A detailed study of Savage's paper "The space of positive definite matrices and Gromov's invariant"

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In this note, we shall examine in detail Savage's paper [Sa82] and show why the proof presented there of the positivity of the simplicial volume of compact locally symmetric spaces covered by $\mathrm{SL}_n\mathbb{R}/SO(n)$, where $n \geq 2$, is not fully correct. It is easy to show that the result would follow from a uniform bound on the volume of straight ideal simplices with rank 1 boundary points as vertices. Although a volume bound is computed in [Sa82], this bound only applies to simplices of the first barycentric subdivision corresponding to a *certain* order of *certain* ideal simplices. This is not enough to prove the asserted theorem. For a proof of the theorem for $n = 3$ see [Bu05] and for $n \geq 4$ and more generally any compact locally symmetric space of noncompact type not covered by $\mathrm{SL}_3\mathbb{R}/SO(3)$, see [LaSch05].

Note that the mistake is easy to see: The statement of Theorem 7.4 in [Sa82] and its proof do not agree. We shall say more on that below.

For the notation, which we have tried to keep as close as possible to Savage's, we invite the reader to consult the first sections of either [Sa82] or [Bu05].

1 General position

We start with an easy lemma providing a useful criterion for knowing when a simplex generated by rank 1 matrices is degenerate.

Lemma 1 *Let $x_0, ..., x_d$ be points in S^{n-1}. Then the simplex of Pos_n^{tr} generated by $x_0x_0^t$, ..., $x_dx_d^t$ is degenerate if and only if there exists i in $\{1, ..., n-1\}$ and $i(i+1)/2+1$ vectors among $x_0, ..., x_d$ spanning an i-dimensional vector subspace of $\mathbb{R}^n$.*

Proof. We will here only need the "if" part of the lemma, so we omit the proof of the "only if" part, which is as easy.

Suppose that $i(i+1)/2+1$ vectors among $x_0, ..., x_d$ span an i-dimensional vector subspace of $\mathbb{R}^n$. This is equivalent to saying that $i(i+1)/2+1$ points among $x_0x_0^t$, ..., $x_dx_d^t$ and hence an $i(i+1)/2$-dimensional face of the simplex generated by the $x_j x_j^t$'s lie in a copy of $\mathrm{Pos}_i^{\mathrm{tr}}$ contained in the boundary $\partial\mathrm{Pos}_n^{\mathrm{tr}}$. But $\mathrm{Pos}_i^{\mathrm{tr}}$ has dimension equal to $i(i+1)/2-1$, so the simplex must be degenerate. ■

The statement corresponding to Lemma 1 in [Sa82] is Theorem 5.7 which gives the maximal dimension of a straight simplex lying in the subset of the boundary $\partial\mathrm{Pos}_n^{\mathrm{tr}}$ consisting of rank i matrices, for every $1 \leq i \leq n-1$.

The next theorem shows how arbitrary simplices are put in general position. It is simple to prove using only the criterion of Lemma 1 for degeneracies. In [Sa82], this is done in the beginning of Section 7.

Theorem 2 *Let $P_0, ..., P_d$ be rank 1 matrices in ∂Pos_n^{tr} generating a nondegenerate simplex. Then there exists g in $SL_n\mathbb{R}$ and integers $0 = \beta_0 < ... < \beta_n \leq d$ satisfying*

$$\beta_i \leq \frac{(i-1)i}{2}$$

such that

$$\rho_g^{tr}(P_{\beta_i}) = E_i,$$

and furthermore, $P_0, ..., P_{\beta_i - 1}$ lie in a copy of Pos_{i-1}^{tr} in ∂Pos_{i-1}^{tr}, for every i in $\{1, ..., n\}$.

Proof. Let $P_0, ..., P_d$ be rank 1 matrices generating a nondegenerate simplex, and let $x_0, ..., x_d$ be the vectors of $\mathbb{R}^n$ (unique up to a sign) satisfying $P_i = x_i x_i^t$, for every i in $\{0, ..., d\}$. We start by proving the following simple fact:

Claim 3 *There exists $0 = \beta_1 < ... < \beta_n \leq d+1$ such that*

1. $\beta_i \leq \frac{(i-1)i}{2}$,
2. $\langle x_0, ..., x_{\beta_i} \rangle$ *is i-dimensional,*
3. $\langle x_0, ..., x_{\beta_i - 1} \rangle$ *is $(i-1)$-dimensional.*

Proof of Claim. We prove the claim by induction: Set $\beta_1 = 0$ and assume there exists $\beta_1, ..., \beta_{i-1}$ as in the claim. Since $P_0, ..., P_d$ generate a nondegenerate simplex, it follows from Lemma 1 that any $(i-1)i/2+1$ vectors among $x_0, ..., x_d$ span an i-dimensional space. This is in particular the case for the vectors $x_0, ..., x_{(i-1)i/2}$. Since the vectors $x_0, ..., x_{\beta_{i-1}}$ span an $(i-1)$-dimensional vector space, and $\beta_{i-1} \leq (i-2)(i-1)/2$, there must exist a β_i as in the claim. ■

It is now easy to prove the theorem: Let $h \in \mathrm{GL}_n\mathbb{R}$ be the unique matrix sending x_{β_i} to the standard vector e_i, for every i in $\{1, ..., n\}$. (Thus, the inverse of h is the matrix whose columns are $x_{\beta_1}, ..., x_{\beta_n}$.) If the sign of $\det(h)$ is not positive, we can without loss of generality replace x_{β_n} by $-x_{\beta_n}$. Set $g = \left(1/\det(h)^{1/n}\right) h$ and note that obviously, g belongs to $\mathrm{SL}_n\mathbb{R}$. We have

$$\frac{g x_{\beta_i}}{\|g x_{\beta_i}\|} = e_i,$$

for i in $\{1, ..., n\}$, since

$$gx_{\beta_i} = \frac{1}{\det(h)^{1/n}} hx_{\beta_i} = \frac{1}{\det(h)^{1/n}} e_i,$$

and hence, taking the norm we obtain

$$\|gx_{\beta_i}\| = \frac{1}{\det(h)^{1/n}}.$$

Finally, it is easy to compute

$$\rho_g^{\mathrm{tr}}(P_{\beta_i}) = \rho_g^{\mathrm{tr}}(x_{\beta_i} x_{\beta_i}^t) = \frac{1}{\mathrm{tr}(gx_{\beta_i} x_{\beta_i}^t g^t)} gx_{\beta_i} x_{\beta_i}^t g^t = e_i e_i^t,$$

since $\mathrm{tr}(yy^t) = \|y\|^2$, for any vector y in $\mathbb{R}^n$. ■

2 Barycentric subdivision

Let $P_0, ..., P_d$ be arbitrary points in $\overline{\mathrm{Pos}_n^{\mathrm{tr}}}$ and

$$\sigma(P_0, ..., P_d) : \Delta^d \longrightarrow \overline{\mathrm{Pos}_n^{\mathrm{tr}}}$$

be the straight singular simplex defined as the convex linear combination of the $d + 1$ points $P_0, ..., P_d$, that is,

$$\sigma(P_0, ..., P_d)(t_0, ..., t_d) = \sum_{i=0}^{d} t_i P_i.$$

The standard simplex Δ^d admits a covering by $(d + 1)!$ simplices of its first barycentric subdivision. The latter simplices are defined as follows: For every order $\prec_\tau$ of the vertices $e_0, ..., e_d$ of Δ^d given by a permutation $\tau \in S_{d+1}$, that is, $e_i \prec_\tau e_j$ if and only if $\tau(i) \leq \tau(j)$, the simplex of the first barycentric subdivision of Δ^d corresponding to the order $\prec_\tau$ is defined as

$$\begin{aligned} \Delta_\tau &= \{(t_0, ..., t_d) \in \Delta^d \mid t_i \leq t_j \text{ if and only if } e_i \prec_\tau e_j\} \\ &= \{(t_0, ..., t_d) \in \Delta^d \mid t_i \leq t_j \text{ if and only if } \tau(i) \leq \tau(j)\}. \end{aligned}$$

A permutation $\tau \in S_{d+1}$ clearly similarly gives an order on the vertices $P_0, ..., P_{d+1}$ and we define the simplex of the first barycentric subdivision of $\sigma(P_0, ..., P_d)$ corresponding to the order $\prec_\tau$,

$$\sigma_{\mathrm{bar}}(P_0, ..., P_d; \prec_\tau) : \Delta_\tau \to \overline{\mathrm{Pos}_n^{\mathrm{tr}}},$$

to be the restriction of $\sigma(P_0, ..., P_d)$ to Δ_τ.

The barycenter of the simplex $\sigma(P_0, ..., P_d)$ is as usual given as

$$\mathrm{bar}(P_0, ..., P_d) = \sigma(P_0, ..., P_d)\left(\frac{1}{d+1}, ..., \frac{1}{d+1}\right) = \frac{1}{d+1}\sum_{i=0}^{d} P_i.$$

Those barycentric subdivisions, which are by nature euclidean constructions, are not invariant under isometries of the symmetric space. To see that, it is enough to observe that barycenters are not invariant. Let $P_0, ..., P_d$ be vertices in $\overline{\mathrm{Pos}_n^{\mathrm{tr}}}$ and let $g \in \mathrm{SL}_n\mathbb{R}$ define the isometry $\rho_g^{\mathrm{tr}} : S \mapsto (1/\mathrm{tr}(gSg^t))\, gSg^t$. We have

$$\rho_g^{\mathrm{tr}}(\mathrm{bar}(P_0, ..., P_d)) = \frac{1}{\mathrm{tr}\left(\sum_{i=0}^{d} gP_ig^t\right)} \sum_{i=0}^{d} gP_ig^t, \tag{1}$$

whereas

$$\mathrm{bar}(\rho_g^{\mathrm{tr}}(P_0), ..., \rho_g^{\mathrm{tr}}(P_d)) = \frac{1}{d+1}\sum_{i=0}^{d} \frac{1}{\mathrm{tr}\,(gP_ig^t)} gP_ig^t. \tag{2}$$

Since $\mathrm{tr}(gP_ig^t)$ is in general far from being constant, the expressions (1) and (2) cannot agree.

3 Savage's unproven Theorem

Let now $P_0, ..., P_d$ be arbitrary rank 1 matrices in $\partial\mathrm{Pos}_n^{\mathrm{tr}}$ and let $\sigma : \Delta^d \to \overline{\mathrm{Pos}_n^{\mathrm{tr}}}$ be the straight ideal simplex given by

$$\sigma(t_0, ..., t_d) = \sum_{i=0}^{d} t_iP_i.$$

Since the volume of a degenerate simplex is zero, we can without loss of generality assume that the simplex σ is nondegenerate. Applying Theorem 2 to the vertices P_i's, we put the simplex σ in general position via an isometry ρ_g^{tr}, for some g in $\mathrm{SL}_n\mathbb{R}$. In accordance with Savage's notation, we now denote this isometry by h and the singular simplex with vertices $h(P_0), ..., h(P_d)$ by $f : \Delta^d \to \overline{\mathrm{Pos}_n^{\mathrm{tr}}}$, so that

$$f(t_0, ..., t_d) = \sum_{i=0}^{d} t_ih(P_i).$$

Of course, since h is an isometry, the volume of σ is equal to the volume of $h \circ \sigma$. Furthermore, f and $h \circ \sigma$ have same image and hence same volume. Therefore, the two simplices f and σ have same volume. But this is not true anymore when one restricts to first barycentric subdivisions, since as we saw in the previous section, barycenters are not invariant under isometries of the symmetric space. This is Savage's mistake. Once observed, it is very easy to point out. Indeed, the main volume bound in [Sa82] is obtained as a consequence of the unproven Theorem 7.4. Before we can state the latter theorem and its true proven version, we need some more notation. Choose w_i on the unit sphere of $\mathbb{R}^n$ such that $h(P_i) = w_i w_i^t$, for $i = 0, ..., d$. Let $\langle ., . \rangle$ be the standard scalar product on $\mathbb{R}^n$. Choose $\alpha_1, ..., \alpha_n$ between 0 and d such that $\langle w_{\alpha_i}, e_n \rangle$ has maximal absolute value. Note that by construction, $\beta_1 < ... < \beta_{n-1} < \alpha_i$, for $i = 1, .., n$.

Unproven Theorem 7.4 of [Sa82] *Notation as above. Let T be a subset of the image of $h \circ \sigma$ and let $\Delta_T \subset \Delta^d$ be its preimage $\Delta_T = (h \circ \sigma)^{-1}(T)$. Then there exists a constant $C(n)$ such that*

$$\mathrm{Vol}(T) \leq C(n) \left| \prod_{i=1}^{n} \langle w_{\alpha_i}, e_n \rangle \right| \int_{\Delta_T} \frac{dt_1 ... dt_d}{\left(\left(\Pi_{k=1}^{n-1} t_{\beta_k} \right) \left(\Sigma_{i=1}^{n} t_{\alpha_i} \langle w_{\alpha_i}, e_n \rangle^2 \right) \right)^{(n+1)/2}}.$$

Wrong proof of [Sa82]. The first equation of the proof - which is correct - just relies on an explicit expression for the volume form. This form is computed in both Theorem 4.3 in [Sa82] and Proposition 7 in [Bu05]. Thus one has

$$\mathrm{Vol}(T) = \int_T \frac{C_0(n)}{(\det(S))^{(n+1)/2}} dx_1 \wedge ... \wedge dx_d,$$

where $C_0(n)$ is a positive constant. The mistake is now that Savage applies the change of variable formula to the map $f : \Delta^d \to \mathrm{Pos}_n^{\mathrm{tr}}$, while he replaces the domain, not by $f^{-1}(T)$ as he should, but by $\Delta_T = (h \circ \sigma)^{-1}(T)$. In this way, he concludes, using his Theorem 5.14 (Lemma 9 in [Bu05]) that

$$\begin{aligned} \mathrm{Vol}(T) &= \int_{\Delta_T} \frac{C_0(n) dt_1 \cdot ... \cdot dt_d}{(\det(f(t_1, ..., t_d)))^{(n+1)/2}} \\ &= \int_{\Delta_T} \frac{|\det f'| \, C_0(n) dt_1 \cdot ... \cdot dt_d}{(\Sigma_{j_1 < ... < j_n} (\Pi_{i=1}^{n} t_{j_i}) \det(w_{j_1}, ..., w_{j_n}))^{(n+1)/2}}, \end{aligned}$$

while he should have concluded that

$$\mathrm{Vol}(T) = \int_{f^{-1}(T)} \frac{|\det f'| \, C_0(n) dt_1 \cdot ... \cdot dt_d}{(\Sigma_{j_1 < ... < j_n} (\Pi_{i=1}^{n} t_{j_i}) \det(w_{j_1}, ..., w_{j_n}))^{(n+1)/2}}. \tag{3}$$

Thus, the true statement is contained in the next theorem. ∎

True Theorem 7.4 *Notation as above. Let T be a subset of the image of $h \circ \sigma$ and let $\overline{\Delta_T} \subset \Delta_0$ be its preimage $\overline{\Delta_T} = (f)^{-1}(T)$. Then there exists a constant $C(n)$ such that*

$$\mathrm{Vol}(T) \leq C(n) \left| \prod_{i=1}^{n} \langle w_{\alpha_i}, e_n \rangle \right| \int_{\overline{\Delta_T}} \frac{dt_1 ... dt_d}{\left(\left(\Pi_{k=1}^{n-1} t_{\beta_k} \right) \left(\Sigma_{i=1}^{n} t_{\alpha_i} \langle w_{\alpha_i}, e_n \rangle^2 \right) \right)^{(n+1)/2}}. \tag{4}$$

Proof of the True Theorem 7.4. It remains to show how the Theorem follows from Equation (3). This will be an easy consequence of the two following rather elementary inequalities.

On the one hand, we have, for the denominator of the integrand, the inequality

$$\Sigma_{j_1<\ldots<j_n}\left(\Pi_{i=1}^{n}t_{j_i}\right)\det(w_{j_1},...,w_{j_n})^2 \geq \left(\Pi_{k=1}^{n-1}t_k\right)\left(\Sigma_{i=1}^{n}t_{\alpha_i}\left\langle w_{\alpha_i},e_n\right\rangle^2\right), \quad (5)$$

which is simply given by restricting the sum on the left hand side to the subsum corresponding to the indices $\beta_1 < ...\beta_{n-1} < \alpha_i$, for $i = 1..n$. In passing, we have of course used the facts that all the summands are positive and that $w_{\beta_j} = e_j$ so that

$$\det(w_{\beta_1},...,w_{\beta_{n-1}},w_{\alpha_i})^2 = \left\langle w_{\alpha_i},e_n\right\rangle^2.$$

Inequality (5) corresponds to the observation following Theorem 7.4 in [Sa82], on page 257.

On the other hand, start by observing that

$$|\det f'| = \det(h(P_0),...,h(P_d)).$$

(For a proof, see Theorem 8 in [Bu05].) Let us now show, as is done in Savage's Theorem 6.1, that there exists a constant $C'(n)$, depending on n solely, such that

$$\det(h(P_1),...,h(P_{d+1})) \leq C'(n)\prod_{i=1}^{n}\left\langle e_n,w_{\alpha_i}\right\rangle. \quad (6)$$

To see that, recall that for every rank 1 vertex $R = ww^t$, where w belongs to $\mathbb{R}^n$, its coordinates have the form w^iw^j, where $1 \leq i \leq j \leq n$ and w^i stands for the i-th coordinate of the vector w. Furthermore, if R belongs to $\partial\mathrm{Pos}_n^{\mathrm{tr}}$, then w has norm 1. To simplify the notation, let us for the proof of the above inequality relabel the vertices $h(P_0),...,h(P_d)$ as R_{ij}, where $1 \leq i \leq j \leq n$ and accordingly renumber the vectors w_i, for $0 \leq i \leq d$, as w_{ij}, where $1 \leq i \leq j \leq n$. In particular, $R_{ij} = w_{ij}w_{ij}^t$. Let us now develop $\det(h(P_0),...,h(P_d)) = \det(R_{ij})$ as a sum over the permutations in S_{d+1}, which is identified with the permutations of the set $\{i \leq j \mid 1 \leq i,j \leq n\}$, of the products of the corresponding entries:

$$\det(R_{ij}) = \sum_{\sigma\in S_{d+1}}\mathrm{sign}(\sigma)\prod_{1\leq i\leq j\leq n}w^i_{\sigma(i\leq j)}w^j_{\sigma(i\leq j)}.$$

Each of the summand clearly satisfies the inequality

$$\left|\prod_{1\leq i\leq j\leq n}w^i_{\sigma(i<j)}w^j_{\sigma(i<j)}\right| \leq \prod_{i=1}^{n}\left\langle e_n,w_{\alpha_i}\right\rangle,$$

since the product of the left hand side of the above contains the n-th coordinate of precisely n distinct vectors, the w_{α_i}'s were chosen so as to have the n biggest n-th coordinate and each $w^i_{k,\ell}$ is smaller or equal to 1 in absolute value. We hence obtain the desired inequality with $C'(n) = 1/\left(d+1\right)!$.

We now plug the inequalities (5) and (6) in the volume formula (3) in order to obtain the correct version of Savage's Theorem 7.4. ■

In the next section, we will discuss the cumbersome computations that Savage makes in order to conclude that the integral appearing in (both versions of) Theorem 7.4 is uniformly bounded when integrated on the simplex

$$\Delta_0^d = \{(t_0, ..., t_d) \in \Delta^d \mid t_0 \geq ... \geq t_d\}.$$

This implies that the volume of $f(\Delta_0^d)$ is uniformly bounded, but not the volume of $(h \circ \sigma)(\Delta_0^d)$ as desired.

Observe that Theorem 7.4 can only be used to prove that the volume of certain simplices of the first barycentric subdivision of f is uniformly bounded. Indeed, the integral of Theorem 7.4 diverges for example for any simplex for which the t_{β_i}'s are the $n-1$ smallest coordinates: In this case, we would have to start by integrating, from 0 to one of the variable t_{β_j}, a function of the form $A/t^{(n+1)/2}$, which can not converge when $n \geq 1$.

4 The rest of the computations

As explained above, the correct version of Savage's Theorem 7.4 is of no use to prove the existence of a uniform bound on the volume of straight ideal simplices, but let us nevertheless finish the description of Savage's proof. Let, as before, Δ_0^d be the following simplex of the first barycentric subdivision of Δ^d:

$$\Delta_0^d = \{(t_0, ..., t_d) \in \Delta^d \mid t_0 \geq ... \geq t_d\}.$$

In order to bound the integral of Theorem 7.4 when integrated on Δ_0^d, Savage decomposes the simplex Δ_0^d into what he calls slices. On the one hand, his Theorems 6.3, 6.6, 6.7, 6.9 and 6.10 provide a succession of inequalities leading to an upper bound on the Euclidean volume of those slices, say $\mathrm{Vol}_E(\text{slice}) \leq f_1(\text{slice})$. On the other hand, his Theorems 7.7 and 7.10 give an upper bound of the form

$$f_2(\text{slice}) \cdot \int_{\Delta_0^d} \frac{dt_{\alpha_1} \cdot ... \cdot dt_{\alpha_n}}{\left(\sum_{i=1}^n t_{\alpha_i} \langle e_n, w_{\alpha_i} \rangle^2\right)^{(n+1)/2}} \tag{7}$$

for the supremum of the expression (4). Savage then shows in Theorem 7.21 that the sum over all possible slices of the values $f_1(\text{slice}) \cdot f_2(\text{slice})$ converges. Thus, the theorem reduces to showing that the integral in (7) converges, which is achieved in Theorems 7.16 and 7.18.

It is in fact equivalent to start from Equation (4) with the estimates of his Theorem 7.16, and then apply the complicated decompositions into slices. Alternatively, one can then observe that those decompositions are nothing else than an approximation by infinite Riemannian sums to an integral which we can as well compute directly. It is unfortunate that Savage did not see this simple argument. We believe it is those lengthy and hard core computations (involving the slices) which are responsible for that the mistake in Savage's proof was not discovered earlier.

Let us now come back to the estimates used by Savage in his Theorem 7.16: By the inequality between arithmetic and geometric mean, we have

$$\left(\sum_{i=1}^{n} t_{\alpha_i} \langle e_n, w_{\alpha_i}\rangle^2\right)^{n/2} \geq n \left(\prod_{i=1}^{n} t_{\alpha_i} \langle e_n, w_{\alpha_i}\rangle^2\right)^{1/2} \tag{8}$$

and by further first restricting to a subsum (recall that all summands are positive), we obtain

$$\begin{aligned}\left(\sum_{i=1}^{n} t_{\alpha_i} \langle e_n, w_{\alpha_i}\rangle^2\right)^{1/2} &\geq \left(t_{\alpha_1} \langle e_n, w_{\alpha_1}\rangle^2 + t_{\alpha_2} \langle e_n, w_{\alpha_2}\rangle^2\right)^{1/2} \\ &\geq 2\left(t_{\alpha_1} t_{\alpha_2} \langle e_n, w_{\alpha_1}\rangle^2 \langle e_n, w_{\alpha_2}\rangle^2\right)^{1/4} \\ &\geq (t_{\alpha_1} t_{\alpha_2})^{1/4}, \end{aligned} \tag{9}$$

where the last inequality comes from that

$$\langle e_n, w_{\alpha_1}\rangle^2 \langle e_n, w_{\alpha_2}\rangle^2 \geq \langle e_n, e_n\rangle^2 \langle e_n, e_{n+1}\rangle^2 = \frac{1}{2}.$$

Since all the t_i's are between 0 and 1 we clearly have that $t_i^{1/2} \geq t_i^{3/4}$. Together with Equations (8) and (9) we now have

$$\begin{aligned}\left(\sum_{i=1}^{n} t_{\alpha_i} \langle e_n, w_{\alpha_i}\rangle^2\right)^{(n+1)/2} &= \left(\sum_{i=1}^{n} t_{\alpha_i} \langle e_n, w_{\alpha_i}\rangle^2\right)^{n/2} \left(\sum_{i=1}^{n} t_{\alpha_i} \langle e_n, w_{\alpha_i}\rangle^2\right)^{1/2} \\ &\geq n \left(\prod_{j=1}^{n} \langle e_n, w_{\alpha_j}\rangle\right) \left(\prod_{i=1}^{n} t_{\alpha_i}^{3/4}\right). \end{aligned} \tag{10}$$

Plugging the two inequalities (10) and (6) in the Equation (3) of Theorem 7.4 we obtain that

$$\mathrm{Vol}(\sigma) \leq C'(n) \int_{\Delta_\tau} \frac{dt_1 \cdot \ldots \cdot dt_d}{\left(\prod_{i=1}^{n} t_{\alpha_i}^{3/4}\right)\left(\prod_{k=1}^{n-1} t_{\beta_k}^{(n+1)/2}\right)}.$$

Instead of Savage's decomposition into slices, we could conclude the proof by showing that this last integral (which is independent of our starting points) converges for our particular order on the vertices (but not for an arbitrary order).

5 Conclusion

The careful reader will have noticed that we have now gone through the whole proof of [Sa82] and nowhere did we find any explanation for the discrepancy between the stated Theorem 7.6 and its proof. We thus have to conclude that Savage's proof of the positivity of the simplicial volume of compact manifold covered by $\mathrm{SL}_n\mathbb{R}/SO(n)$ is incomplete.

We do not see how to save this proof: Theorem 7.4 is the starting point for the only volume bound given in [Sa82] and it can not be used to prove that the volume of $h \circ \sigma$ (and hence σ) is bounded, since in fact it diverges when integrated on the whole simplex Δ^d.

References

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