell,m)=A_{r,s}(k,\ell,m)$, where $\mathcal{M}$ denotes the class of matroids?*

Furthermore the estimate in Theorem 4.2 is not sharp. Ideally we would like to find an explicit characterization of the extremal polymatroids, but more likely improve the bounds via for example probabilistic methods.

The inflexibility of the parameter k is partly the reason we have not been able to prove any interesting recursive inequalities (other than monotonicity).

Question 2. *Do the numbers $A_{r,s}(k,\ell,m)$ satisfy any reasonable recursions or recursive inequalities?*

Computing exact answers to some of these numbers for small or specialized parameters is also helpful. Closest to hand are the numbers $A_{r,s}(k,1,m)$ and the diagonal numbers $A_{r,1}(r+1,r+1,r+1)$. Finally we would like to remark that the problem remains interesting for many subclasses of polymatroids (see Example 3.1). The additional structure in a more restricted class may render the problem more manageable.

Question 3. *Are there any interesting classes $\mathcal{C}$ for which $A^{\mathcal{C}}_{r,s}(k,\ell,m)$ have better estimates?*

Especially good would be to understand these numbers for the class of representable matroids. For $\ell=\binom{m}{r}$ this is a statement about the existence of an $r\times m$ submatrix with r-wise independent columns. In information theory literature such matrices are also known as parity-check matrices for linear codes with minimal distance at least $r+1$.

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