Plücker Correction Problem: Analysis and Improvements in Efficiency*

João Cardoso[1,2], Pedro Miraldo[3], and Helder Araujo[1]

[1]*Institute for Systems and Robotics, Universidade de Coimbra, Portugal*
[2]*Coimbra Institute of Engineering, Instituto Politécnico de Coimbra, Portugal*
[3]*Institute for Systems and Robotics, Instituto Superior Técnico, Universidade de Lisboa, Portugal*
jocar@isec.pt, pmiraldo@isr.tecnico.ulisboa.pt, and helder@isr.uc.pt

Abstract

A given six dimensional vector represents a 3D straight line in Plücker coordinates if its coordinates satisfy the Klein quadric constraint. In many problems aiming to find the Plücker coordinates of lines, noise in the data and other type of errors contribute for obtaining 6D vectors that do not correspond to lines, because of that constraint. A common procedure to overcome this drawback is to find the Plücker coordinates of the lines that are closest to those vectors. This is known as the Plücker correction problem. In this article we propose a simple, closed-form, and global solution for this problem. When compared with the state-of-the-art method, one can conclude that our algorithm is easier and requires much less operations than previous techniques (it does not require Singular Value Decomposition techniques).

1 Introduction

In many 3D vision problems that range from camera calibration to robot navigation, it is required to represent 3D straight lines. Plücker coordinates are one of the most used formulations [4]. Vectors of Plücker coordinates are built-up by stacking both direction and moment of the respective lines (both 3D vectors), giving a 6D vector. Direction and moment vectors must satisfy the so-called Klein quadric constraint, that is, they need to be orthogonal to each other. In this paper we address the problem of estimating Pücker coordinates from general (unconstrained) $\mathbb{R}^6$ vectors. This is called the Plücker correction problem.

For many reasons, especially when considering data with noise, it is hard to include the orthogonal constraint

in the estimator (frequently, it requires non-linear procedures). Some authors do not consider this constraint on their methods or propose optimization techniques involving this constraint but, to avoid unnecessary computational effort, non-linear procedures are stopped before fulfilling the respective constraint. Several examples can be found in the literature, e.g. camera models (mapping between pixels and 3D straight lines) [12]; triangulation of 3D lines [10]; structure-from-motion using lines [6, 8]; and 3D reconstruction of lines using a single image of a non-central catadioptric camera [8]. Thus, to get Plücker coordinates on these cases, a Plücker correction needs to be applied.

The state-of-the-art method for solving the Plücker correction problem is due to Bartoli and Sturm at [6], hereafter called BS method. They find the closest Plücker coordinates (in Euclidean sense) from an unconstrained six-dimensional vector by orthogonally projecting the input vector onto the Klein quadric (and so it verifies the orthogonal constraint). Their approach involves SVD decompositions and can be found in [6, p. 425]. It should be noticed that in the description of the BS Plücker correction algorithm (Table 2 in [6]) there is a typo in the entries of the matrix T. The correct matrix T is given by

$$T = \begin{bmatrix} z_{12} & z_{22} \\ z_{21} & -z_{11} \end{bmatrix}. \tag{1}$$

We performed a detailed analysis of the proof of the BS method ([6, p. 425]) and propose some clarifications. For the proof of BS method the following identity is used:

$$\|\mathbf{U}\mathbf{A} - \mathbf{B}\| = \|\mathbf{A} - \mathbf{U}^T\mathbf{B}\| \tag{2}$$

where $\mathbf{A}$ and $\mathbf{B}$ are matrices of sizes $n \times n$ and $m \times n$, respectively, with $m > n$, and $\mathbf{U}$ is an $m \times n$ matrix with orthonormal columns (i.e . $\mathbf{U}^T\mathbf{U} = \mathbf{I}$, but $\mathbf{U}\mathbf{U}^T \neq \mathbf{I}$). This identity is related with the invariance of the Frobenius norm for left multiplication by matrices with orthonormal columns.

However, (2) does not hold in general. It is valid if, and only if, the space of columns of $\mathbf{B}$ coincides with the space of columns of $\mathbf{U}$, in which case $\mathbf{B} = \mathbf{UF}$, for some matrix $\mathbf{F}$ of size $n \times n$. We notice that, in the proof of the BS method, this requirement is not explicitly mentioned.

Paper [15] also addresses the same problem. However they do not address the problem of the computational efficiency and the proposed approach is quite different from the method described in this paper. Moreover, even if the solution proposed in [15] is indeed a global minimum, there is no formal proof of such a fact. We shall recall that Lagrange multipliers yield only local minima, unless the existence of a global minimum is guaranteed.

1.1 Notations and Problem Definition

Column vectors are represented by bold small letters (*e.g.* $\mathbf{a} \in \mathbb{R}^n$ for an n-dimensional vector). Bold capital letters denote matrices (*e.g.* $\mathbf{A} \in \mathbb{R}^{n \times m}$ for an $n \times m$ matrix). Regular small letters denoted zero dimensional elements (*e.g.* a). $\|.\|$ denotes the Frobenius norm for matrices or the 2-norm for vectors. Recall that, for any matrix $\mathbf{A}$, the Frobenius norm is given by $\|\mathbf{A}\|^2 = \text{trace}(\mathbf{A}^T\mathbf{A})$ and, for any vector $\mathbf{u}$, the 2-norm satisfies $\|\mathbf{u}\|^2 = \mathbf{u}^T\mathbf{u}$.

The Plücker coordinates of a 3D straight line $\mathcal{G}$ can be represented by a six dimensional vector:

$$\mathcal{G} \sim (\mathbf{u}^T, \mathbf{v}^T) \in \mathbb{R}^6, \tag{3}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are, respectively, the direction and moment of the line, verifying the Klein quadric constraint

$$\mathbf{u}^T\mathbf{v} = \mathbf{0}, \tag{4}$$

see [4].

Let $\mathbf{a}$ and $\mathbf{b}$ be given vectors in $\mathbb{R}^3$ (not necessarily satisfying the orthogonality constraint), and assume that $\mathbf{x}$ and $\mathbf{y}$ denote vectors in $\mathbb{R}^3$. Mathematically, the Plücker correction problem corresponds to solving the nonlinear constrained optimization problem formulated as

$$\min_{\mathbf{x}^T\mathbf{y}=\mathbf{0}} \|[\mathbf{a}\ \mathbf{b}] - [\mathbf{x}\ \mathbf{y}]\|^2. \tag{5}$$

While the objective function

$$f(\mathbf{x}, \mathbf{y}) \doteq \|\mathbf{a} - \mathbf{x}\|^2 + \|\mathbf{b} - \mathbf{y}\|^2, \tag{6}$$

is convex, the Klein quadric constraint $\mathbf{x}^T\mathbf{y} = 0$ is not. The optimization problem (5) belongs to a class of non-convex problems known in the literature as quadratically constrained quadratic programs. This means, in particular, that the existence of a global minimum may be a non trivial problem.

The main goals of this paper are to prove the existence of a global minimum for (5) and to give an explicit formula for computing such a minimum. Our approach is essentially based on the application of the classical Lagrange multipliers to the constrained problem (5). It does not involve singular value decompositions which turns it faster than the BS method. Our results are supported by mathematical proofs and numerical experiments. In addition our method is designed to deal with general n-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$, and not exclusively with 3-dimensional vectors.

1.2 The Explicit Formula

Let us consider two general 3D vectors $\mathbf{a}$ and $\mathbf{b}$, such that $\mathbf{a} \neq \pm\mathbf{b}$ and $\mathbf{b} \neq \mathbf{0}$. As it will be shown in the following sections, the proposed solution $(\mathbf{x}_*, \mathbf{y}_*)$ for the global minimum of (5) is given by

$$\mathbf{x}_* = \frac{1}{1-\alpha^2}(\mathbf{a} - \alpha\mathbf{b}) \quad \text{and} \quad \mathbf{y}_* = \frac{1}{1-\alpha^2}(\mathbf{b} - \alpha\mathbf{a}), \tag{7}$$

where

$$\alpha \doteq \frac{2p}{q + \sqrt{q^2 - 4p^2}},$$
$$\text{with } p \doteq \mathbf{a}^T\mathbf{b} \quad \text{and} \quad q \doteq \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2. \tag{8}$$

2 Plücker Correction using Lagrange Multipliers

Suppose again that $\mathbf{a}$ and $\mathbf{b}$ satisfy $\mathbf{a} \neq \pm\mathbf{b}$ and $\mathbf{b} \neq \mathbf{0}$, and let $\mathcal{L}(\mathbf{x}, \mathbf{y}, \lambda)$ denote the Lagrangian function associated to (5). Some calculation yields

$$\mathcal{L}(\mathbf{x}, \mathbf{y}, \lambda) = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\,\mathbf{a}^T\mathbf{x} - $$
$$ - 2\,\mathbf{b}^T\mathbf{y} + \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\lambda\,\mathbf{x}^T\mathbf{y}, \tag{9}$$

where the real number λ is the Lagrange multiplier, $\mathbf{x}$ and $\mathbf{y}$ are the aimed solutions. The partial derivatives of the Lagrangian with respect to $\mathbf{x}$, $\mathbf{y}$ and λ are (for formulae of derivatives with respect to vectors see [1])

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}} = -2\mathbf{a} + 2\mathbf{x} + 2\lambda\mathbf{y} \tag{10}$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{y}} = -2\mathbf{b} + 2\mathbf{y} + 2\lambda\mathbf{x} \tag{11}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 2\mathbf{x}^T\mathbf{y}. \tag{12}$$

Equating these partial derivatives to zero, one obtains the first order optimality conditions (also known as

Karush, Kuhn, Tucker conditions; see [9, 2]):

$$\mathbf{a} - \mathbf{x} = \lambda \mathbf{y} \qquad (13)$$

$$\mathbf{b} - \mathbf{y} = \lambda \mathbf{x} \qquad (14)$$

$$\mathbf{x}^T \mathbf{y} = 0. \qquad (15)$$

From (13) and (14),

$$\mathbf{x} = \frac{\mathbf{a} - \lambda \mathbf{b}}{1 - \lambda^2}, \quad \text{and} \quad \mathbf{y} = \frac{\mathbf{b} - \lambda \mathbf{a}}{1 - \lambda^2} \qquad (16)$$

(a geometric interpretation of (15) and (16) can be found in Figure 1). Replacing $\mathbf{x}$ and $\mathbf{y}$ in (15), leads to the quadratic equation in λ

$$p\lambda^2 - q\lambda + p = 0, \qquad (17)$$

where p and q are

$$p \doteq \mathbf{a}^T \mathbf{b} \quad \text{and} \quad q \doteq \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2. \qquad (18)$$

Hence, the solutions of (17) are

$$\lambda_1 \doteq \frac{q + \sqrt{q^2 - 4p^2}}{2p} \quad \text{and} \quad \lambda_2 \doteq \frac{q - \sqrt{q^2 - 4p^2}}{2p}. \qquad (19)$$

Denoting, for $i = 1, 2$,

$$(\mathbf{x}_i, \mathbf{y}_i) = \left(\frac{\mathbf{a} - \lambda_i \mathbf{b}}{1 - \lambda_i^2}, \frac{\mathbf{b} - \lambda_i \mathbf{a}}{1 - \lambda_i^2} \right), \qquad (20)$$

we know that $(\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2)$ are the candidates to be a local minimum of the Lagrangian $\mathcal{L}(\mathbf{x}, \mathbf{y}, \lambda)$. However, since the gradient of the constraint $\mathbf{x}^T \mathbf{y} = \mathbf{0}$ involved in (5) annihilates at $(\mathbf{0}, \mathbf{0})$, one also needs to consider this non regular and non stationary point. Thus a local minimum of (5) must be attained at one of the following three points: $(\mathbf{x}_1, \mathbf{y}_1)$, $(\mathbf{x}_2, \mathbf{y}_2)$ or $(\mathbf{0}, \mathbf{0})$.

Now we shall note the following facts:

- The assumption of $\mathbf{a}$ and $\mathbf{b}$ being such that $\mathbf{a} \neq \pm \mathbf{b}$, with $\mathbf{b} \neq \mathbf{0}$, guarantees, in particular, that λ cannot be ± 1, which ensures that $\mathbf{x}$ and $\mathbf{y}$ in (16) are well defined.

- In (19), λ_i is always a real number because $q \geq 2p$. Indeed,

$$q - 2p = \mathbf{a}^T \mathbf{a} + \mathbf{b}^T \mathbf{b} - 2\mathbf{a}^T \mathbf{b} =$$
$$(\mathbf{a} - \mathbf{b})^T (\mathbf{a} - \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|^2 \geq 0. \quad (21)$$

Moreover, if $p = 0$, then $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, which means that the global minimum of the objective function is attained at $(\mathbf{a}, \mathbf{b})$.

We end this section by showing that the objective function (6) satisfies

$$f(\mathbf{x}_2, \mathbf{y}_2) \leq f(\mathbf{x}_1, \mathbf{y}_1) \leq f(\mathbf{0}, \mathbf{0}), \qquad (22)$$

which proves that $(\mathbf{x}_2, \mathbf{y}_2)$ given in (20) is the local minimum where the objective function f attains the smallest value. In Sec. 2.1 it is shown that $(\mathbf{x}_2, \mathbf{y}_2)$ is also the global minimum. Substituting $\mathbf{x}$ and $\mathbf{y}$ given in (16) in the objective function (6), some calculation yields the following real function depending on the real variable λ:

$$g(\lambda) \doteq \left(\frac{\lambda}{1 - \lambda^2} \right)^2 (q\lambda^2 - 4p\lambda + q). \qquad (23)$$

Now (22) can be rewritten as

$$g(\lambda_2) \leq g(\lambda_1) \leq q. \qquad (24)$$

Let us write the function g in the form

$$g(\lambda) = [\phi(\lambda)]^2 \, \psi(\lambda), \qquad (25)$$

with $\phi(\lambda) \doteq \lambda/(1 - \lambda^2)$ and $\psi(\lambda) \doteq q\lambda^2 - 4p\lambda + q$.

For λ_1 and λ_2 given in (19), it is not hard to check that $\phi(\lambda_1) = -\phi(\lambda_2)$, which implies $[\phi(\lambda_1)]^2 = [\phi(\lambda_2)]^2$. Hence, to prove the first inequality of (24), it remains to show that $\psi(\lambda_2) \leq \psi(\lambda_1)$.

In fact, because λ_i $(i = 1, 2)$ satisfies the quadratic equation (17), one has $\lambda_i^2 = (q/p)\lambda_i - 1$. Therefore,

$$\psi(\lambda_i) = \frac{q^2 - 4p^2}{p} \lambda_i,$$

and, consequently,

$$\psi(\lambda_2) = \frac{q(q^2 - 4p^2) - (q^2 - 4p^2)^{3/2}}{2p^2} \leq \psi(\lambda_1) =$$
$$\frac{q(q^2 - 4p^2) + (q^2 - 4p^2)^{3/2}}{2p^2}. \qquad (26)$$

Finally, considering the equality

$$\lambda_1^2 = (q/p)\lambda_1 - 1 \qquad (27)$$

and the fact that the quadratic function $\psi(\lambda)$ is non negative, for all real numbers λ, the second inequality of (24) arises as a consequence of the equivalences

$$
\begin{aligned}
g(\lambda_1) \leq q \quad &\Leftrightarrow \quad -4p\lambda_1^3 + 3q\lambda_1^2 \leq q \\
&\Leftrightarrow \quad \lambda_1(-4p\lambda_1^2 + 3q\lambda_1) \leq q \quad \text{(by (27))} \\
&\Leftrightarrow \quad \lambda_1(4p - q\lambda_1) \leq q \\
&\Leftrightarrow \quad 4p\lambda_1 - q\lambda_1^2 \leq q \\
&\Leftrightarrow \quad \frac{4p^2 - q^2}{p}\lambda_1 \leq 0 \quad \text{(by (27) again)} \\
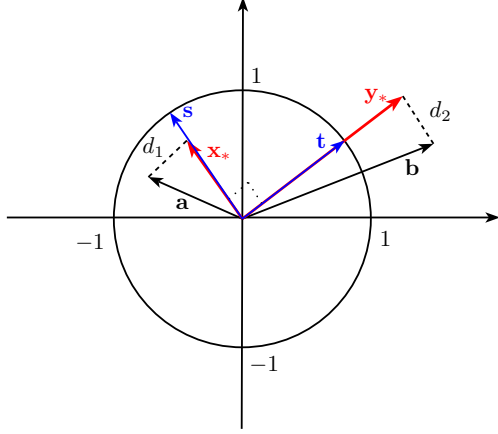&\Leftrightarrow \quad -\psi(\lambda_1) \leq 0.
\end{aligned}
$$

Figure 1: Geometric interpretation of the Plücker correction problem. Given vectors $\mathbf{a}$ and $\mathbf{b}$, the goal is to find two perpendicular vectors $\mathbf{x}_*$ and $\mathbf{y}_*$ that minimize the sum $d_1 + d_2$ representing the sum of the distance between $\mathbf{a}$ and the line through the origin with the direction of the unit vector $\mathbf{s}$ with the distance between $\mathbf{b}$ and the line through the origin with the direction of the unit vector $\mathbf{t}$.

2.1 Existence of a Global Minimum

Given a unit vector $\mathbf{s}$ in the Euclidean space $\mathbb{R}^n$, consider the line ℓ through the origin with the direction of $\mathbf{s}$. Given another vector $\mathbf{a} \in \mathbb{R}^n$, it is well-known that the unique vector of ℓ that is closest to $\mathbf{a}$ (in Euclidean sense) is the orthogonal projection of $\mathbf{a}$ onto ℓ, which is the vector $\mathbf{x} \doteq (\mathbf{a}^T\mathbf{s})\mathbf{s}$ (see, for instance, [3, p.435]).

Hence, the constrained problem (5) corresponds to find orthonormal vectors $\mathbf{s}$ and $\mathbf{t}$ minimizing the sum of the distance d_1, between $\mathbf{a}$ and the line through the origin with the direction of $\mathbf{s}$, with the distance d_2, between $\mathbf{b}$ and the line through the origin with the direction of $\mathbf{t}$ (see Fig. 1). This means that (5) can be reformulated as

$$\min_{\substack{\mathbf{s}^T\mathbf{t} = \mathbf{0}, \\ \|\mathbf{s}\| = 1, \ \|\mathbf{t}\| = 1}} \left\| [\mathbf{a} \ \mathbf{b}] - \left[(\mathbf{a}^T\mathbf{s})\mathbf{s} \quad (\mathbf{b}^T\mathbf{t})\mathbf{t} \right] \right\|^2 .$$
(28)

This formulation of the Plücker correction problem is apparently more complicated and less practical than (5), but it is helpful to show that (5) has in fact a global minimum. To see this, one just needs to observe that the constraints in (28) define a closed and bounded (and, consequently, compact) set in $\mathbb{R}^{2n}$ endowed with the Euclidean metric. Thus, by the classical Weierstrass theorem (see, for instance, [9, Appendix A.6]), there exists at least a global minimum. This proves that the analysis carried out in Section 2, using the Lagrange multipliers, guarantees that the objective function (6) attains a global minimum at

$$(\mathbf{x}_2, \mathbf{y}_2) = \left(\frac{\mathbf{a} - \lambda_2 \mathbf{b}}{1 - \lambda_2^2}, \ \frac{\mathbf{b} - \lambda_2 \mathbf{a}}{1 - \lambda_2^2} \right),$$

where λ_2 is defined as in (19).

2.2 Cases $\mathbf{a} = \pm\mathbf{b}$

As far as we know, the cases when $\mathbf{a} = \pm\mathbf{b}$ rarely occur in practical problems of computer vision. However, it is worth to make some comments on this particular case.

If $\mathbf{a} = \mathbf{b}$ with $\mathbf{b} = \mathbf{0}$, the solution of (5) is obviously $\mathbf{x} = \mathbf{0}$ and $\mathbf{y} = \mathbf{0}$. If $\mathbf{a} = \mathbf{b}$ but $\mathbf{b} \neq \mathbf{0}$ then replacing $\mathbf{b}$ by $\mathbf{a}$ in the Lagrangian (9), we get a simpler expression. Finding the first order optimality conditions and solving them, one easily concludes that any pair of vectors of the form $(\mathbf{a}+\mathbf{y}, \mathbf{y})$, with $(\mathbf{a}+\mathbf{y})^T\mathbf{y} = \mathbf{0}$ and $\mathbf{y}$ arbitrary, gives a local minimum. The value of the Lagrangian at all these local minima is $\|\mathbf{a}\|^2$ and does not depend on $\mathbf{y}$. Choosing, for instance, $\mathbf{y} = \mathbf{0}$, it follows that the pair $(\mathbf{a}, \mathbf{0})$ is a local minimum of (5). Using a similar argument to that of Sec. 2.1, this pair is also a global minimum.

Similarly, if $\mathbf{a} = -\mathbf{b}$, with $\mathbf{b} \neq \mathbf{0}$, it can be concluded that $(\mathbf{a}, \mathbf{0})$ provides also a global minimum for (5).

3 Experiments

In this section, we compare the method based on the explicit formula (7), with the method of Bartoli and Sturm [6], in terms of computational effort. Both methods were implemented using MATLAB. We consider three different algorithms (the codes are shown in the Appendix A):

- LMPC which denotes the method derived in this paper;
- BS which corresponds to the Bartoli and Sturm's approach; and
- BS-LSVD which denotes to the method proposed by Bartoli and Sturm, where the SVD is computed using closed-form techniques.

To compare the methods we use the following procedure: we randomly generated unconstrained 10^6 vectors $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^3$ (Klein quadric $\mathbf{a}^T\mathbf{b} = 0$ is not enforced). For each trial, we apply both the Plücker correction algorithms to the respective six-dimensional vectors, storing the values of the corresponding objective functions (6). The results in terms of computational time required for each algorithm are shown in Tab. 1.

Both LMPC and BS-LSVD methods can be implemented using only closed-form steps. However, while LMPC can be computed with a few steps (it only requires six lines of code), BS-LSVD requires more algebraic operations and takes a longer time to execute, see Tab. 1. Indeed, BS-LSVD is about two times slower

Table 1: Evaluation of the computational time for the three algorithms, for general unconstrained 10^6 trials.

Algorithm	For all trials	For each trial (median)
LMPC	6.5797 s	5.3940 μs
BS	43.688 s	34.827 μs
BS-LSVD	14.0058 s	11.805 μs

than LMPC. On the other hand, if we consider the original approach BS, our experiments have shown that it is about six times slower than our method. We recall that it requires the computation of two singular value decompositions and some iterative steps.

3.1 Discussion of the Experimental Results

Firstly, as one can see from the code of the three algorithms shown in Appendix A, the number of operations required by the method proposed in this paper is significantly lower. With this information, one can conclude that from a computational complexity perspective, our method is more efficient.

In many applications there are estimates of 3D line coordinates that are not obtained from points. 3D lines can be estimated from intersections of planes, for example, or they can be obtained from non-conventional sensors, such as non-central generic cameras. In non-central generic cameras calibration, it consists in estimating a 3D line for each pixel [11, 12]. If we represent 3D lines using Plücker coordinates, the Klein quadric constraint has to be enforced — in this case this correction has to be applied to all and each pixel, which corresponds to call the Plücker correction algorithm a large number of times. Let us consider a camera system, with a standard image size of 1280×1024, which contains a total of 1310720 pixels. For this case and from the experimental results, one can conclude that the Plücker correction step using the method proposed in this paper will be, at least, twice as fast as the BS method, which consists in saving more than eight seconds.

Other example is, using perspective cameras, the estimation of 3D lines from the intersections of the back-projecting planes from two or more images.

4 Conclusions

In this paper we have addressed the Plücker correction problem, by minimizing the Frobenius norm between the estimated and input vectors. By solving the corresponding optimization problem using Lagrange multipliers, a simple solution, that can be computed in closed-form with a very small number of oper-

ations, was proposed. Contrarily to the state-of-the-art method, we have proved theoretically that our method computes always the global minimum. In addition, the special cases (where a solution cannot be computed) are analysed. As the experimental results show, the proposed method is faster.

References

[1] H. Lütkepohl, *Handbook of Matrices*, Jonh Wiley and Sons, 1996.

[2] J. Nocedal and S. Wright, *Numerical Optimization*, Springer-Verlag New York, 1999.

[3] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia, 2000.

[4] H. Pottmann and J. Wallner, "Computational Line Geometry", Springer-Verlag Berlin Heidelberg, 2001.

[5] R. Hartley and A. Zisserman, *Multiple View Geometry in Computer Vision*, 2nd Edition, Cambridge University Press, 2004.

[6] A. Bartoli and P. Sturm, "Structure-from-Motion Using Lines: Representation, Triangulation and Bundle Adjustment", Computer Vision and Image Understanding, 2005.

[7] D. Lanman, M. Wachs G. Taubin, and F. Cukierman, "Reconstructing a 3D Line from a Single Catadioptric Image", IEEE Int'l Symposium on 3D Data Processing, Visualization, and Transmission (3DPVT), 2006

[8] T. Lemaire and S. Lacroix, "Monocular-vision based SLAM using Line Segments", IEEE Proc. Int'l Conf. Robotics and Automation (ICRA), 2007.

[9] D. Luenberger and Y. Ye, *Linear and Nonlinear Programming*, Third Ed., Springer, 2008.

[10] K. Josephson and F. Kah, "Triangulation of Points, Lines and Conics", J Math Imaging Vis, 2008.

[11] P. Sturm, S. Ramalingam, "A Generic Concept for Camera Calibration", European Conf. Computer Vision (ECCV), 2011

[12] P. Miraldo and H. Araujo, "Calibration of Smooth Camera Models", IEEE Trans. Pattern Analisys and Machine Intelligence, 2013.

[13] P. Miraldo and H. Araujo, "Planar Pose Estimation for General Cameras using Known 3D Lines", IEEE/RSJ Proc. Int'l Conf. Intelligent Robots and Systems (IROS), 2014.

[14] P. Miraldo, H. Araujo, and N. Gonçalves, "Pose Estimation for General Cameras using Lines", IEEE Trans. Cybernetics, 2015.

[15] F. Wu, M. Zhang, G. Wang, and Z. Hu, "Algebraic Error Based Triangulation and Metric of Lines", PLoS ONE, 2015.

A Appendix

Considering two general vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ that do not verify the *Klein constraint*, using our method, the closest orthogonal vectors are given by $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ such that:

—————————————————— LMPC ——————————————————

```
p = a1*b1 + a2*b2 + a3*b3;
q = a1*a1 + a2*a2 + a3*a3 + b1*b1 + b2*b2 + b3*b3;
mu = 2*p/(q+sqrt(q*q-4*p*p));
u_ = 1/(1-mu*mu);
x1 = (a1-mu*b1)*u_; x2 = (a2-mu*b2)*u_; x3 = (a3-mu*b3)*u_;
y1 = (b1-mu*a1)*u_; y2 = (b2-mu*a2)*u_; y3 = (b3-mu*a3)*u_;
```

The Bartoli and Sturm' algorithm is based on the singular value decomposition (SVD) and can be implemented as:

—————————————————— BS ——————————————————

```
[U,S,V] = svd(A,0);
Z = S*V'; z11 = Z(1,1); z21 = Z(2,1); z22 = Z(2,2); z12 = Z(1,2);
T = [z12, z22; z21, -z11];
[~,St,V_] = svd(T);
hv = V_(:,2);
hV = [hv(1),-hv(2);hv(2),hv(1)];
R = U*hV*diag(diag(hV'*S*V'));
x = R(:,1);
y = R(:,2);
```

Since in this case we are dealing with 3×3 matrices, there is a closed form solution for this decompositions. Thus, in our experiments we also implemented a closed form solution for the Bartoli and Sturm method:

—————————————————— BS−LSVD ——————————————————

```
s1 = sqrt((a11*a11*a11*a11 + 2*a11*a11 *a12*a12 + 2*a11*a11 *a21*a21 - ... )
s2 = sqrt(a11*a11 /2 - (a11*a11*a11*a11 + 2*a11*a11 *a12*a12 + ... );
v11 = -((a11*a12 + a21*a22 + a31*a32))/(a11^2 + a21^2 + a31^2 - s1*s1); v21 = 1;
v11 = v11/(v11*v11 + v21*v21)^(1/2); v21 = v21/(v11*v11 + v21*v21)^(1/2);
v12 = v21; v22 = -v11;
u11 = (a12*v21 + a11*v11)/s1; u12 = (a12*v22 + a11*v12)/s2;
u21 = (a21*v11 + a22*v21)/s1; u22 = (a21*v12 + a22*v22)/s2;
u31 = (a31*v11 + a32*v21)/s1; u32 = (a31*v12 + a32*v22)/s2;
z11 = s1*v11; z12 = s1*v21;
z21 = s2*v12; z22 = s2*v22;
t11 = z12; t12 = z22; t21 = z21; t22 = -z11;
st1 = t11/2 + t22/2 - (t11*t11 - 2*t11*t22 + t22*t22 + 4*t12*t21)^(1/2)/2;
st2 = t11/2 + t22/2 + (t11*t11 - 2*t11*t22 + t22*t22 + 4*t12*t21)^(1/2)/2;
if st1 < st2
v1 = (t12*1)/(st1 - t11); v2 = 1; nv = (v1*v1 + v2*v2)^(1/2);
v1 = v1/nv; v2 = v2/nv;
else
v1 = (t12*1)/(st2 - t11); v2 = 1; nv = (v1*v1 + v2*v2)^(1/2);
v1 = v1/nv; v2 = v2/nv;
end
h11 = v1; h12 = -v2; h21 = v2; h22 = v1;
x1 = (u11*h11 + u12*h21)*(h11*s1*v11 + h21*s2*v12);
y1 = (u11*h12 + u12*h22)*(h12*s1*v21 + h22*s2*v22);
x2 = (u21*h11 + u22*h21)*(h11*s1*v11 + h21*s2*v12);
y2 = (u21*h12 + u22*h22)*(h12*s1*v21 + h22*s2*v22);
x3 = (u31*h11 + u32*h21)*(h11*s1*v11 + h21*s2*v12);
y3 = (u31*h12 + u32*h22)*(h12*s1*v21 + h22*s2*v22);
```