Stability and stabilization of hybrid systems

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Goals and class structure

Goal: After these lectures, you should

- Know some basic theory for stability and stabilization of hybrid systems
- Be familiar with the computational methods for piecewise linear systems
- Understand how the tools can be applied to (relatively) practical systems

Three lectures:

1. Stability theory
2. Computational tools for piecewise linear systems
3. Applications

Part I – Stability theory

Outline:
- A hybrid systems model and stability concepts
- Lyapunov theory for smooth systems
- Lyapunov theory for stability and stabilization of hybrid systems

Acknowledgements: M. Heemels, TU/e

A hybrid systems model

We consider hybrid systems on the form

$$\dot{x}(t) = f(x(t), i(t))$$
$$i(t^+) = \nu(x(t), i(t))$$

where

$x(t) \in \mathbb{R}^n$ is the continuous state vector

$i(t) \in \{1, 2, \ldots, M\}$ is the discrete state

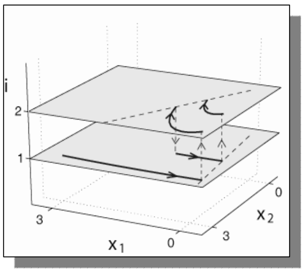
The discrete state indexes vector fields $f(x, i) = f_i(x)$ while $\nu(x, i)$ is the transition function describing the evolution of the discrete state.

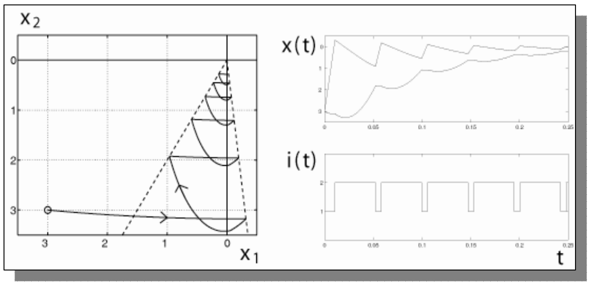
Unless stated otherwise, we will assume that $i(t)$ is piecewise continuous (i.e., that there is only a finite number of mode changes per unit time)

For now, disregard issues with sliding modes, zeno, … (see refs for details)

Example: a switched linear system

$$\dot{x}(t) = A_{i(t)}x(t)$$
$$i(t^+) = \begin{cases} 2 & \text{if } i(t) = 1 \text{ and } x_2 = -10x_1 \\ 1 & \text{if } i(t) = 2 \text{ and } x_2 = 2x_1 \end{cases}$$





(numerical values for matrices A_i are given in notes for Lecture 2)

Stability concepts

Focus: stability of equilibrium point (in continuous state-space) $x = 0$

Global asymptotic stability (GAS): ensure that

$$\lim_{t\to\infty} x(t) = 0 \qquad \text{for all initial states } (x(0), i(0))$$

Global uniform asymptotic stability (GUAS): ensure that

$$\lim_{t\to\infty} x(t) = 0 \qquad \text{for all initial states } (x(0), i(0)) \text{ and for all piecewise continuous } i(t)$$

(i.e., uniformly in $i(t)$)

Three fundamental problems

Problem P1: Under what conditions is

$$\dot{x}(t) = f(x(t), i(t))$$

GAS for all (piecewise continuous) switching signals $i(t)$?

Problem P2: Given vector fields $f(x,i) = f_i(x)$, design strategy $\nu(x,i)$:

$$\begin{aligned} \dot{x}(t) &= f(x(t), i(t)) \\ i(t^+) &= \nu(x(t), i(t)) \end{aligned}$$

is globally asymptotically stable.

Problem P3: determine if a given switched system

$$\begin{aligned} \dot{x}(t) &= f(x(t), i(t)) \\ i(t^+) &= \nu(x(t), i(t)) \end{aligned}$$

is globally asymptotically stable.

Part I – Stability theory

Outline:

- A hybrid systems model and stability concepts
- Lyapunov theory for smooth systems
- Lyapunov theory for stability and stabilization of hybrid systems

Aim: establishing common grounds by reviewing fundamentals.

Lyapunov theory for smooth systems

Theorem. *Let* $x = 0$ *be an equilibrium point of* $\dot{x} = f(x)$, *and let* $V : \mathbb{R}^n \mapsto \mathbb{R}$ *be a continuously differentiable function such that*

(i) $V(x) \to \infty$ *as* $\|x\| \to \infty$ *(radially unbounded)*

(ii) $V(0) = 0$ *and* $V(x) > 0$ *if* $x \neq 0$ *(positive definite)*

(iii) $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0$ *for all* $x \neq 0$ *(decreasing)*

then $x = 0$ *is globally asymptotically stable.*

Interpretation: Lyapunov function is abstract measure of system energy, system energy should decrease along all trajectories.

Converse theorem

Under appropriate technical conditions (mainly smoothness of vector fields)

Theorem. *If* $x = 0$ *is a GAS equilibrium of* $\dot{x} = f(x)$, *then there exists a radially unbounded Lyapunov function* $V(x)$

Consequence: worthwhile to search for Lyapunov functions

Remaining challenge: how to perform Lyapunov function search?

Stability of linear systems

Theorem. *The following statements are equivalent:*

(i) The linear system $\dot{x} = Ax$ is asymptotically stable

(ii) There is a quadratic Lyapunov function

$$V(x) = x^T P x$$

for some positive definite matrix $P > 0$ such that

$$A^T P + PA < 0$$

Moreover, for every asymptotically stable A and for any $Q > 0$ there is a $P > 0$ such that the following *Lyapunov equality* holds

$$A^T P + PA = -Q$$

Partial proof

(ii)→(i): Assume that there is $P > 0$ such that $A^T P + PA < 0$. Then there exists an $\epsilon > 0$ such that

$$A^T P + PA + \epsilon P < 0$$

Letting $V(x) = x^T P x$, then for all $t \in \mathbb{R}$

$$\begin{aligned}\frac{d}{dt}V(x(t)) + \epsilon V(x(t)) &= x^T(t)(A^T P + PA)x(t) + \epsilon x^T(t) P x(t) \\ &= x^T(t)(A^T P + PA + \epsilon P)x(t) \leq 0\end{aligned}$$

After integration, this yields for all $t \leq t_0$,

$$x^T(t) P x(t) \leq x^T(t_0) P x(t_0) e^{-\epsilon t}$$

Now use that $\lambda_{\min}(P)\|x\|^2 \leq x^T P x \leq \lambda_{\max}(P)\|x\|^2$ to infer

$$\|x(t)\|^2 \leq \|x(t_0)\|^2 \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} e^{-\epsilon t}$$

Stability of discrete-time systems

Theorem. *Let $x = 0$ be an equilibrium point of $x(t_{k+1}) = f(x(t_k))$, and let $V : \mathbb{R}^n \mapsto \mathbb{R}$ be a continuously differentiable function s.t.*

(i) $V(x) \to \infty$ *as* $\|x\| \to \infty$

(ii) $V(0) = 0$ *and* $V(x) > 0$ *if* $x \neq 0$

(iii) $\Delta V(x) = V(f(x(t_k))) - V(x(t_k)) < 0$ *for all* $x \neq 0$

then $x = 0$ is globally asymptotically stable.

Interpretation: energy should decrease at each sampling instant (event)

Performance analysis

Lyapunov techniques also useful for estimating system performance.

Theorem. *If there exists a radially unbounded, positive definite storage function $V(x)$ satisfying*

$$\frac{\partial V(x)}{\partial x} f(x, w) \leq \gamma^2 \|w\|^2 - \|y\|^2 \qquad \forall x,\, w$$

then the smooth nonlinear system

$$\begin{aligned} \dot{x}(t) &= f(x(t), w(t)) \\ y(t) &= g(x(t)) \end{aligned}$$

has L_2-gain less than γ (i.e., $\int_0^t \|y(s)\|^2 \, ds \leq \gamma^2 \int_0^s \|w(s)\|^2 \, ds \;\; \forall t$)

Part I – Stability theory

Outline:
- A hybrid systems model and stability concepts
- Lyapunov theory for smooth systems
- Lyapunov theory for stability and stabilization of hybrid systems

Content:
- Guaranteeing stability independent of switching strategy
- Design a stabilizing switching strategy
- Prove stability for a given switching strategy

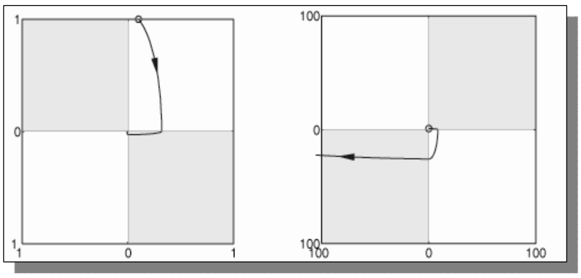
Switching between stable systems

Q: does switching between stable dynamics always create stable motions?

A: no, not necessarily.

$$\dot{x} = A_{i(x)}x \text{ for } x \in X_i \text{ with } A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}$$

Subsystems are stable and share the same eigenvalues,
but stability depends on switching!



P1: Stability for arbitrary switching signals

Problem: when is the switched system

$$\dot{x}(t) = f(x(t), i(t)) = f_{i(t)}(x(t))$$

GAS for all (piecewise continuous) switching signals $i(t)$?

Claim: only if each subsystem

$$\dot{x}(t) = f_i(x(t))$$

admits a radially unbounded Lyapunov function.

(can you explain why?)

The common Lyapunov function approach

In fact, if the submodels are smooth, the following results hold.

Theorem. *If all submodels share a common positive definite radially unbounded Lyapunov function, then the switched system is GUAS.*

Theorem. *If the switched system is GUAS, then all submodels share a positive definite radially unbounded common Lyapunov function.*

Hence, common Lyapunov functions necessary and sufficient.

Switched linear systems

For switched linear systems

$$\dot{x}(t) = A_{i(t)}(x(t))$$

it is natural to look for a common quadratic Lyapunov function

$$V(x) = x^T P x \qquad \text{with } P > 0$$

$V(x)$ is a common Lyapunov function if

$$\dot{V}(x) = x^T(A_i^T P + P A_i)x < 0 \quad \text{for all } i = 1, 2, \dots, M$$

Such a Lyapunov function can be found by solving linear matrix inequalities

$$P > 0 \qquad A_i^T P + P A_i < 0 \quad \text{for all } i = 1, 2, \dots, M$$

(systems that admit quadratic V(x) are called *quadratically stable*)

Infeasibility test

It is also possible to prove that there is no common quadratic Lyapunov fcn:

Theorem. *If there exist positive definite matrices $R_i > 0$ such that*

$$\sum_{i=1}^{M} R_i A_i^T + A_i R_i > 0$$

then there is no $P > 0$ such that

$$A_i^T P + P A_i < 0 \qquad \forall i \in \{1, \dots, M\}$$

Example

Question: Does GUAS of switched linear system imply existence of a common quadratic Lyapunov function?

Answer: No, the system given by

$$A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -1 & -10 \\ 0.1 & -1 \end{pmatrix}$$

is GUAS, but does not admit any common quadratic Lyapunov function since

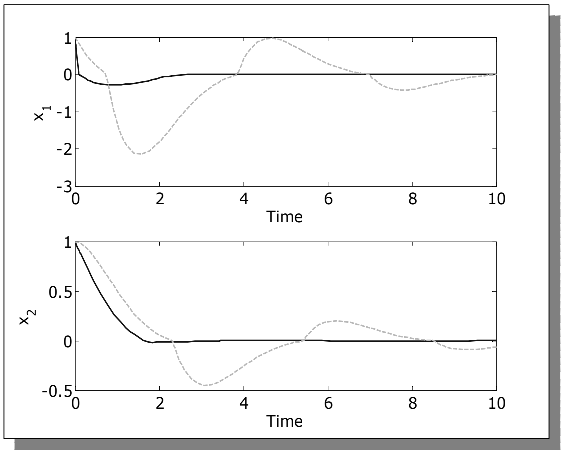
$$R_1 = \begin{pmatrix} 0.2996 & 0.7048 \\ 0.7048 & 2.4704 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 0.2123 & -0.5532 \\ -0.5532 & 1.9719 \end{pmatrix}$$

satisfy the infeasibility condition.

(there is, however, a common *piecewise quadratic* Lyapunov function)

Example

Sample trajectories of switched system
(under two different switching strategies)



Even if solutions are very different, all motions are asymptotically stable

P2: Stabilization

Problem: given matrices A_i, find switching rule $\nu(x,i)$ such that

$$\begin{aligned} \dot{x}(t) &= A_{i(t)}x(t) \\ i(t^+) &= \nu(x(t), i(t)) \end{aligned}$$

is asymptotically stable.

Stabilization of switched linear systems

Theorem. *If there exist* $\alpha_i > 0$ *with* $\sum_i \alpha_i = 1$ *such that*

$$\dot{x}(t) = \sum_i \alpha_i A_i x(t) := A_{\mathrm{eq}} x(t)$$

is globally asymptotically stable, then there exists a switching strategy that makes the switched system globally asymtotically stable.

Note: if only two subsystems, then condition is also necessary.

Stabilizing switching rules (I)

State-dependent switching strategy designed from Lyapunov function for A_{eq}

Solve Lyapunov equality $A_{\text{eq}}^T P + P A_{\text{eq}} = -Q$. It follows that

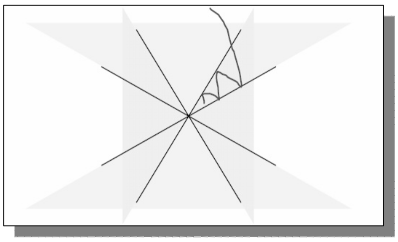
$$\sum_i \alpha_i\, x^T (A_i^T P + P A_i) x = x^T (A_{\text{eq}}^T P + P A_{\text{eq}}) x = -x^T Q x < 0$$

Thus, for each x, at least one mode satisfies $x^T (A_i^T P + P A_i) x(t) < 0$

This implies, in turn, that the switching rule

$$\nu(x) = \arg\min x^T (A_i^T P + P A_i) x$$

is well-defined for all x and that it generates globally asymptotically stable motions.



Stabilizing switching rules (II)

Alternative switching strategy: activate mode i fraction α_i of the time, e.g.,

$$i(t^+) = \begin{cases} 1 & \text{if } 0 \le t < \alpha_1 T \\ 2 & \text{if } \alpha_1 T \le t < (\alpha_1 + \alpha_2) T \\ \vdots & \\ N & \text{if } \sum_{i=1}^{N-1} \alpha_i T \le t < T \end{cases}$$

(strategy repeats after duty cycle of T seconds). “Average dynamics” is

$$\dot{x} = A_{\text{eq}} x$$

and for sufficiently small T the spectral radius of

$$\exp(A_1 \alpha_1 T) \exp(A_2 \alpha_2 T) \cdots \exp(A_N \alpha_N T)$$

is less than one (i.e., state at beginning of each duty cycle will tend to zero)

Example

Consider the two subsystems given by

$$A_1 = \begin{pmatrix} -0.5 & 1 \\ 100 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -1 & -100 \\ -0.5 & -1 \end{pmatrix}$$

Both subsystems are unstable, but the matrix $A_{eq}=0.5A_1+0.5A_2$ is stable.

State-dependent switching: set Q=I, solve Lyapunov equation to find

$$P = \begin{pmatrix} 0.5700 & 0.0015 \\ 0.0015 & 0.5728 \end{pmatrix}$$

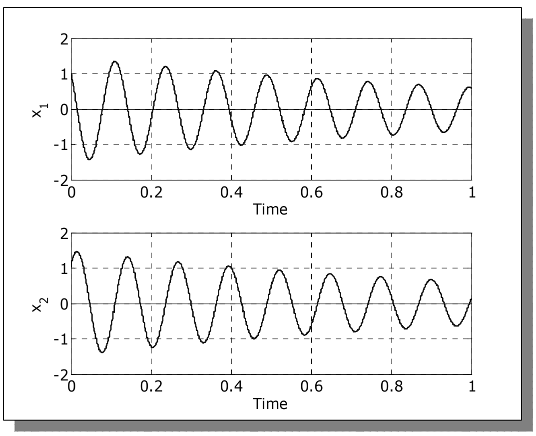
Time-dependent switching: choose duty cycle T such that spectral radius of

$$\exp(A_i T/2)\exp(A_2 T/2)$$

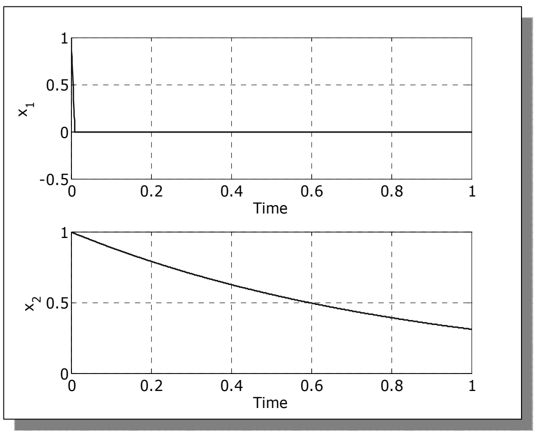
is less than one. Alternate between modes each T/2 seconds.

Example cont'd

Time-driven switching



State-dependent switching



P3: Stability for a given switching strategy

Problem: how can we verify that the switched system

$$\dot{x}(t) = f(x(t), i(t))$$
$$i(t^+) = \nu(x(t), i(t))$$

is globally asymptotically stable?

Stability for given switching strategy

For simplicity, consider a system with two modes, and assume that

$$\dot{x}(t) = f_i(x(t)) \qquad i = 1,\ 2$$

are globally asymptotically stable with Lyapunov functions V_i

Even if there is no common Lyapunov function, stability follows if

$$V_{i(t_{k-1})}(x(t_k)) = V_{i(t_k)}(x(t_k)) \qquad \forall k = 1, 2, \ldots$$

where t_k denote the switching times.

Reason: V_i is continuous Lyapunov function for the switched system.

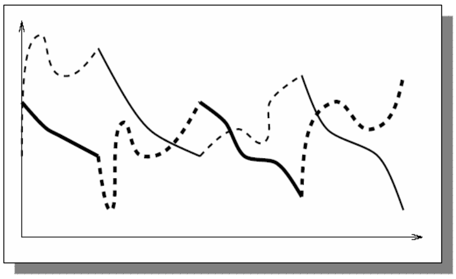
Multiple Lyapunov function approach

Theorem. *Consider the switched system where all submodels* $\dot{x} = f_i(x)$ *are globally asymptotically stable with Lyapunov functions* V_i.

Suppose that for each pair of swtiching times (t_k, t_l), $k < l$ *with* $i(t_k) = i(t_l) = \hat{\imath}$ *and* $i(t_m) \neq \hat{\imath}$ *for* $t_k < t_m < t_l$, *we have*

$$V_i(x(t_l)) \leq V_i(x(t_k)) - \rho(|x(t_k)|)$$

then the swtiched system is globally asymptotically stable.



Multiple Lyapunov function approach

Weaker versions exist:

– No need to require that submodels are stable, sufficient to require that all submodels admit *Lyapunov-like* functions:

$$V_i(x) > 0 \qquad \text{for } x \in X_i$$

$$\frac{\partial V_i(x)}{\partial x} f_i(x) < 0 \qquad \text{for } x \in X_i$$

where X_i contains all x for which submodel f_i can be activated.

– Can weaken requirement that V_i should decrease along trajectories of f_i

See the references for details and precise statements.

Summary

A whirlwind tour:

- *selected* results on stability and stabilization of hybrid systems

Three specific problems

- Guaranteeing stability independent of switching signal
- Design a stabilizing switching strategy (stabilizability)
- Prove stability for a given switching strategy

Focus has been on Lyapunov-function techniques

- Alternative approaches exist!

Strong theoretical results, but hard to apply in practice

- Can be overcome by developing automated numerical techniques (Lecture 2!)

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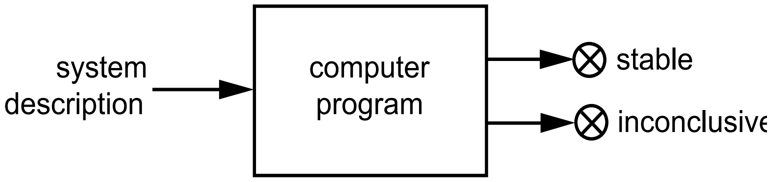
Part II – Computational tools

- Piecewise linear systems
- Well-posedness and solution concepts
- Linear matrix inequalities
- Piecewise quadratic stability
- Extensions

Computational stability analysis: philosophy

Aim: develop analysis tools that

- are computationally efficient (e.g. run in polynomial time)
- work for *most* practical problem instances
- produce guaranteed results (when they work)



Piecewise linear systems

Piecewise linear system:

1. a subdivision of $\mathbb{R}^n$ into regions X_i

$$\bigcup_{i=1}^{M} X_i \subseteq \mathbb{R}^n$$

we will assume that X_i are polyhedral and disjoint
(i.e. that cells only share common boundaries)

2. (possibly different) affine dynamics in each region

$$\begin{cases} \dot{x}(t) = A_i x(t) + a_i + B_i u(t) \\ y(t) = C_i x(t) + c_i + D_i u(t) \end{cases} \qquad \text{for } x(t) \in X_i \quad i \in I$$

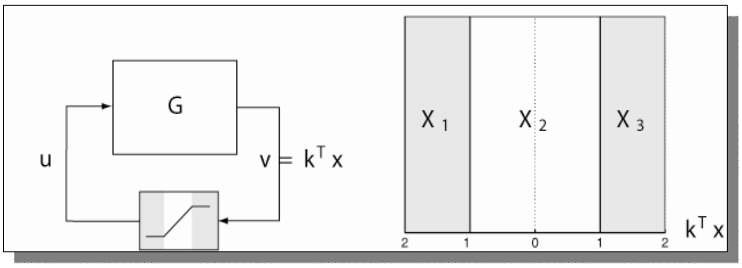
Example

Saturated linear system: $\dot{x} = Ax + b\,\mathsf{sat}(v),\ v = k^T x$

Three regions: negative saturation, linear operation, positive saturation

$$\dot{x} = \begin{cases} Ax - b & x \in X_1 \\ (A - bk^T)x & x \in X_2 \\ Ax + b & x \in X_3 \end{cases}$$

Cells are polyhedral (i.e., can be described by a set of linear inequalities)

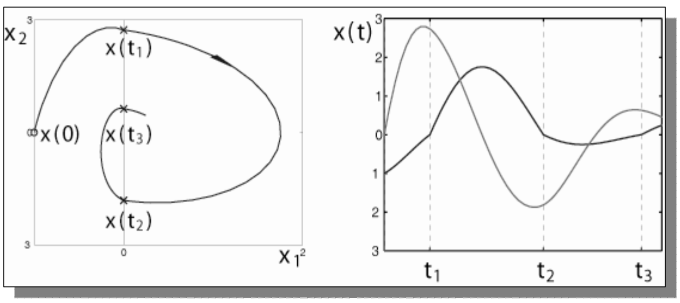


Well-posedness and solutions

Definition. *Let* $x(t) \in \cup_{i\in I} X_i$ *be an absolutely continuous function. We say that* $x(t)$ *is a* trajectory *of the system*

$$\begin{cases} \dot{x}(t) = A_i x(t) + a_i + B_i u(t) \\ y(t) = C_i x(t) + c_i + D_i u(t) \end{cases} \qquad \text{for } x(t) \in X_i \quad i \in I$$

on $[t_0, t_f]$ *if, for allmost all* $t \in [t_0, t_f]$*, the equation* $\dot{x}(t) = A_i x(t) + a_i + B_i u(t)$ *holds for all* i *with* $x(t) \in X_i$*.*

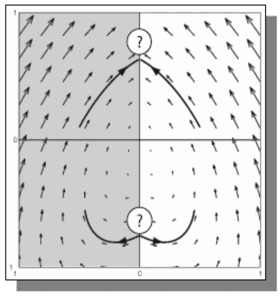


Trajectories: existence and uniqueness

Observation: trajectories may not be unique, or may not exist.

Example:

$$\begin{cases}\dot{x}_1 = -2x_1 - 2x_2\,\mathrm{sgn}(x_1)\\ \dot{x}_2 = x_2 + 4x_1\,\mathrm{sgn}(x_1)\end{cases}$$



Initial values in $\mathcal{S}_1^- = \{x \mid x_1 = 0 \wedge x_2 \le 0\}$ create non-unique trajectories.

Trajectories that reach $\mathcal{S}_1^+ = \{x \mid x_1 = 0 \wedge x_2 \ge 0\}$ cannot be continued

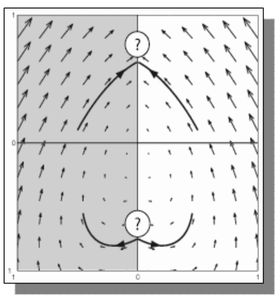
Attractive sliding modes

Would like to single out situations with non-existence of solutions.

Definition. *The system*

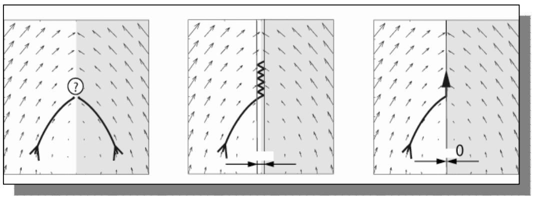
$$\begin{cases}\dot{x}(t) = A_i x(t) + a_i + B_i u(t)\\ y(t) = C_i x(t) + c_i + D_i u(t)\end{cases} \qquad \textit{for } x(t) \in X_i \quad i \in I$$

is said to have an attractive sliding mode *at* x_s *if there exists a trajectory with final state* x_s *but no trajectory with initial state* x_s.



Generalized solutions

Solution concepts for systems with sliding modes typically averages dynamics in neighboring cells



Definition. *Let* $x(t) \in \cup_{i \in I} X_i$ *be an absolutely continuous function. We say that* $x(t)$ *is a* Filippov solution *of (1) on* $[t_0, t_f]$ *if*

$$\dot{x}(t) \in \overline{\text{co}}_{k \in K(t)} \{A_k x(t) + a_k + B_k u(t)\}$$

for almost all t*, where* K *is the set of indicies such that* $x(t) \in X_k$*.*

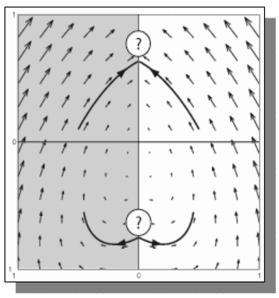
Note: Filippov solutions may remain on cell boundaries, and are not necessarily unique.

Equivalent dynamics on sliding modes

Example: Piecewise linear system

$$\begin{cases} \dot{x}_1 = -2x_1 - 2x_2 \text{sgn}(x_1) \\ \dot{x}_2 = x_2 + 4x_1 \text{sgn}(x_1) \end{cases}$$

on $\mathcal{S}_1^+ = \{x \mid x_1 = 0 \wedge x_2 \geq 0\}$



Filippov solutions satisfy $\dot{x}(t) \in \alpha A_1 x(t) + (1-\alpha) A_2 x(t)$ for some $\alpha \in [0, 1]$

If x(t) should stay on $\mathcal{S}_1^+$, we must have $\dot{x}_1(t) = 0$, i.e.,

$$\alpha \cdot 2x_2 + (1-\alpha) \cdot (-2x_2) = x_2(4\alpha - 2) = 0$$

The only solution is given by α=1/2, resulting in the unique sliding dynamics

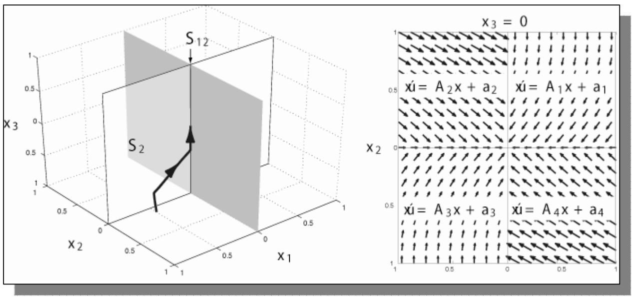
$$\dot{x}_1 = 0, \qquad \dot{x}_2 = x_2$$

Non-uniqueness of sliding dynamics

Observation: sliding dynamics on intersecting boundaries often non-unique

Example:

$$\dot{x}_1 = x_2 - \text{sgn}(x_1)$$
$$\dot{x}_2 = x_3 - \text{sgn}(x_2)$$
$$\dot{x}_3 = -2x_1 - 4x_2 - 4x_3 - x_3\text{sgn}(x_2)\text{sgn}(x_1 + 1)$$



Filippov solutions on $\mathcal{S}_{12} = \{x \mid x_1 = 0 \wedge x_2 = 0 \wedge |x_3| \leq 1\}$ are not unique.
(can you explain why?)

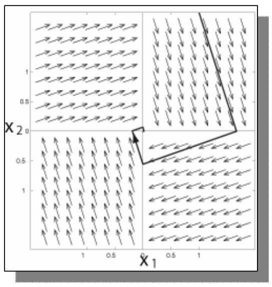
Valid Filippov solutions on $\mathcal{S}_{12}$ differ in time constants of a factor four or more.

Establishing attractivity of sliding modes

Note: non-trivial to detect that a pwl system has attractive sliding modes

Example: The piecewise linear system

$$\dot{x}_1 = -\text{sgn}(x_1) + 2\text{sgn}(x_2)$$
$$\dot{x}_2 = -2\text{sgn}(x_1) - \text{sgn}(x_2)$$



has a sliding mode at the origin.

However, determining that it is attractive is not easy

- Vector field inspection or quadratic Lyapunov functions can't be used (why?)
- Finite-time convergence to the origin can be established by noting that

$$\frac{d}{dt}(|x_1| + |x_2|) = -2$$

Key points

Piecewise linear systems: polyhedral partition and locally affine dynamics

$$\begin{cases} \dot{x}(t) = A_i x(t) + a_i + B_i u(t) \\ y(t) = C_i x(t) + c_i + D_i u(t) \end{cases} \qquad \text{for } x(t) \in X_i \quad i \in I$$

For general piecewise linear systems, solution concepts are non-trivial
- Trajectories may not be unique, or may not exist (unless continuous)
- Meaningful solution concepts for attractive sliding modes exist (e.g. Filippov solutions)

Introducing “new modes” on cell boundaries with sliding dynamics not easy
- Sliding modes may occur on any intersection of cell boundaries
- Hard to determine if potential sliding mode is attractive
- Dynamics of sliding modes may be non-unique and non-linear

Part II – Computational tools

- Piecewise linear systems
- Well-posedness and solution concepts
- Linear matrix inequalities
- Piecewise quadratic stability
- Extensions

Linear matrix inequalities

Linear matrix inequality (LMI): An inequality on the form

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i > 0$$

where F_i are symmetric matrices, X>0 denotes that X is positive definite.

Example: The condition $P > 0$ on standard form:

$$p_{11}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + p_{12}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + p_{22}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} > 0$$

LMI features

- Optimization under LMI constraints is a *convex* optimization problem
  - Strong and useful theory, e.g. duality
    (we have already used it once – when?)
- Multiple LMIs is an LMI
  - Example: Lyapunov inequalities $P > 0,\ A^T P + PA < 0$
    equivalent to single LMI

$$\begin{bmatrix}P & 0\\0 & -A^T P - PA\end{bmatrix} > 0$$

- Efficient software and convenient user interfaces publicly available
  - Example: YALMIP interface by J. Löfberg at ETHZ
- S-procedure, Shur complements, … and much more!

Example: Quadratic stabilization

Recall from Lecture 1 that $V(x) = x^T P x$ guarantees that

$$\dot{x}(t) = A_{i(t)}(x(t))$$

is GAS for all switching signals i(t) (i.e., GUAS) if P satisfies

$$P > 0$$
$$A_i^T P + P A_i < 0 \qquad \forall i \in \{1, 2, \ldots, M\}$$

an LMI condition!

Consequence: quadratic Lyapunov function found efficiently (if it exists)!

Quadratic stability of PwL systems

$V(x) = x^T P x$ is a Lyapunov function for the piecewise linear system

$$\dot{x} = A_i x \qquad x \in X_i$$

if we have

$$x^T P x > 0 \qquad \forall x \neq 0$$
$$x^T (A_i^T P + P A_i) x < 0 \qquad \forall x \in X_i \backslash 0$$

Note: not necessary to require that $A_i^T P + P A_i < 0$

How can we bring the restricted conditions into the LMI framework?

S-procedure

When does it hold that, for all x,

$$x^T R x \geq 0 \Rightarrow x^T P x \geq 0$$

(i.e., non-negativity of quadratic form $x^T R x$ implies non-neagivity of $x^T P x$)

Simple condition: there exists $\tau \in \mathbb{R}_+$ satisfying the LMI $P \geq \tau R$

Extension to multiple quadratic forms: if there exist $\tau_i \geq 0$ such that

$$P - \sum_i \tau_i R_i \geq 0$$

then $\boldsymbol{(x^T R_1 x \geq 0) \wedge (x^T R_2 x \geq 0) \cdots \Rightarrow x^T P x \geq 0}$

(non-trivial fact: simple condition is necessary if there exists u: $u^T R u > 0$)

Bounding polyedra by quadratic forms

Example: The polyhedron

$$X = \{x \mid |x| \leq 1\} = \{x \mid (x \geq -1) \wedge (x \leq 1)\} = \{x \mid (x+1 \geq 0) \wedge (1-x \geq 0)\}$$

can be described by the quadratic form

$$q(x) = \tau(x+1)(1-x) = \tau(1-x^2) \geq 0$$

for $\tau \geq 0$

In general: for polyhedra $X_i = \{x \mid E_i x + e_i \succeq 0\}$ the quadratic form

$$q(x) = \left(\begin{bmatrix} E_i & e_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \right)^T U_i \left(\begin{bmatrix} E_i & e_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \right) \quad = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \bar{E}_i^T U_i \bar{E}_i \begin{bmatrix} x \\ 1 \end{bmatrix}$$

is non-negative for all $x \in X_i$ if W_i has non-negative entries

Quadratic stability cont'd

Consider the piecewise linear system

$$\dot{x} = A_i x \qquad \text{for } x \in X_i = \{x \mid E_i x \succeq 0\}$$

(no affine terms, all regions contain the origin). Then,

Theorem. *If there exists a positive definite matrix P and matrices U_i with non-negative entries such that*

$$A_i^T P + P A_i + E_i^T U_i E_i < 0$$

then every Filippov solution tends to zero exponentially.

Example

Recall the switched system

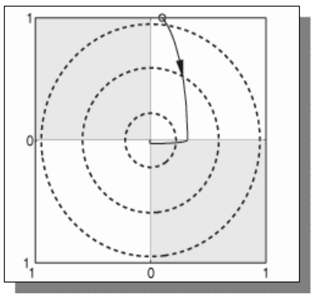
$$\dot{x} = A_1 x \text{ for } x_1 x_2 \geq 0, \quad \dot{x} = A_2 x \text{ for } x_1 x_2 \leq 0$$

with

$$A_1 = \begin{pmatrix} -0.1 & 1 \\ -10 & -0.1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -0.1 & 10 \\ -1 & -0.1 \end{pmatrix}$$

from Lecture 1. Applying the above procedure, we find

$P = I$, *e.g.*, $V(x) = x^T x$.



(stability cannot be verified without S-procedure – can you explain why?)

Piecewise quadratic Lyapunov functions

Natural to consider continuous, *piecewise quadratic*, Lyapunov functions

$$V(x) = x^T P_i x + 2q_i^T x + r_i = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \qquad \text{for } x \in X_i$$

Surprisingly, such functions can also be computed via optimization over LMIs.

Relation to multiple Lyapunov functions:

- Local expressions for V(x) are Lyapunov-like functions for associated dynamics (stronger relationship will emerge in the extensions)

Convenient notation

Use the augmented state vector

$$\bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}$$

and re-write system dynamics as

$$\begin{bmatrix} \dot{\bar{x}} \\ y \end{bmatrix} = \left[\begin{array}{cc|c} A_i & a_i & B_i \\ 0_{1\times n} & 0 & 0_{1\times m} \\ \hline C_i & c_i & D_i \end{array}\right] \begin{bmatrix} \bar{x} \\ u \end{bmatrix} = \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array}\right] \begin{bmatrix} x \\ 1 \end{bmatrix}$$

When analyzing properties of the equilibrium $x = 0$ we let

$I_0 \subseteq I$ be the set of indices for regions containing origin

$I_1 \subseteq I$ be the set of indices for regions that do not contain origin

and assume that $a_i = c_i = 0$ for $i \in I_0$

Enforcing continuity

How to ensure that the Lyapunov function candidate

$$V(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \bar{x}^T \bar{P}_i \bar{x} \qquad \text{for } x \in X_i$$

is continuous across cell boundaries?

Proposition. $\bar{x}^T \bar{P}_i \bar{x} = \bar{x}^T \bar{P}_j \bar{x}$ *for all* $x \in X_i \cap X_j = \{x \mid \bar{h}_{ij}^T \bar{x} = 0\}$
if and only if there exists $\bar{t}_{ij} \in \mathbb{R}^{n+1}$ *such that*

$$\bar{P}_i = \bar{P}_j + \bar{h}_{ij}^T \bar{t}_{ij} + \bar{t}_{ij}^T \bar{h}_{ij}$$

Enforce one linear equality for each cell boundary.

Enforcing continuity (II)

Alternative: direct parameterization

For each region, construct continuity matrices $\bar{F}_i = [F_i \quad f_i]$ such that

$$\bar{F}_i \bar{x} = \bar{F}_j \bar{x} \text{ for all } x \in X_i \cap X_j$$

and consider Lyapunov functions on the form

$$V(x) = \bar{x}^T \bar{F}_i^T T \bar{F}_i \bar{x} \text{ for } x \in X_i$$

(the free variables are now collected in the symmetric matrix T)

To make Lyapunov function quadratic in regions that contain origin, we also require

$$f_i = 0 \text{ for } i \in I_0$$

(construction automated in, for example, Pwltools)

Piecewise quadratic stability

Theorem (Piecewise Quadratic Stability). *Consider symmetric matrices T, U_i and W_i such that U_i and W_i have nonnegative entries, while $P_i = F_i^T T F_i$ and $\bar{P}_i = \bar{F}_i^T T \bar{F}_i$ satisfy*

$$\begin{cases} 0 > A_i^T P_i + P_i A_i + E_i^T U_i E_i \\ 0 < P_i - E_i^T W_i E_i \end{cases} \qquad i \in I_0$$

$$\begin{cases} 0 > \bar{A}_i^T \bar{P}_i + \bar{P}_i \bar{A}_i + \bar{E}_i^T U_i \bar{E}_i \\ 0 < \bar{P}_i - \bar{E}_i^T W_i \bar{E}_i \end{cases} \qquad i \in I_1$$

Then every trajectory $x(t) \in \cup_{i \in I} X_i$ satisfying

$$\dot{x} = A_i x + a_i \qquad \text{for } x \in X_i$$

tends to zero exponentially.

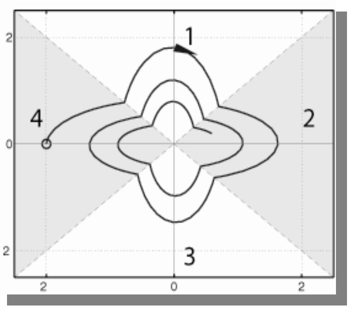
Example

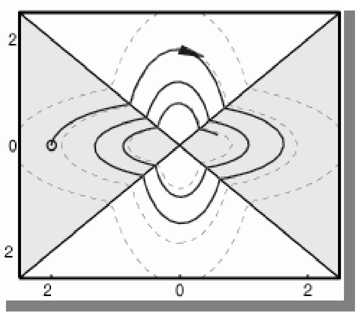
Piecewise linear system with partition shown below,

$$A_1 = A_3 = \begin{bmatrix} -\epsilon & \omega \\ -\alpha\omega & -\epsilon \end{bmatrix}, \qquad A_2 = A_4 = \begin{bmatrix} -\epsilon & \alpha\omega \\ -\omega & -\epsilon \end{bmatrix}.$$

and $\alpha = 5$, $\omega = 1$, $\epsilon = 0.1$

(Clearly) not quadratically stable, but pwQ Lyapunov function readily found.





Potential sources of conservatism

1. Quadratic Lyapunov functions necessary and sufficient for linear systems, *but* piecewise quadratic Lyapunov functions *not necessary* for stability of PWL systems.

2. S-procedure terms $\bar{E}_i^T W_i \bar{E}_i$ effectively the sum of several quadratic forms

$$\bar{x}^T \bar{E}_i^T W_i \bar{E}_i \bar{x} = \sum_i \sum_j w_{ij} (\bar{e}_i^T \bar{x})^T (\bar{e}_j^T \bar{x})$$

   hence, S-procedure is not guaranteed to be loss-less (better tools exist)

3. Use of affine terms and strict inequalities can also be conservative.

$\vdots$

Extensions

Many extensions possible:

- determining regions of attraction (i.e. non-global stability properties)
- Lyapunov functions that guarantee stability of potential sliding modes
- nonlinear and uncertain dynamics in each region
- performance analysis (e.g. L_2-gains)
- (some) control synthesis
- hybrid systems (overlapping regions) and discontinuous Lyapunov fncs.
- Lyapunov functionals and Lagrange stability
- stability of limit cycles
- similar tools for discrete-time hybrid systems

$\vdots$

(too much to be covered in this lecture!)

We will sketch a couple of extensions

Performance analysis

Theorem (Upper Bound on L_2 Gain). *Suppose there exist symmetric matrices T, U_i and W_i such that U_i and W_i have non-negative entries, while $P_i = F_i^T T F_i$ and $\bar{P}_i = \bar{F}_i^T T \bar{F}_i$ satisfy*

$$0 > \begin{bmatrix} P_i A_i + A_i^T P_i + C_i^T C_i + E_i^T U_i E_i & P_i B_i \\ B_i^T P_i & -\gamma^2 I \end{bmatrix} \qquad \textit{for } i \in I_0$$

$$0 > \begin{bmatrix} \bar{P}_i \bar{A}_i + \bar{A}_i^T \bar{P}_i + \bar{C}_i^T \bar{C}_i + \bar{E}_i^T U_i \bar{E}_i & \bar{P}_i \bar{B}_i \\ \bar{B}_i^T \bar{P}_i & -\gamma^2 I \end{bmatrix} \qquad \textit{for } i \in I_1$$

Then for every trajectory with $x(0) = 0$, $\int_0^\infty (\|x\|_2^2 + \|u\|_2^2)\, dt < \infty$

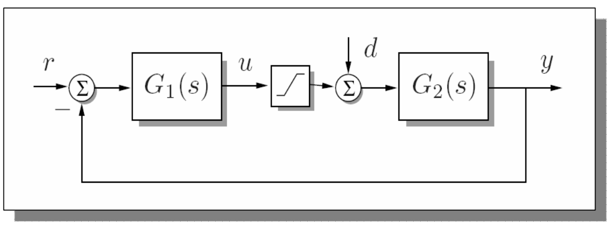
$$\int_0^\infty \|y\|_2^2 \, dt \le \gamma^2 \int_0^\infty \|u\|_2^2 \, dt.$$

The best upper bound on the L_2 induced gain is achieved by minimizing γ^2 subject to the constraints defined by the inequalities.

Proof. Pre/postmultiply with (x, u), note that LMIs imply dissipation inequality

Example

Saturated linear system (unit saturation)



$$G_1(s) = \frac{s-3}{16s^2 + s + 2}$$

$$G_2(s) = \frac{s+7}{4s^2 + 3s + 12}$$

Quadratic storage functions fail to bound L_2-gain.

Piecewise quadratic storage function yields bounds

$$5.52 \le \gamma \le 5.54$$

Linear hybrid dynamical systems

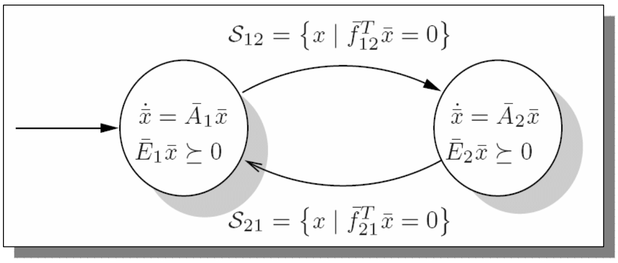
Linear hybrid dynamical system (LHDS)

$$\dot{x}(t) = A_{i(t)}x(t) + a_{i(t)}$$
$$i(t^+) = \nu(x(t), i(t))$$

ν described by finite automaton whose state changes when x hits *transition surfaces*

$$\mathcal{S}_{ij} = \{x \mid \bar{f}_{ij}\bar{x} = 0\}$$

and for each i, the feasible x bounded by a polyhedron $X_i = \{x \mid \bar{E}_i\bar{x} \succeq 0\}$



Discontinuous Lyapunov functions

Multiple quadratic (discontinuous, pwq) Lyapunov function via LMIs

Theorem. *Consider symmetric matrices U_i, W_i with non-negative entries, symmetric matrices P_i, $\bar{P}_i$, and vectors t_{jk}, $\bar{t}_{jk}$ such that*

$$\begin{cases} 0 > A_i^T P_i + P_i A_i + E_i^T U_i E_i \\ 0 < P_i - E_i^T W_i E_i \end{cases} \qquad i \in I_0 \tag{1}$$

$$\begin{cases} 0 > \bar{A}_i^T \bar{P}_i + \bar{P}_i \bar{A}_i + \bar{E}_i^T U_i \bar{E}_i \\ 0 < \bar{P}_i - \bar{E}_i^T W_i \bar{E}_i \end{cases} \qquad i \in I_1 \tag{2}$$

$$0 < \bar{P}_j - \bar{P}_k + \bar{f}_{jk}\bar{t}_{jk}^T + \bar{t}_{jk}\bar{f}_{jk}^T \qquad (j,k) \in T, \quad j \in I_1 \text{ or } k \in I_1 \tag{3}$$

$$0 < P_j - P_k + f_{jk}t_{jk}^T + t_{jk}f_{jk}^T \qquad (j,k) \in T, \quad j,k \in I_0 \tag{4}$$

Then every trajectory of the LHDS tends to zero exponentially.

Note: conditions (3,4) imply that V(t) decreases at points of discontinuity

Example

Linear hybrid system

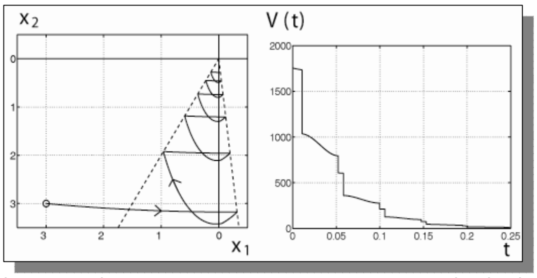
$$\dot{x}(t) = A_{i(t)}x(t)$$

$$i(t^+) = \begin{cases} 2 & \text{if } i(t) = 1 \text{ and } x_2 = -10x_1 \\ 1 & \text{if } i(t) = 2 \text{ and } x_2 = 2x_1 \end{cases}$$

with

$$A_1 = \begin{pmatrix} -1 & -100 \\ 10 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 10 \\ -100 & 1 \end{pmatrix}$$

Trajectories and multiple Lyapunov function found by LMI formulation



Discrete-time versions

Discrete-time piecewise linear systems

$$x[k+1] = A_i x[k] + a_i + B_i u[k] \qquad x[k] \in X_i$$

and piecewise quadratic Lyapunov (not necessarily continuous) functions

$$V(x[k]) = x[k]^T P_i x[k] + 2q_i^T x[k] + r_i \qquad x[k] \in X_i$$

We have

$$\begin{aligned} \Delta V(x[k]) &= V(x[k+1]) - V(x[k]) \\ &= x[k]^T A_i^T P_j A_i x[k] + 2(a_i^T P_j A_i x + q_j^T A_i)x[k] + a_i^T P_j a_i + 2q_j^T a_i + r_j \\ &- x[k]^T P_i x[k] + 2q_i^T x[k] + r_i \\ &= \begin{bmatrix} x[k] \\ 1 \end{bmatrix}^T \begin{bmatrix} A_i^T P_j A_i - P_i & A_i^T P_j a_i + q_j - q_i \\ (\cdot)^T & a_j^T P_i a_j + 2q_j a_i + r_j - r_i \end{bmatrix} \begin{bmatrix} x[k] \\ 1 \end{bmatrix} \end{aligned}$$

for $x[k] \in X_{ij} = \{x \mid x \in X_i \wedge A_i x + a_i \in X_j\} = \{x \mid \bar{E}_i \bar{x} \succeq 0 \wedge \bar{E}_j \bar{A}_i \bar{x} \succeq 0\}$

Discrete-time versions

Discrete-time globally asymptotically stable if there exist matrices P_i, q_i, r_i, $\underline{U}_{ij}$ where W_{ij} has non-negative entries, and a non-negative scalar ε>0, such that

$$\begin{bmatrix} A_i^T P_j A_i - P_i & A_i^T P_j a_i + q_j - q_i \\ (\cdot)^T & a_j^T P_i a_j + 2 q_j a_i + r_j - r_i \end{bmatrix} + \bar{E}_{ij}^T U_{ij} \bar{E}_{ij} \leq \begin{bmatrix} -\epsilon I & 0 \\ 0 & 0 \end{bmatrix}$$

(note: in most solvers, you will need to treat $X_i,\ i \in I_0$ separately)

Observations:

- Again, LMI conditions, hence efficiently verified!
- Potentially one LMI for every pair (i,j) of modes.

Comparison with alternatives

Biswas *et al.* generated optimal hybrid controllers for randomly generated linear systems, and compared performance of several computational methods

Typical results:

Partitions obtained for 3^{rd} order LTI systems, 2 norm objective						
	50 Stable Systems, $N = 1$			50 Unstable Systems, $N = 1$		
Method	Success	Solution Time	Setup Time	Success	Solution Time	Setup Time
Quadratic	45/50	0.7 sec.	0.4 sec.	43/50	1.1 sec.	0.5 sec.
Piecewise Quadratic	50/50	0.7 sec.	1.3 sec.	50/50	1.9 sec.	2.5 sec.
Common SOS order 4	42/50	7.6 sec.	82.2 sec.	32/50	11.4 sec.	141.9 sec.
Piecewise SOS order 4	35/50	12.1 sec.	100.0 sec.	31/50	80.6 sec.	263.8 sec.

Table 2. The number of regions were between 9 and 15 with 9-47 transitions.

Very strong performance, but computational effort increases rapidly

Summary

Computational tools for stability analysis of *one* class of hybrid systems

Piecewise linear systems
- Partition of state space into polyhedra with locally affine dynamics
- Solution concepts: trajectories and Flippov solutions
- Given a pwl model, it is non-trivial to detect attractive sliding modes

Piecewise quadratic Lyapunov functions
- Efficiently computed via optimization over linear matrix inequalities
- Potentially conservative, but strong practical performance

Many extensions, but much work remains!

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Stability and stabilization of hybrid systems

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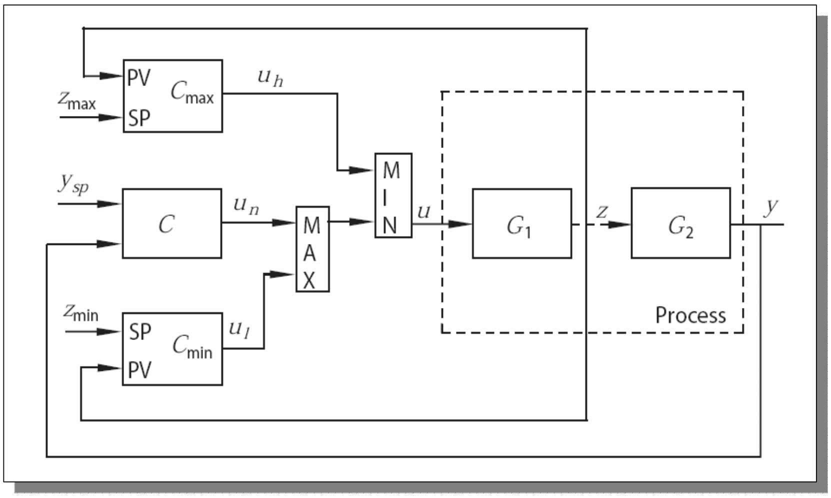
Part III – Examples

- Constrained control via min-max selectors
- Substrate feeding control
- Automatic gear-box control
- A simple relay system

Constrained control via min-max selectors

Common “pre-HYCON” approach for constrained control

Aim: tracking primary variable (y), while
keeping secondary variable (z) within limits



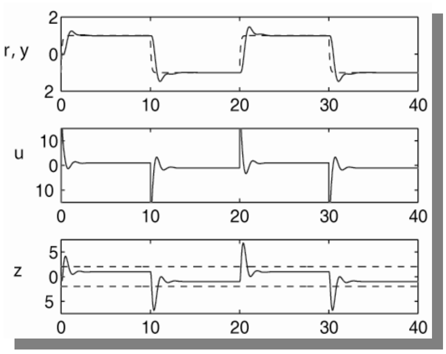
Numerical example

Specific example with

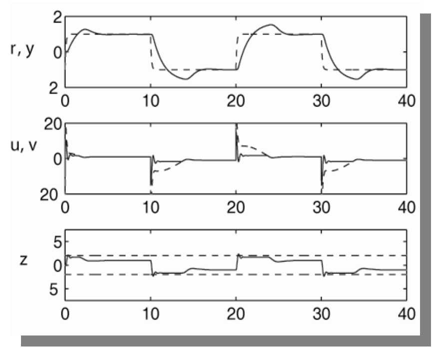
$$G_1(s) = \frac{40}{0.5s^3 + 2s^2 + 22s + 40} \qquad G_2(s) = \frac{5}{s^2 + 7s + 5} \qquad C(s) = \frac{s^2 + 3s + 3}{0.02s^2 + s + 0.01}$$

and proportional constraint controllers.

Control without constraint handling



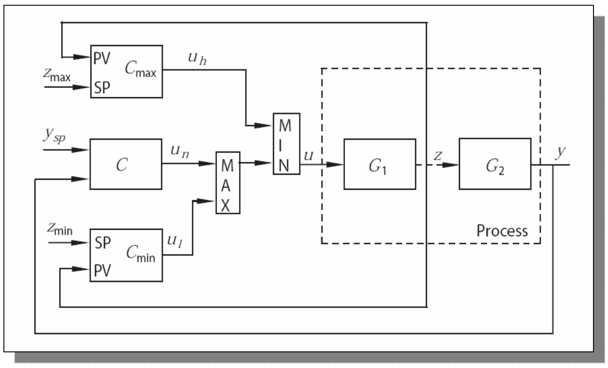
Control with constraint handling

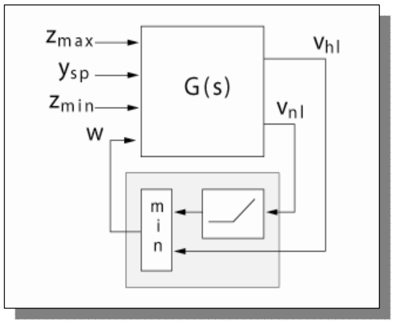


A loop transformation

Linear system interconnected with 3-input/1-output nonlinearity

Loop transformation reduces dimension of nonlinearity by one:

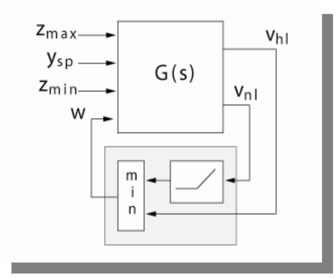


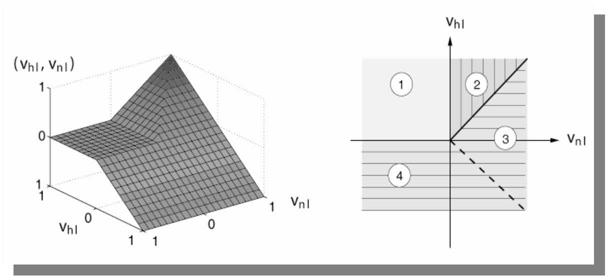


still, few techniques apply to such systems
(e.g. small gain and LDI do not work)

Stability analysis

However, nonlinearity (and hence system) is piecewise linear:





LMI computations return quadratic Lyapunov function
(but S-procedure needed)

Part III – Examples

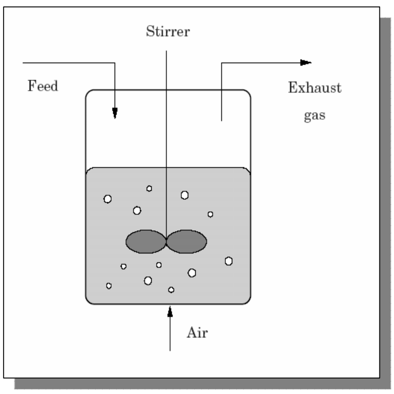
- Constrained control via min-max selectors
- Substrate feeding control
- Automatic gear-box control
- A simple relay system

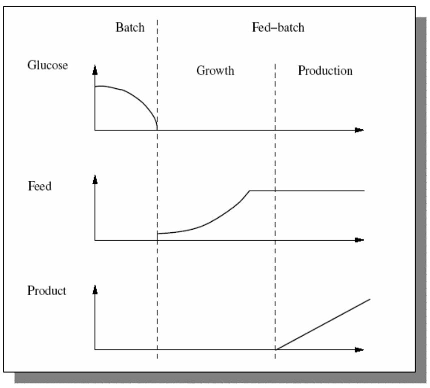
Fed-batch cultivation of *E. coli*

Recombinant (genetically modified) *E. coli* bacteria used to produce proteins.

Bioreactor control: Add feed (nutrition) and oxygen to maximize cell growth.

Fed-batch: feed added continuously, at limiting rate



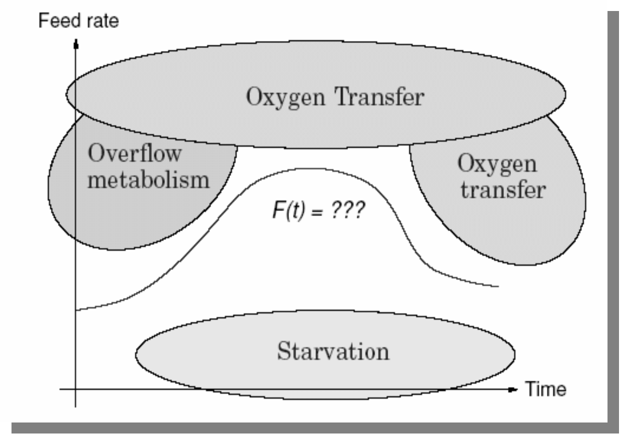


[Velut, 2005]

Control objective

Objective: maximize feed rate while ensuring that

- oxygen level does not drop too low (acetate production, inhibited growth)
- glucose is not in excess ("overflow metabolism")

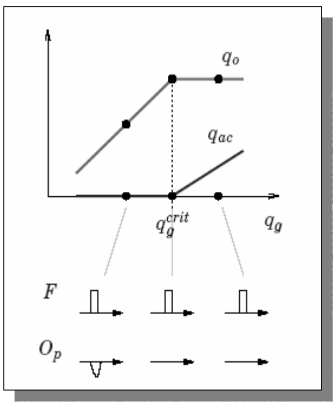


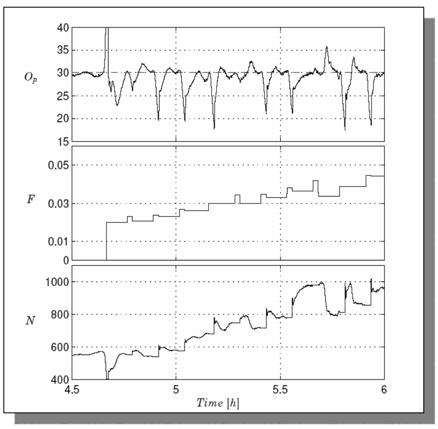
Probing control

Control strategy: increase feed while no acetate formed, decrease otherwise

Acetate formation detected by probing:

– add pulse in feed, observe if oxygen consumed





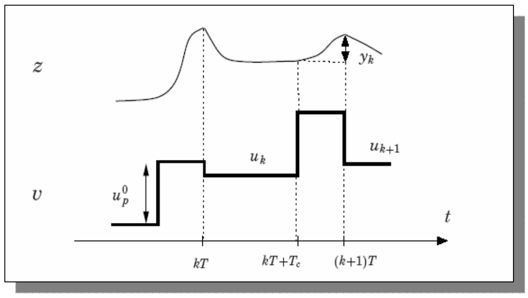
A piecewise linear abstraction

Simplified model of reactor dynamics

$$\dot{x} = ax + bf(v)$$
$$y = cx$$

where $f(v)$ is a piecewise linear function

$$v(t) = u_k + u_p(t) \quad t \in [kT, (k+1)T]$$



Integrating the response over a pulse period, we find the discrete-time model

$$x[k+1] = Ax[k] + B \begin{bmatrix} f(u_k) \\ f(u_k + u_p^0) \end{bmatrix}$$
$$y[k] = Cx[k] + D \begin{bmatrix} f(u_k) \\ f(u_k + u_p^0) \end{bmatrix}$$

Piecewise linear if u_k is a linear in x.

Control strategy

Assume a linear integral control

$$u[k+1] = u[k] + K(y_{\text{ref}}[k] - y[k])$$

fixed length of probing cycle T and probing pulse $T-T_c$

To model saturation in glucose uptake, consider

$$f(v) = \min(v, r^\star)$$

This results in a piecewise linear systems with three regions (why not two?)

Control objective is now to drive system towards saturation.

Control to saturation

The formulation in Lecture 2 does not return any feasible solution
- integrator dynamics in unbounded regions → not exponentially stable

Two potential approaches:

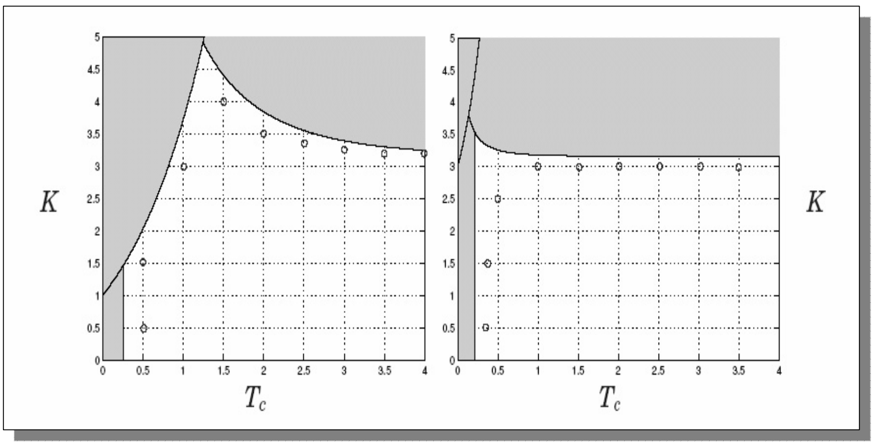
- Prove convergence for initial values within (large but bounded) region (can be done by adding S-procedure terms)

- Remove implicit equality constraints by state-transformation (more satisfying, but more complex; see Velut)

With modifications, stability can (often) be proven VIA pwq Lyapunov fncs.

Numerical results

Stability regions for one specific problem instance (reactor parameters)
- red dots bound region where stability can be established numerically
- shaded regions are shown to be unstable (via local analysis)



Performance analysis

Stability often not enough with stability – would like to optimize performance

- for example, the ability to track time-varying saturation level

Can compute bound γ on performance

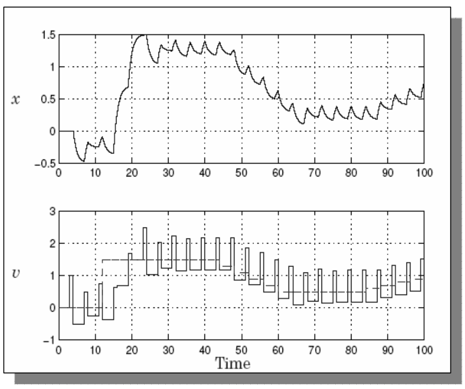
$$\sum_k (\bar{x}[k] - \bar{x}_{\text{ref}}[k])^T \bar{Q} (\bar{x}[k] - \bar{x}_{\text{ref}}[k]) \leq \gamma^2 \sum_k (r[k+1] - r[k])^2$$

for all reference trajectories r[k] via LMI computations.

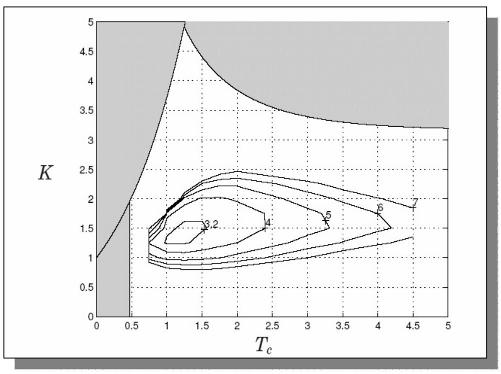
Note: typically large system descriptions…

Numerical example

Simulations for specific r[k]



γ for all rate-limited references



$K \approx 1.4,\ T_c \approx 1.3$

Parameter contours suggest optimal parameters

Tuning rules

Similar behavior observed for various parameter values of the process.

Based on this observation, Velut suggests the following tuning rules

$$K = \frac{1}{\sigma(T - T_c)}$$
$$y_{\text{ref}} = \frac{\sigma(T - T_c)}{2} u_p^0$$
$$1 < aT_c < 2$$

where $\sigma(t)$ is the unit step response of the linear dynamics.

Part III – Examples

- Constrained control via min-max selectors
- Substrate feeding control
- Automatic gear-box control
- A simple relay system

A simple model for car dynamics

Simple model:

$$
\begin{aligned}
M\dot{v} &= F - F_l && \text{car dynamics}\\
F_l &= kv^2\,\mathrm{sign}(v) - Mg\sin\alpha && \text{load force}\\
F &= p/rT && \text{gear box relations}\\
v &= r/p\omega && \text{gear box relations}
\end{aligned}
$$

Inputs: motor torque T and road incline α; output ω

$$
\begin{aligned}
\dot{v} &= \frac{1}{M}Tu - \frac{k}{M}v^2\,\mathrm{sign}v - g\sin\alpha\\
\omega &= vu
\end{aligned}
$$

where $u = p/r$ is the discrete input, determined by the current gear

To emphasize this dependence, we write

$u = u_i := p_i/r_i$ when using gear i

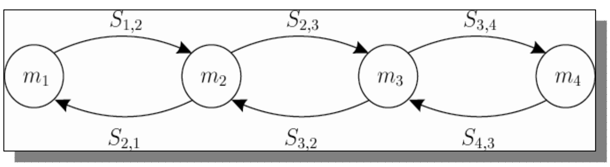
[Pettersson, 1999]

Gear-switching

Gear-switching strategy:

$$
\begin{aligned}
i(t^+) &= i(t) + 1 && \text{if } \omega > \omega_{i(t)}^{\text{hi}}\\
i(t^+) &= i(t) - 1 && \text{if } \omega < \omega_{i(t)}^{\text{lo}}
\end{aligned}
$$

Can be represented by hybrid automaton with four discrete states



Torque control and bumpless transfer

Base controller: non-linear PI

$$
\begin{aligned}
T &= P + I + \frac{k}{u_i} v^2 \mathsf{sign}\, v \\
P &= K_{i(t)}(v_{\mathsf{ref}} - v) \\
\frac{d}{dt} I &= \frac{K_{i(t)}}{T_{i(t)}}(v_{\mathsf{ref}} - v)
\end{aligned}
$$

Changes in acceleration when shifting gears avoided via bumpless transfer:

$$
\begin{aligned}
u_i K_i &= u_j K_j \\
I(t^+) &= \frac{u_{i(t)}}{u_{i(t^+)}} I(t)
\end{aligned}
$$

for all feasible gear changes i→j.
(compatible values of K_i, changes in integral state)

Hybrid system model

Need extended hybrid model that allows for jumps in the continuous state

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), i(t)) \\
x(t^+) &= \rho(x(t), i(t)) \\
i(t^+) &= \nu(x(t), i(t))
\end{aligned}
$$

LMI formulation possible if jump map is affine in x.

Numerical example

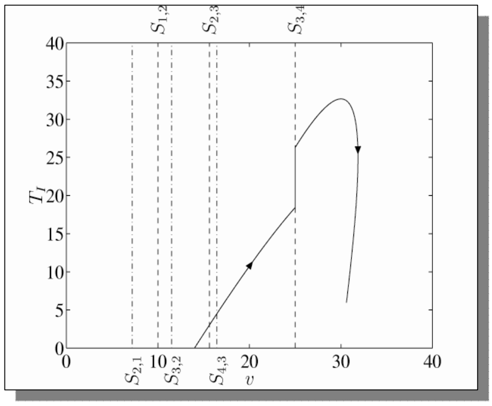
Closed loop system is switched linear system

$$\frac{d}{dt}\begin{bmatrix} e \\ I \end{bmatrix} = \begin{bmatrix} -u_i K_i/M & -p_i/M \\ K_i/T_i & 0 \end{bmatrix} \begin{bmatrix} e \\ I \end{bmatrix} \text{ for } e \in X_i$$

where $e = v_{\text{ref}} - v$ and

$$u_i = \{50, 32, 20, 14\}$$
$$K_i = \{3.75, 5.86, 9.37, 13.39\}$$
$$M = 1500,\ T_i = 40,\ T_i K_i = 187.5$$

Simulation for $v_{\text{ref}} = 30$



Stability

If affine reset maps

$$x(t^+) = H_{i(t)i(t^+)}\bar{x}(t)$$

then, $\bar{x}(t^+)^T \bar{P}_j \bar{x}(t^+) < \bar{x}(t)^T \bar{P}_i \bar{x}(t)$ is guaranteed by solution to LMI

$$0 < \bar{P}_i - H_{jk}^T \bar{P}_k \bar{H}_{jk} + \bar{f}_{jk}^T \bar{t}_{jk} + \bar{t}_{jk}^T \bar{f}_{jk}$$

Can extend discontinuous Lyapunov function computations from Lecture 2

Gear-box example: solution found → exponential convergence to v_{ref}

Remark: analysis needs to be repeated for each value of v_{ref}
(as in bioreactor example)

Part III – Examples

- Constrained control via min-max selectors
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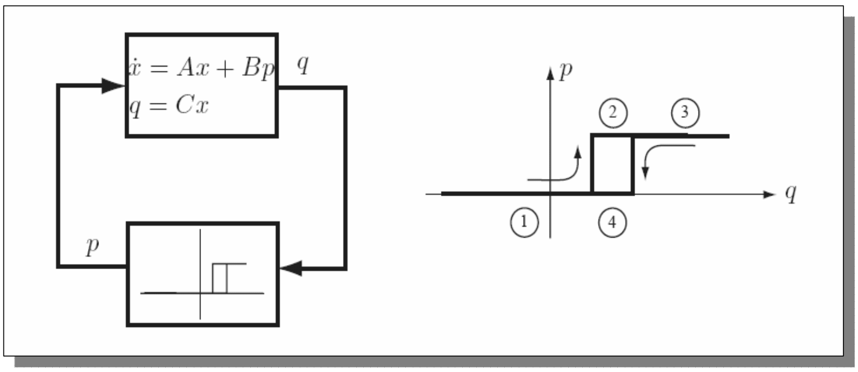
More of a theoretical challenge…

Consider a linear control system under hysteresis relay feedback…



[Hassibi, 2000]

$$A = \begin{pmatrix} -0.1 & -1 \\ 0 & -0.2 \end{pmatrix}, \quad B = \begin{pmatrix} 0.2 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Simulations suggest system is stable, yet no pwq Lyapunov function found.

The challenge

Q: why do piecewise quadratic methods fail, how can they be improved?

The more general challenge:

Put the methods to the test of challenging engineering problems, and help to contribute to the development to improved analysis tools!

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