MILNOR–WOOD INEQUALITIES FOR MANIFOLDS WHICH ARE LOCALLY A PRODUCT OF SURFACES

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ABSTRACT. The classical Milnor–Wood inequalities for the Euler number of flat oriented vector bundles over surfaces are generalized for higher dimensional closed manifolds that admit a local structure of products of planes. These include products of surfaces as well as closed Hilbert–Blumenthal modular varieties. As a consequence, the Chern Conjecture is confirmed for such manifolds. For manifolds admitting a rigid locally symmetric structure, a complete classification is given for the flat vector bundles with nonzero Euler number. Some of the main results were announced in C.R. Acad. Sci. Paris, Ser. I 346 (2008) 661-666.

1. INTRODUCTION

Let ξ be a principal $\mathrm{GL}^+(m, \mathbb{R})$–bundle, or equivalently, an oriented vector bundle with fiber $\mathbb{R}^m$ over a closed m-dimensional oriented manifold M. The (real) *Euler class* $\varepsilon_m(\xi) \in H^m(M, \mathbb{R})$ of ξ is the image under the inclusion of coefficients $\mathbb{Z} \hookrightarrow \mathbb{R}$ of the Poincaré dual of the zero locus of a generic section in the associated vector bundle (the first obstruction to the existence of a nowhere vanishing section). The *Euler number* of ξ is the natural pairing of the Euler class with the (real) fundamental class $[M] \in H_m(M, \mathbb{R})$ of M:

$$\chi(\xi) = \langle \varepsilon_m(\xi), [M] \rangle.$$

A principal $\mathrm{GL}^+(m, \mathbb{R})$–bundle ξ over a smooth manifold M is called *flat* if it admits a flat structure, i.e. a connection on ξ with zero curvature. Equivalently, a $\mathrm{GL}^+(m, \mathbb{R})$–bundle is flat if it is induced by a representation of the fundamental group $\pi_1(M)$. The latter characterization could be used as a definition of flat bundles (and flat structures) in the general setup of topological spaces (rather than differentiable manifolds).

Lusztig and Sullivan [Su76] observed that there are only finitely many isomorphy classes of $GL^+(m, \mathbb{R})$–bundles admitting a flat structure, and hence a bound depending only on M for the possible Euler numbers of flat $GL^+(m, \mathbb{R})$–bundles over M. Indeed, two representations in the same connected component of the space of representations $\text{Rep}(\pi_1(M), GL^+(m, \mathbb{R}))$ induce isomorphic bundles, and the space of representations is a real algebraic variety, hence it has finitely many connected components.[*]

In his celebrated paper [Mi58], Milnor proved that a $GL^+(2, \mathbb{R})$–bundle ξ over a surface Σ_g of genus $g \geq 1$ admits a flat structure if and only if its Euler number $\chi(\xi)$ satisfies the inequality $|\chi(\xi)| = |\langle \varepsilon_2(\xi), [\Sigma_g] \rangle| \leq g - 1$. In particular this shows that contrary to the Chern and Pontrjagin classes, the Euler class is a nontrivial characteristic class for flat bundles. Indeed, every integer can be realized as an Euler number of some $GL^+(2, \mathbb{R})$–bundle over a surface. Milnor's inequality was later generalized to circle bundles by Wood [Wo75].

In dimension greater than 2, up to now there were few examples where explicit bounds were given for $\chi(\xi)$, or more generally for any primary characteristic number. In an unpublished work, Smillie gave some explicit bounds for the Euler number of flat bundles over hyperbolic manifolds of even dimension > 2. However, it is not known whether nontrivial flat bundles over such manifolds exist at all. In a more geometric direction, Besson, Courtois and Gallot [BCG07] proved sharp Milnor–Wood type inequalities for the pullback of the volume form under representations $\Gamma \to \text{Isom}_+(X)$ of cocompact lattices Γ in $\text{Isom}_+(X)$, where X is a product of symmetric spaces of strictly negative curvature. Note however that in dimension > 2, the volume form is in general not a primary characteristic class. One purpose of the current paper is to prove sharp generalizations of Milnor's inequality for a family of higher dimensional manifolds and use these inequalities to analyze the possible flat bundles, and in particular to prove the Chern Conjecture.

Bounded cohomology. In his groundbreaking essay [Gr82], Gromov naturally puts Milnor's inequality in the context of bounded cohomology. Canonical L^1 and L^∞ norms can be defined on the spaces of singular chains and cochains of a closed oriented n-dimensional manifold M. These in turn induce seminorms on the respective real valued homologies and cohomologies. It follows from the Hahn–Banach theorem (see [Gr82, Corollary, page 7] or [BePe92, Proposition F.2.2]) that

$$(1) \qquad |\langle \beta, [M] \rangle| = \|\beta\|_\infty \cdot \|M\|, \ \forall \beta \in H^n(M) \text{ with } \|\beta\|_\infty < \infty,$$

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[*]Note that for a given bundle there could still be uncountably many flat structures – corresponding to nonconjugate representations lying in the same component.

where $\|M\|$ denotes the L^1 seminorm of the fundamental class of M, the *simplicial volume* of M. Thus, if β is a characteristic class, a bound on the characteristic number $|\langle\beta,[M]\rangle|$ can be obtained by bounding both $\|\beta\|_\infty$ and $\|M\|$. Unfortunately, estimating each of these terms is usually very difficult.

Nonzero exact simplicial volume computations are rare. For oriented surfaces Σ_g of genus $g \geq 1$, it is not difficult to show that $\|\Sigma_g\| = 2\,|\chi(\Sigma_g)| = 4(g-1)$. In particular, if $g \geq 2$ and Σ_g is endowed with a hyperbolic structure, then $\|\Sigma_g\| = \mathrm{Vol}(\Sigma_g)/\pi$. More generally, if M is an m–dimensional closed hyperbolic manifold, then $\|M\| = \mathrm{Vol}(M)/v_m$ [Gr82, Th78], where v_m denotes the supremum of the volumes of geodesic simplices in m-dimensional hyperbolic space, and is known explicitly in low dimensions only. The only further computation of a nonzero simplicial volume is given in [Bu08] for manifolds locally isomorphic to the product of two copies of the hyperbolic plane. In this case, one has $\|M\| = 6 \cdot \chi(M) = 3/(2\pi^2) \cdot \mathrm{Vol}(M)$.

Gromov proved [Gr82] (see also [Bu07]) that characteristic classes of flat G-bundles have finite L^∞ seminorm when G is a real algebraic subgroup of $\mathrm{GL}(m,\mathbb{R})$, but actual upper bounds for their norms are only known in special cases. For the Euler class ε_m, Gromov [Gr82] obtained from Sullivan–Smillie's [Su76] corresponding simplicial results that $\|\varepsilon_m(\xi)\|_\infty \leq 1/2^m$, whenever ξ is a $\mathrm{GL}^+(m,\mathbb{R})$-bundle admitting a flat structure. Independently, Ivanov and Turaev [IvTu82] exhibited an explicit bounded cocycle representing the Euler class of flat bundles, producing the same bound. In degree 2, sharp upper bounds for the Kähler class were computed by Domic and Toledo [DoTo87] in terms of the rank of the associated symmetric space. Clerc and Ørsted [ClOr03] later generalized this to include all Hermitian symmetric spaces.

In view of the (im)possible seminorm computations, sharp generalizations of Milnor's inequality were essentially carried out in degree 2 only. Note however that in dimension 2, bounded cohomology not only naturally leads to Milnor–Wood type inequalities, but can further be used to study rigidity properties of representations of surface groups. We refer the reader to the work of Burger, Iozzi and coauthors for more details [BIW08], [BuIo07], [BILW05]. In higher dimensions, Smillie's (unpublished) inequalities for the Euler number of flat bundles over hyperbolic manifolds of even dimension $m = 2n$ are obtained by using the upper bound $\|\varepsilon_{2n}(TM)\|_\infty \leq 1/2^{2n}$ for the Euler class and the value $\|M\| = \mathrm{Vol}(M)/v_{2n} = ((2\pi)^n/(1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)v_{2n}))|\chi(M)|$ for the simplicial volume. Combined with Equation (1), this leads to

$$|\chi(\xi)| \leq \frac{\pi^n}{2^n \cdot 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)v_{2n}}|\chi(M)|.$$

Observe that the constant is strictly greater than 1 when $n > 1$, so that this inequality does not imply the nonexistence of flat or affine structures on M.

In this paper, we prove sharp upper bounds for the Euler number in degree $2n$ of every flat $\mathrm{GL}^+(2n, \mathbb{R})$-bundle ξ over a closed manifold M admitting a Riemannian structure locally isometric to a product of n hyperbolic planes. However our approach avoids estimating the simplicial volume $\|M\|$ as well as the norm of the Euler class. Instead, since by (1), we have $\|M\| = |\chi(M)| / \|\varepsilon_{2n}(TM)\|_\infty$, we prove Theorem 1.1 by showing that $\|\varepsilon_{2n}(\xi)\|_\infty \leq (1/2^n) \|\varepsilon_{2n}(TM)\|_\infty$ when ξ is flat.

Statement of the results. Denote by $\mathcal{H}$ the real hyperbolic plane. We consider closed manifolds M admitting a complete Riemannian structure locally isometric to $\mathcal{H}^n$ for some $n \geq 1$. It is well known that the universal cover of M is isometric to the symmetric space $\mathcal{H}^n$ and the fundamental group $\pi_1(M)$ acts on $\mathcal{H}^n$ by deck transformations. This produces an embedding of $\pi_1(M)$ as a torsion free uniform lattice in $\mathrm{Isom}(\mathcal{H}^n)$. We call such M an $\mathcal{H}^n$–manifold. We prove Milnor–Wood type inequalities for $\mathcal{H}^n$–manifolds:

Theorem 1.1. *Let M be a closed $\mathcal{H}^n$–manifold and ξ a $\mathrm{GL}^+(2n, \mathbb{R})$–bundle over M. If ξ admits a flat structure, then*

$$|\chi(\xi)| = |\langle \varepsilon(\xi), [M] \rangle| \leq \frac{1}{(-2)^n} \chi(M).^\dagger$$

The case $n = 1$ is Milnor's celebrated inequality. For $n = 2$, a weaker upper bound is obtained in [Bu07] by combining the explicit computation of the simplicial volume for such manifolds and Ivanov–Turaev's upper bound [IvTu82] for the Euler class of flat oriented vector bundles.

It is an old conjecture of Chern, formulated independently by Milnor [Mi58] who proved it in dimension 2, that a closed manifold of even dimension with nonzero Euler characteristic cannot admit an affine structure. As a consequence of Theorem 1.1, the conjecture is confirmed for $\mathcal{H}^n$–manifolds:

Corollary 1.2. *A closed $\mathcal{H}^n$–manifold does not admit an affine structure.*

Proof of Corollary 1.2. If M admitted an affine structure, the $\mathrm{GL}^+(2n, \mathbb{R})$-bundle associated to its tangent bundle TM would admit a flat structure. So by Theorem 1.1,

$$|\chi(M)| = |\chi(TM)| = |\langle \varepsilon(\xi), [M] \rangle| \leq \frac{1}{2^n} |\chi(M)|,$$

which is impossible since $\chi(M) \neq 0$. $\qquad\qquad\square$

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†Note that the right hand side of the inequality is always strictly positive, since the Euler characteristic $\chi(M)$ is nonzero, and its sign is $(-1)^n$. This can for example be seen from Hirzebruch's Proportionality Principle [Hi58] recalled below.

This result is new for $n \geq 3$. For $n = 1$ and $n = 2$ it follows from the corresponding inequalities of [Mi58] and [Bu07] respectively. The nonexistence of a *complete* affine structure was proved by Kostant and Sullivan [KoSu75]. However, proving the nonexistence of a *non*-complete affine structure is usually much harder. Note that even in the case of products of hyperbolic surfaces, the nonexistence of affine structures on the product cannot be directly deduced from the nonexistence of affine structure on the factors (cf. Example 3.6 below).

We really prove the stronger statement that the tangent bundle of manifolds as in Theorem 1.1 do not admit a flat structure. In general, manifolds with nonzero Euler characteristic may admit a flat structure (see Smillie [Sm77b]).

It is a famous question whether a closed manifold can admit both an affine and a hyperbolic structure. Our result shows that it can never admit simultaneously an affine and an $\mathcal{H}^n$–structure. It is natural to ask whether a closed manifold can admit simultaneously an affine and some nonpositively curved locally symmetric Riemannian structure with no local Euclidean de-Rahm factors. If one allows local Euclidean factors, simple examples can be constructed: for instance $\Sigma_g \times S^1$ admits both an affine and an $\mathcal{H} \times \mathbb{R}^1$ structures where Σ_g is a closed surface of genus $g \geq 2$. Similarly, when dropping the closed assumption one may give simple examples: the punctured torus admits both an affine structure and a complete hyperbolic structure of finite volume.

The $\mathcal{H}^n$–manifolds are of particular interest among all locally symmetric manifolds of even dimension. Indeed, while one can deduce from superrigidity theorems that some rigid locally symmetric manifolds M admit no nontrivial flat bundle of dimension $\dim(M)$, $\mathcal{H}^n$–manifolds do admit (in many cases a unique) flat bundle with nonzero Euler number (see Theorems 1.7 and 1.10 and Corollaries 1.9 and 1.12). By Corollary 1.2 these bundles cannot be isomorphic to the tangent bundle TM.

In general, a product of closed surfaces may admit an affine structure, for instance $\Sigma_g \times S^2 \times T^2$, where T^2 is the two dimensional torus, admits such a structure. On the other hand this forces the Euler characteristic to vanish. More generally, suppose M is a compact Riemannian manifold whose universal cover is a product of symmetric planes, i.e. $\tilde{M} \cong \mathcal{H}^{n_1} \times (S^2)^{n_2} \times (\mathbb{R}^2)^{n_3}$. If $n_3 \neq 0$ we have $\chi(M) = 0$ while if $n_2 \neq 0$ we have that M is an S^2 bundle over a smaller dimensional closed manifold, in which case the Euler class of any flat oriented $\mathbb{R}^{\dim(M)}$–bundle over M vanishes. Thus the general case reduces to the case $n_2 = n_3 = 0$ and we derive:

Corollary 1.3. *Let M be a closed manifold admitting a Riemannian structure with respect to which the universal cover is a direct product of symmetric planes. If $\chi(M) \neq 0$ then the tangent bundle TM admits no flat structure.*

Note that $\mathrm{Isom}(\mathcal{H}^n) \cong S_n \ltimes \mathrm{PGL}(2, \mathbb{R})^n$ where S_n denotes the full n–th symmetric group. The orientation-preserving isometries form a subgroup of index 2 denoted $\mathrm{Isom}^+(\mathcal{H}^n)$. Set $G_n = S_n \ltimes \prod_{i=1}^n \mathrm{GL}^1(2, \mathbb{R})$, where $\mathrm{GL}^1(2, \mathbb{R})$ denotes the group of 2×2 real matrices with determinant ± 1. Then G_n is an order 2^n sheeted cover of $\mathrm{Isom}(\mathcal{H}^n)$. Let G_n^+ be the preimage of $\mathrm{Isom}^+(\mathcal{H}^n)$ in G_n. Note that G_n admits a natural faithful $2n$–dimensional real representation, where the image of G_n^+ is the intersection of G_n with $\mathrm{SL}(2n, \mathbb{R})$. We denote by $s : \mathrm{Isom}^+(\mathcal{H}^n) \to S_n$ the canonical (surjective) homomorphism, and by $f : G_n^+ \to \mathrm{Isom}^+(\mathcal{H}^n)$ the covering map.

Definition 1.4. A discrete subgroup $\Gamma \leq \mathrm{Isom}^+(\mathcal{H}^n)$ will be called *cofaithful* if it admits a faithful lift to G_n^+, i.e. if there exists a subgroup $\tilde{\Gamma} \leq G_n^+$ which intersects $\ker(f)$ trivially and such that $\Gamma = f(\tilde{\Gamma})$. In that case, the isomorphism $\Gamma \to \tilde{\Gamma}$ inverse to the restriction of f to $\tilde{\Gamma}$ will be called a *cofaithful lift* (or a cofaithful map). An $\mathcal{H}^n$–manifold M will be called *cofaithful* if its fundamental group is cofaithful.

Remark 1.5. Milnor proved [Mi58] that every compact hyperbolic surface is cofaithful. Therefore a direct product of surfaces is also cofaithful. However, we do not know if for $n > 1$ every compact $\mathcal{H}^n$–manifold is cofaithful.

If $\tilde{\Gamma} \leq G_n^+$ has no elements of order two, then $\Gamma = f(\tilde{\Gamma})$ is cofaithful. Since every finitely generated subgroup of G_n^+ admits a subgroup of finite index with this property (for instance by [R72, 6.11]), it follows that any complete $\mathcal{H}^n$–manifold admits a cofaithful finite cover.

The next result gives the precise value of the Euler number of flat vector bundles induced by a cofaithful map.

Theorem 1.6. *The flat $\mathrm{GL}^+(2n, \mathbb{R})$-bundle ξ on a closed cofaithful orientable manifold $M = \Gamma \backslash \mathcal{H}^n$ induced by a cofaithful map $\Gamma \to G_n^+$ composed with the natural embedding of G_n^+ in $\mathrm{GL}_{2n}^+(\mathbb{R})$ satisfies*

$$|\chi(\xi)| = \frac{1}{2^n} |\chi(M)|.$$

Since every closed $\mathcal{H}^n$–manifold admits a cofaithful finite cover, one deduces that, up to finite cover, the inequality of Theorem 1.1 is sharp for every such manifold. However, not every integer in the interval $[\frac{-|\chi(M)|}{2^n}, \frac{|\chi(M)|}{2^n}]$ is in general an Euler number of a flat bundle, and Theorem 1.1 can be refined as in Theorem 1.7 below.

We will say that an $\mathcal{H}^n$–manifold is *rigid* if it has no finite cover which decomposes as a product manifold with a 2–dimensional factor. This terminology is motivated by the (local, Mostow and Margulis) rigidity theorems which apply for such manifolds. For instance, Hilbert–Blumental modular varieties are rigid. By a closed Hilbert–Blumental[‡] manifold we mean a manifold of the form $\Gamma\backslash\mathcal{H}^n$ where Γ is an anisotropic Hilbert–Blumental group, i.e. a subgroup of finite index of $\mathbb{G}(\mathcal{O}_{\mathbb{K}})$ where $\mathbb{K}$ is a totally real number field of degree $m \geq n > 1$, $\mathcal{O}_{\mathbb{K}}$ its ring of integers and $\mathbb{G}$ an anisotropic $\mathbb{K}$ almost simple algebraic group such that $\prod_{\gamma\in\mathrm{Gal}(\mathbb{K}/\mathbb{Q})}\mathbb{G}^\gamma(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{R})^n \times K$ for some compact group K. The induced embedding of Γ in $\mathrm{SL}_2(\mathbb{R})^n$ is discrete cocompact and irreducible. By Margulis' arithmeticity theorems, every rigid $\mathcal{H}^n$–manifold admits a finite cover which is a product of Hilbert–Blumental modular manifolds.

Recall (see [R72, Theorem 5.22]) that every closed $\mathcal{H}^n$–manifold M admits a finite cover N of the form

$$N = \Sigma_{g_1} \times \ldots \times \Sigma_{g_k} \times N',$$

where N' is rigid, $k \geq 0$, and the Σ_{g_i}'s are surfaces of genus $g_i \geq 2$. The pullback of a flat $\mathrm{GL}^+(2n,\mathbb{R})$-bundle ξ_M over M is a flat $\mathrm{GL}^+(2n,\mathbb{R})$-bundle over N.

Theorem 1.7. *Let N be as above, and let ξ_N be a flat $GL^+(2n,\mathbb{R})$-bundle over N. Then*

$$\chi(\xi_N) \in \left\{ \pm\frac{\chi(N')}{2^{\dim(N')/2}}\prod_{i=1}^{k}\ell_i : |\ell_i| \leq g_i - 1 \right\} \cup \{0\}.$$

Moreover, if N is cofaithful, all the integers are actually attained as Euler numbers of flat bundles.

Rewriting Theorem 1.7 in two special interesting cases, namely when M is a product of surfaces, or when M is rigid, we obtain:

Corollary 1.8. *Let ξ be a $GL^+(2n,\mathbb{R})$-bundle over the product of closed oriented surfaces $M_1,\ldots,M_n$ of respective genus $g_i \geq 2$, for $i = 1,\ldots,n$. If ξ is flat, then*

$$\chi(\xi) \in \left\{ \prod_{i=1}^{n}\ell_i : |\ell_i| \leq g_i - 1 \right\}.$$

[‡]Note that the varieties considered in the original work of Blumental and Hilbert were isotropic, i.e. noncompact while here the unisotropic (or closed) ones are considered. These are sometimes referred as quaternionic Shimura varieties or Picard modular manifolds.

Corollary 1.9. *Let M be a compact rigid $\mathcal{H}^n$–manifold, and let ξ be a $\mathrm{GL}^+(2n, \mathbb{R})$–bundle over M. If ξ is flat and $\chi(\xi) \neq 0$, then*

$$|\chi(\xi)| = \left| \frac{1}{2^n} \chi(M) \right|.$$

For rigid manifolds we show that the flat $\dim(M)$–dimensional vector bundles with positive Euler number are parameterized by the (finite) first cohomology group in the n–dimensional vector space over the field $\mathbb{F}_2$ of two elements. More precisely, let M be a closed oriented $\mathcal{H}^n$–manifold with fundamental group $\Gamma \leq \mathrm{Isom}(\mathcal{H}^n) \cong S_n \ltimes (\mathrm{Isom}(\mathcal{H}))^n$. We denote by $H^1_{\mathrm{Sym}}(M, \mathbb{F}_2^n) = H^1_{\mathrm{Sym}}(\Gamma, \mathbb{F}_2^n)$ the first cohomology group of Γ in $\mathbb{F}_2^n$ with respect to the action of Γ on $\mathbb{F}_2^n$ that permutes the coordinates according to the map $s : \Sigma \to S_n$.

Theorem 1.10. *Let M be a compact rigid $\mathcal{H}^n$-manifold. Then there exists a flat vector bundle over M with nonzero Euler number if and only if M is cofaithful. In that case, there is a one-to-one correspondence between the set of flat vector bundles over M with positive Euler number and the elements of the finite group $H^1_{\mathrm{Sym}}(M, \mathbb{F}_2^n)$. Moreover M admits a canonical normal finite cover N such that all these bundles induce the same vector bundle with the same flat structure over N.*

The reader is referred to the proof of 1.10 below for a description of the representations corresponding to these bundles and the correspondence with the first $\mathbb{F}_2^n$ cohomology classes.

When M is a Hilbert–Blumenthal modular manifold or a direct product of such, the fundamental group Γ is imbedded in the direct product $\prod_{i=1}^n \mathrm{PSL}(2, \mathbb{R})$ and in particular the map $s : \Gamma \to S_n$ is trivial. In that case $H^1_{\mathrm{Sym}}(M, \mathbb{F}_2^n) \cong \mathrm{Hom}(\Gamma, \mathbb{Z}/(2))^n$. Hence we have:

Corollary 1.11. *Let M be a closed cofaithful Hilbert–Blumenthal modular manifold. Then M admits exactly 2^{nk+1} flat $2n$–dimensional vector bundles with nonzero Euler number, where $k = \dim H^1(M, \mathbb{Z}/(2))$ and $n = \dim(M)/2$. The Euler number for each of these vector bundles is either $\pm \chi(M)/2^n$, where the sign is determined according to the choice of orientation.*

Note that $\dim H^1(M, \mathbb{Z}/(2))$ is always finite since $\pi_1(M)$ is finitely generated. On the other hand, there are many examples of closed Hilbert–Blumenthal modular manifolds with nontrivial first $\mathbb{Z}/(2)$–cohomology. In fact if $\Gamma \leq \prod_{i=1}^n \mathrm{SL}_2(\mathbb{R})$ is an arithmetic lattice, one can map it to $\mathrm{SL}_2(\mathbb{F}_q)$ for some finite field $\mathbb{F}_q$, and take Γ' to be the preimage of the 2–Sylow subgroup. Then $H^1(\Gamma', \mathbb{Z}/(2)) \neq \langle 0 \rangle$. More generally, Lubotzky [Lu87] showed that

every finitely generated linear group admits a finite index subgroup which is mapped onto $\mathbb{Z}/(2)$. When $k = 0$ we have:[§]

Corollary 1.12. *Let M be a compact cofaithful Hilbert–Blumenthal modular $\mathcal{H}^n$–manifold with trivial first $\mathbb{F}_2$–cohomology. Then there exists a unique flat $\mathrm{GL}^+(2n, \mathbb{R})$–bundle over M with positive Euler number.*

In contrast to Milnor's characterization of flat bundles over surfaces [Mi58] which states that a $\mathrm{GL}^+(2, \mathbb{R})$-bundle ξ over a surface of genus g is flat *if and only if* $|\chi(\xi)| \leq g - 1$, the converse of Theorems 1.1 and 1.7 do not hold in general; there exist nonflat bundles whose Euler numbers are allowed as a possible Euler numbers of flat bundles. Examples are given in Section 3.

Remark 1.13. Our results also hold for unoriented manifolds. Indeed, the Euler number of an oriented vector bundle can also be defined over an unoriented manifold M, for example as one half of the Euler number of the pullback of the bundle to an oriented double cover of M. Similarly, the simplicial volume of an unoriented manifold is simply one half of the simplicial volume of an oriented double cover. All our proofs apply with minor changes to this wider setting.

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[§]This uniqueness result was stated incorrectly in the announcement [BuGe08], where the condition $k = 0$ is missing.

2. Background

2.1. **Simplicial volume and bounded cohomology.** Let M be an n-dimensional closed oriented manifold. The L^1–norm on the space $C_*(M)$ of real-valued chains on M, associated to the canonical basis of singular simplices,

$$\left\| \sum_{i=1}^{r} a_i \sigma_i \right\|_1 = \sum_{i=1}^{r} |a_i|,$$

for $\sum_{i=1}^{r} a_i \sigma_i$ in $C_q(M)$, induces a seminorm, which we still denote by $\|-\|_1$, on the real valued homology $H_*(M)$ of M. The seminorm of a homology class is defined as the infimum of the norms of its representatives. The *simplicial volume* of M, denoted by $\|M\|$, is the seminorm of the real valued fundamental class $[M] \in H_n(M)$ of M.

The dual L^∞-norm (or *Gromov norm*) on the space $C^*(M)$ of real valued cochains on M is given, for every cochain c in $C^q(M)$, by

$$\begin{aligned} \|c\|_\infty &= \sup\{|c(z)| : z \in C_q(M) \text{ with } \|z\|_1 = 1\} \\ &= \sup\{|c(\sigma)| : \sigma : \Delta^q \to M \text{ continuous}\}. \end{aligned}$$

The subspace of bounded cochains $C_b^*(M) \subset C^*(M)$ consists of those cochains for which the Gromov norm is finite. Since the boundary operators on $C_*(M)$ are bounded with respect to the L_1–norm, the coboundary operators of $C^*(M)$ restrict to $C_b^*(M)$. The *bounded cohomology* $H_b^*(M)$ of M is, by definition, the cohomology of the cocomplex $C_b^*(M)$. The inclusion of cocomplexes $C_b^*(M) \subset C^*(M)$ induces a *comparison map* $c : H_b^*(M) \to H^*(M)$. The Gromov norm on the space of (bounded) cochains induces a seminorm on $H_b^*(M)$ and on $H^*(M)$, which we still denote by $\|-\|_\infty$ (allowing the value $+\infty$ on $H^*(M)$). For α in $H^q(M)$, we have

$$\|\alpha\|_\infty = \inf\{\|\alpha_b\|_\infty : \ \alpha_b \in H_b^q(M), c(\alpha_b) = \alpha\},$$

where, over the empty set, the infimum is considered to be infinity.

For $\beta \neq 0 \in H^n(M)$, it follows from the Hahn-Banach theorem that

$$\frac{|\langle \beta, [M] \rangle|}{\|\beta\|_\infty} = \|M\|.$$

The continuous bounded cohomology of a topological group G is defined similarly. Recall first that the continuous cohomology $H_c^*(G)$ of G is the cohomology of the cocomplex $C_c^*(G)^G$ endowed with its natural homogeneous coboundary operator δ: The space of continuous cochains is given as

$$C_c^q(G) = \{c : G^{q+1} \longrightarrow \mathbb{R} : c \text{ is continuous}\},$$

and $C_c^q(G)^G$ is the subspace of left G-invariant cochains, where G acts diagonally on G^{q+1}. The coboundary operator $\delta : C_c^q(G)^G \to C_c^{q+1}(G)^G$ is defined

as

$$\delta f(g_0, \ldots, g_q) = \sum_{i=0}^{q} (-1)^q f(g_0, \ldots, \hat{g}_i, \ldots, g_q),$$

for $f \in C_c^q(G)^G$ and $(g_0, \ldots, g_q) \in G^{q+1}$. For c in $C_c^q(G)$, let

$$\|c\|_\infty = \sup\{|c(g_0, \ldots, g_q)| : (g_0, \ldots, g_q) \in G^{q+1}\}.$$

Set

$$C_{c,b}^q(G) = \{c \in C_c^q(G) : \|c\|_\infty < +\infty\},$$

and let $C_{c,b}^*(G)^G$ be the cocomplex of continuous bounded G–invariant cochains. Clearly, the coboundary operator restricts to $C_{c,b}^*(G)^G$, and the *continuous bounded cohomology* $H_{c,b}^*(G)$ of G is defined as the cohomology of this co-complex.

The inclusion of cocomplexes $C_{c,b}^*(G)^G \subset C_c^*(G)^G$ induces a *comparison map* $c : H_{c,b}^*(G) \to H_c^*(G)$. As in the singular case, the sup norm induces seminorms both on $H_{c,b}^*(G)$ and on $H_c^*(G)$ (again allowing the value $+\infty$ in the latter case) and we have, for any α in $H_c^q(G)$, that

$$\|\alpha\|_\infty = \inf\{\|\alpha_b\|_\infty : \alpha_b \in H_b^q(M), c(\alpha_b) = \alpha\}.$$

If Γ is a discrete group, then the continuity condition is void and we omit the term "continuous" and the subscript "c" in the corresponding terminology and notations. Note that the group cohomology $H^*(\Gamma)$ is then nothing but the Eilenberg-MacLane cohomology of Γ [McL63, Chapter 4, paragraph 5].

Let $f : H \to G$ be a continuous homomorphism between topological groups. The induced maps $f^* : H_c^*(G) \to H_c^*(H)$ and $f^* : H_{c,b}^*(G) \to H_{c,b}^*(H)$ on cohomology are defined at the cochain level by precomposing any cochain $c : G^{q+1} \to \mathbb{R}$ with $(f, \ldots, f) : H^{q+1} \to G^{q+1}$. The sup norm is not increased at the cochain level, hence for any α in $H_c^q(G)$ or in $H_{c,b}^q(G)$, we have $\|f^*(\alpha)\|_\infty \leq \|\alpha\|_\infty$.

Let Γ now be the fundamental group of a closed manifold M. As for standard singular cohomology, the natural map $M \to B\Gamma$, where $B\Gamma$ denotes the classifying space of Γ-bundles, induces a natural map in bounded cohomology

$$H_b^*(\Gamma) \cong H_b^*(B\Gamma) \to H_b^*(M).$$

In contrast to the standard case, Gromov [Gr82, Section 3.1] (see also [Iv85]) proved the remarkable theorem that this map is an isometric isomorphism. For aspherical manifolds (which we will exclusively be dealing with) this theorem is easy to prove. Indeed, in that case it is classical that there is an isomorphism $H^*(\Gamma) \cong H^*(M)$ between the standard cohomology groups. Furthermore, it is easy to exhibit explicit cochain maps $C^*(\Gamma)^\Gamma \to C^*(M)$

and $C^*(M) \to C^*(\Gamma)^\Gamma$ realising the isometric isomorphisms $H^*(\Gamma) \cong H^*(M)$ and $H_b^*(\Gamma) \cong H_b^*(M)$.

2.2. **The Euler class.** The classical theorem of Hopf stating that the index of a nondegenerate vector field over a manifold M is equal to the Euler characteristic $\chi(M)$ can be reformulated by saying that the Euler number of the tangent bundle is equal to the Euler characteristic of M [MiSt79, Corollary 11.12]:

$$\chi(TM) = \chi(M).$$

(In fact, this is the origin of the name "Euler class".) By Poincaré duality, $\chi(M) = 0$ when the dimension of M is odd. More generally, the real Euler class of an odd dimensional vector bundle always vanishes [MiSt79, Property 9.4].

The universal Euler class ε_m lives in the m-th cohomology group of the classifying space $B\mathrm{GL}^+(m, \mathbb{R})$. Denoting by $\mathrm{GL}^+(m, \mathbb{R})^\delta$ the group $\mathrm{GL}^+(m, \mathbb{R})$ endowed with the discrete topology, recall that the classifying space $B\mathrm{GL}^+(m, \mathbb{R})^\delta$ classifies flat $\mathrm{GL}^+(m, \mathbb{R})$–bundles and that its cohomology is by definition isomorphic to the Eilenberg–Maclane cohomology of its fundamental group $\mathrm{GL}^+(m, \mathbb{R})$ [McL63, Chapter 4, paragraph 5].

We shall further see that it is natural to consider the Euler class as a continuous cohomology class in $H_c^m(\mathrm{GL}^+(m, \mathbb{R}))$. More generally, let G be a reductive Lie group and K a maximal compact subgroup in G. We have a commutative diagram

$$
\begin{array}{ccc}
H^*(BG) & \longrightarrow & H^*(BG^\delta) \\
\downarrow & & \downarrow{\scriptstyle \cong} \\
H_c^*(G) & \longhookrightarrow & H^*(G).
\end{array}
$$

The left vertical arrow can be described as follows: On the one hand, the embedding $K \hookrightarrow G$ is a homotopy equivalence and hence $H^*(BG) \cong H^*(BK)$. Since the Chern–Weil homomorphism is an isomorphism for compact groups [Bott73], $H^*(BK)$ is isomorphic to the ring of $\mathrm{Ad}(K)$–invariant polynomials on the Lie algebra of K. On the other hand, by the Van Est isomorphism [Ve55] (see also [Gui80, Corollary 7.2]), the continuous cohomology $H_c^*(G)$ is isomorphic to the G–invariant differential forms on G/K. The quotient map $G \to G/K$ is a K–bundle. This bundle is endowed with the Maurer–Cartan connection. Chern–Weil theory now naturally assigns to any $\mathrm{Ad}(K)$–invariant polynomial a differential form on G/K, which is G–invariant by the G–equivariance of the Maurer–Cartan connection. For more details and explicit formulas for primary characteristic classes of flat bundles viewed as continuous cohomology classes, see [Du76].

To see that the lower horizontal arrow is injective one can argue as follows: Since G is reductive, by Borel's Theorem [Bor63] it admits a cocompact lattice Γ, and the map $H_c^*(G) \to H^*(G)$ is factorized by the restriction $H_c^*(G) \hookrightarrow H^*(\Gamma)$, which is injective, having as left inverse the transfer map (given by integration over a fundamental domain for $\Gamma \setminus G$).

For $G = \mathrm{GL}^+(m, \mathbb{R})$ and m even, the image of the Euler class in any of the above cohomology groups, which we still denote by ε_m, is nontrivial. Since a $\mathrm{GL}^+(m, \mathbb{R})$-bundle ξ over M admitting a flat structure is induced from a representation $\rho : \pi_1(M) \to \mathrm{GL}^+(m, \mathbb{R})$ of the fundamental group of M, the Euler class $\varepsilon_m(\xi)$ is just the image under the natural homomorphism $H^m(\pi_1(M)) \to H^m(M)$ of $\rho^*(\varepsilon_m)$, where $\rho^* : H_c^m(G) \to H^m(\pi_1(M))$ is the homomorphism induced by ρ. The fact that the Euler class $\varepsilon_m(\xi)$ is the pullback of the continuous (bounded) cohomology class $\varepsilon_m \in H_c^m(\mathrm{GL}^+(m, \mathbb{R}))$ is fundamental in our approach.

Remark. The Euler class is the only characteristic class (for $\mathrm{GL}^+(m, \mathbb{R})$) surviving the passage to the classifying space of flat bundles. All other generators of $H^*(B\mathrm{GL}^+(m, \mathbb{R}))$ are Pontrjagin classes and vanish on flat bundles since they are in the image of the Chern-Weil homomorphism [KoNo69, Chapter XII]. For a topological proof of this classical consequence of Chern-Weil theory in the differential context see [KaTo68, Section 6].

Recall [IvTu82] that the Euler class can be represented by a bounded cocycle, i.e. it lies in the image of the comparison map $H_{c,b}^m(\mathrm{GL}^+(m, \mathbb{R})) \to H_c^m(\mathrm{GL}^+(m, \mathbb{R}))$.

Define $\mathrm{Bl}_{2,n}^+$ to be the subgroup of $\mathrm{GL}^+(2n, \mathbb{R})$ consisting of the diagonal embedding of the product of n copies of $\mathrm{GL}^+(2, \mathbb{R})$, i.e. the image of the injective homomorphism

$$\rho_\Delta : \prod_{i=1}^n \mathrm{GL}^+(2, \mathbb{R}) \longrightarrow \mathrm{Bl}_{2,n}^+ < \mathrm{GL}^+(2n, \mathbb{R})$$
$$(A_1, \ldots, A_n) \longmapsto \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{pmatrix}.$$

It follows from the Whitney product formula for the Euler class that the induced cohomology map

$$\rho_\Delta^* : H_c^{2n}(\mathrm{GL}^+(2n, \mathbb{R})) \to H_c^{2n}(\mathrm{Bl}_{2,n}^+) \cong H_c^{2n}(\prod_{i=1}^n \mathrm{GL}^+(2, \mathbb{R}))$$

sends ε_{2n} to the cup product $\varepsilon_2 \cup \ldots \cup \varepsilon_2$ of n copies of ε_2.

Let us end this preliminary section with two basic facts regarding ε_m:

Conjugation by an element $g \in \mathrm{GL}(m, \mathbb{R})$ induces an isomorphism on the underlying (universal) unoriented vector bundles, which preserves orientation if $\det(g) > 0$ and reverses orientation if $\det(g) < 0$. This yields:

Lemma 2.1. *For $g \in GL(m, \mathbb{R})$ let $\rho_g : GL^+(m, \mathbb{R}) \to GL^+(m, \mathbb{R})$ be the corresponding conjugation $\rho_g(A) = gAg^{-1}$, $A \in GL^+(m, \mathbb{R})$. Then*

$$\rho_g^*(\varepsilon_m) = \left\{ \begin{array}{ll} \varepsilon_m & \text{if } \det(g) > 0, \\ -\varepsilon_m & \text{if } \det(g) < 0. \end{array} \right.$$

Since the Euler class $\varepsilon_m \in H_c^m(\mathrm{GL}^+(m, \mathbb{R}))$ lies in the image of the composition

$$H_c^m(\mathrm{PSL}(m, \mathbb{R})) \longrightarrow H_c^m(\mathrm{SL}(m, \mathbb{R})) \longrightarrow H_c^m(\mathrm{GL}^+(m, \mathbb{R}))$$

induced by the natural projections, for $\rho : \Gamma \to \mathrm{GL}^+(m, \mathbb{R})$, the Euler class $\rho^*(\varepsilon_m)$ is determined by the projection of ρ to $\mathrm{PSL}(m, \mathbb{R})$. Similarly, we have:

Lemma 2.2. *Let $\rho_1, \rho_2 : \Gamma \to \prod_{i=1}^n GL^+(2, \mathbb{R})$ be two representations whose projections to $\prod_{i=1}^n PSL(2, \mathbb{R})$ coincide. Then*

$$\rho_1^*(\varepsilon_2 \cup \ldots \cup \varepsilon_2) = \rho_2^*(\varepsilon_2 \cup \ldots \cup \varepsilon_2).$$

3. Some simple examples

The purpose of this section is to answer several questions which naturally arise and to give examples illustrating some issues regarding the main results.

3.1. **Nonflat bundles with even Euler number.**

Claim 3.1. *Let M be a closed oriented manifold of even dimension $2n$, and let $k \in 2\mathbb{Z}$ be an arbitrary even integer. Then there exists an oriented $\mathbb{R}^{2n}$–vector bundle ξ over M with $\chi(\xi) = k$.*

Since any manifold admits a degree 1 mapping onto the sphere of the same dimension, obtained by sending an open disk U in M diffeomorphically onto $S^{2n} \setminus \{x_0\}$ and its complement $M \setminus U$ to some fixed point $x_0 \in S^{2n}$, Claim 3.1 follows from its validity for S^{2n}. The tangent bundle TS^{2n} over S^{2n} has Euler number $\chi(TS^{2n}) = 2$, and an oriented vector bundle over S^{2n} with Euler number equal to $2d$ is obtained by taking the pullback of TS^{2n} by a self-map of degree d.

Moreover for $n = 1, 2, 4$ the assertion holds for odd $k \in \mathbb{Z}$ as well: Letting F be the complex numbers, the quaternions or the octonions, the projective space $\mathbb{P}(F^2)$ (which is a sphere of dimension 2, 4 or 8) carries a canonical F-bundle (the total space consists of pairs $([V], v)$ where $[V] \in \mathbb{P}(F^2)$ and $v \in$

V) with Euler number equal to 1. For more details about the 4–dimensional case, see [MiSt79, Lemma 20.9].

Since only finitely many isomorphy classes of bundles of a fixed dimension over a given manifold M can admit a flat structure, most of the bundles constructed above will not be flat. We will now show that in certain cases, and in particular when the base is an $\mathcal{H}^n$–manifold, a nonflat vector bundle can attain every (or every even) integer as an Euler number.

Proposition 3.2. *Let M be a closed oriented 4–dimensional manifold, and let $k \in \mathbb{Z}$ be an arbitrary integer. Then there exists an oriented $\mathbb{R}^4$–vector bundle ξ over M not admitting a flat structure and such that $\chi(\xi) = k$.*

Proof. Suppose first that $k \neq 0$. Let ξ_1 be the oriented $\mathbb{R}^4$–vector bundle over S^4 underlying the canonical F–bundle over the projective space $\mathbb{P}(F^2) \simeq S^4$ considered earlier, for F the quaternions. It is proven in [MiSt79, Lemma 20.9] that $\chi(\xi_1) = 1$ and the first real Pontrjagin class of ξ_1 is nonzero. As above, one constructs the bundle ξ over M by taking the pullback of ξ_1 by a map $f : M \to S^4$ of degree k. It follows that $\chi(\xi) = k$ and the first real Pontrjagin class of ξ is nonzero, which implies that the bundle cannot admit a flat structure. The case $k = 0$ is treated similarly based on the existence of bundles over S^4 with vanishing Euler number and nonvanishing first real Pontrjagin class shown in [MiSt79, Lemma 20.10]. $\qquad\square$

Lemma 3.3. *Let $k, n \in \mathbb{Z}$ be integers and suppose that $n \geq 2$. Then there exists an oriented $\mathbb{R}^{2n}$–vector bundle ξ over the product $\prod_{i=1}^{n} S^2$ of n copies of the sphere S^2 such that ξ does not admit a flat structure and $\chi(\xi) = k$.*

Proof. Let η_1 be the oriented $\mathbb{R}^2$–vector bundle over S^2 underlying the canonical $\mathbb{C}$–bundle over the projective space $\mathbb{P}(\mathbb{C}^2) \simeq S^2$. It is easy to check that $\chi(\eta_1) = 1$. As explained in the proof of Proposition 3.2, there exists an oriented $\mathbb{R}^4$–vector bundle ξ_k over $S^2 \times S^2$ such that $\chi(\xi_k) = k$ and the first real Pontrjagin class of ξ_k does not vanish: $p_1(\xi_k) \neq 0 \in H^4(S^2 \times S^2, \mathbb{R})$. For $\xi = \xi_k \times \Pi_{i=3}^n \eta_1$, we have

$$\chi(\xi) = \chi(\xi_k) = k.$$

Furthermore, since, for dimension reasons, all the Pontrjagin classes of $\Pi_{i=3}^n \eta_1$ vanish,

$$p_1(\xi) = p_1(\xi_k \times \Pi_{i=3}^n \eta_1) = p_1(\xi_k) \neq 0 \in H^4(\Pi_{i=1}^n S^2, \mathbb{R}),$$

so that ξ does not admit a flat structure. $\qquad\square$

Proposition 3.4. *Let M be a closed oriented $\mathcal{H}^n$–manifold with $n \geq 2$. There exists a finite cover N of M such that for every even integer $k \in 2\mathbb{Z}$ there exists an oriented $\mathbb{R}^{2n}$–vector bundle ξ over N such that ξ does not admit a flat structure and $\chi(\xi) = k$.*

This implies in particular that upon passing to a finite cover, the converses of Theorems 1.1 and 1.7 are wrong in dimension greater or equal to 4. That is, in contrast to the case $n = 1$, the Euler number not only does not determine the isomorphy class of an oriented $\mathbb{R}^{2n}$–vector bundle, it does not even determine whether a bundle admits a flat structure or not.

Proof. For $n = 2$, this is a consequence of Proposition 3.2 even without assuming that k is even and without passing to a finite cover. Let us thus suppose that $n \geq 3$.

It is shown in [Ok01] that there exists a finite cover N of any closed oriented $\mathcal{H}^n$–manifold which admits a tangential map of nonzero degree to the dual compact symmetric space. That is, there exists a map $f : N \to \prod_{i=1}^{n} S^2$ which furthermore induces Matsushima's map on the corresponding singular real cohomology groups, and consequently has degree $\chi(N)/2^n$. Thus for $k \in (\chi(N)/2^n)\mathbb{Z}$ the result follows immediately from Lemma 3.3.

In order to treat general $k \in 2\mathbb{Z}$, start with ξ_0, the pullback by $f : N \to \prod_{i=1}^{n} S^2$ of an oriented $\mathbb{R}^{2n}$–vector bundle over $\prod_{i=1}^{n} S^2$ with $\chi(\xi_0) = 0$ and nonvanishing first Pontrjagin class, established in (the proof of) Lemma 3.3. Then modify ξ_0 inside a closed disk U of N as follows, in order to obtain a bundle with any given even Euler number: Let $V \subset U$ be a smaller disk contained in the interior of U. Let x_0 be an interior point in V and y_0 be a point in S^{2n}, and consider the wedge $N \vee S^{2n}$ obtained by identifying x_0 and y_0. Let $\phi : N \to N \vee S^{2n}$ be defined as the identity on $N \setminus U$, a diffeomorphism $U \setminus V \to U \setminus \{x_0\}$ restricting to the identity on ∂U and extending to the constant map $\partial V \to y_0$, and a map $V \to S^{2n}$ wrapping V around the sphere and sending ∂V to y_0. Let ξ_k be the pullback through ϕ of the bundle over $N \vee S^{2n}$ obtained by gluing the bundle ξ_0 to some bundle with Euler number k over the sphere. Clearly, $\chi(\xi_k) = k$.

To see that the bundle ξ_k cannot admit a flat structure, note that there exists a 4–dimensional CW–complex X in N which has nonzero first Pontrjagin class ξ_0. Since X has codimension at least 2 in N, the closed disk U can be chosen disjoint from X, so that the restrictions of ξ_0 and ξ_k to X will agree, and the first Pontrjagin class of ξ_k being nontrivial on X will be nonzero on N (and in fact equal to $p_1(\xi_0)$ as a cohomology class on N, not only on X). In particular, $p_1(\xi_k) \neq 0 \in H^4(N, \mathbb{R})$, and the bundle ξ_k cannot admit a flat structure. $\square$

Remark 3.5. In the case of a product of surfaces, examples of nonflat bundles with any given Euler number are easily constructed using the fact that there exists a degree 1 map from any surface onto S^2.

3.2. **Flat bundles with zero Euler number.** Let M be an $\mathcal{H}^n$–manifold. For simplicity we will assume that its fundamental group is contained in the

identity connected component $\mathrm{Isom}(\mathcal{H}^n)^\circ \cong \prod_{i=1}^n \mathrm{PSL}(2,\mathbb{R})$ and admits a cofaithful representation $\rho : \pi_1(M) \to \prod_{i=1}^n \mathrm{SL}(2,\mathbb{R})$. Let I be any subset of $\{1,2,\ldots,n\}$ and denote by $\mathrm{proj}_I : \prod_{i=1}^n \mathrm{SL}(2,\mathbb{R}) \to \Pi_{i\in I}\mathrm{SL}(2,\mathbb{R})$ the canonical projection. Denote by $\rho_I : \pi_1(M) \to \mathrm{SL}(2n,\mathbb{R})$ the composition of ρ with proj_I followed by the natural inclusion $\prod_{i\in I}\mathrm{SL}(2,\mathbb{R}) \hookrightarrow \prod_{i=1}^n \mathrm{SL}(2,\mathbb{R})$ and the diagonal embedding $\prod_{i=1}^n \mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{SL}(2n,\mathbb{R})$. Let ξ_I be the corresponding flat bundle over M. For $I = \{1,2,\ldots,n\}$, we have $\chi(\xi_I) = 2^{-n}|\chi(M)|$, while for $I \neq \{1,2,\ldots,n\}$, we have $\chi(\xi_I) = 0$ since the representation ρ_I commutes through a Lie group whose associated symmetric space has dimension strictly smaller than $2n$.

For $I \neq J$ the representations ρ_I and ρ_J cannot be conjugated since we can chose a sequence $\gamma_m \in \pi_1(M)$ such that $\rho_I(\gamma_m) \to 1$ while $\rho_J(\gamma_m) \to \infty$.

For noncofaithful manifolds, examples can be constructed by taking linear representations of some finite quotient or, if $\pi_1(M)$ lies in $\mathrm{Isom}(\mathcal{H}^n)^\circ$, considering its embedding in $\prod_{i=1}^n \mathrm{PSL}(2,\mathbb{R})$ as a cocompact lattice composed with the projections on factors followed with some linear (oriented) representation of $\mathrm{PSL}(2,\mathbb{R})$.

3.3. The variety of flat structures.

For a given oriented closed manifold M, the space of flat m $(= \dim M)$ oriented vector bundles over M coincides with the character variety $\mathrm{Hom}(\pi_1(M), \mathrm{GL}^+(m,\mathbb{R}))/G$ since two flat bundles are isomorphic as flat bundles if and only if the corresponding representations of $\pi_1(M)$ are conjugated. On the other hand two representations $\rho_1, \rho_2 : \pi_1(M) \to \mathrm{GL}^+(m,\mathbb{R})$ that lie in the same connected component of $\mathrm{Hom}(\pi_1(M), \mathrm{GL}^+(m,\mathbb{R}))$ induce abstractly isomorphic bundles.

If M is a rigid $\mathcal{H}^n$–manifold, it can be deduced from Margulis' superrigidity theorem that two representations that lie in the same connected component are conjugate, and moreover that the character variety is finite. However, if M is a nonrigid closed $\mathcal{H}^n$–manifold, connected components of $\mathrm{Hom}(\pi_1(M), \mathrm{GL}^+(2n,\mathbb{R}))$ in general have dimension larger than the corresponding conjugacy classes, and one deduces that there are continuously many nonequivalent flat structures over some fixed given vector bundle.

3.4. Affine product of nonaffine manifolds.

Example 3.6 (E. Ghys). There exist two closed manifolds M_1, M_2 whose product $M_1 \times M_2$ admits an affine structure while neither M_1 nor M_2 do.

Let M be a compact quaternionic hyperbolic surface, i.e. an 8–dimensional manifold of the form $\Gamma\backslash X$ where Γ is a torsion free cocompact lattice of $\mathrm{Sp}(2,1)$ and X is the associated symmetric space. Take

$$M_1 = (M \times S^1) \qquad \text{and} \qquad M_2 = S^3.$$

To give an affine structure on $(M \times S^1) \times S^3$ one can argue as follows: Let Q denote the quaternions. Let V be the open "light cone" in $Q^{2,1}$ (i.e. Q^3 equipped with the standard $(2,1)$ form). $\mathrm{Sp}(n,1)$ acts on Q^3 preserving this form and $\Gamma \backslash V$ is homeomorphic to $M \times Q^*$, where Q^* denotes the space of nonzero quaternions and is homeomorphic to $\mathbb{R}^+ \times S^3$. Denote by t the affine homothety of multiplying by 2. Then $(\Gamma \times \langle t \rangle) \backslash V$ is homeomorphic to $M \times S^1 \times S^3$ and it inherits the affine structure from V, as the $\Gamma \times \langle t \rangle$ action is affine.

Now S^3 is compact simply connected and hence cannot admit an affine structure. Consider $M \times S^1$. Suppose it admits an affine structure, and let $\rho : \Gamma \times \mathbb{Z} \to \mathrm{GL}_9^+(\mathbb{R})$ be the associated representation. Since $\mathrm{Sp}(2,1)$ has no nontrivial 9-dimensional representations, it follows from the Corlette and Gromov–Schoen superrigidity theorems that $\rho(\Gamma)$ is finite. Hence up to replacing M by a finite cover, we may assume that $\rho(\Gamma)$ is trivial, and so up to replacing S^1 by a finite cover, we have that either ρ has trivial image, which is impossible since $M \times S^1$ has no flat Riemannian structure (for instance by Bieberbach's theorem), or that the image of ρ is infinite cyclic. The last possibility cannot hold as well; in fact the closed affine manifolds with cyclic holonomy were classified by Smillie (in his unpublished thesis [Sm77a]) who showed that they are all Hopf manifolds.

4. VANISHING RESULTS FOR THE EULER CLASS

In this section we assemble several statements indicating the vanishing of the Euler class for certain representations to be considered later.

4.1. Vanishing of the Euler class for tensor representations. Identifying $\mathbb{R}^2 \otimes \mathbb{R}^2$ with $\mathbb{R}^4$ we obtain a representation of $\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R})$ which we call the tensor representation and denote by $\rho_\otimes$ (we will also use this symbol to denote the restriction of this representation to subgroups of $\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R})$). Note that with respect to the basis $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$ of $\mathbb{R}^2 \otimes \mathbb{R}^2$ induced by the tensor of standard basis vectors $\{e_1, e_2\}$ of $\mathbb{R}^2$, $\rho_\otimes$ takes the form

$$\rho_\otimes(A,B) = A \otimes B = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Lemma 4.1. *The pullback of the Euler class vanishes under the tensor representation $\rho_\otimes : SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \to SL(4, \mathbb{R})$:*

$$\rho_\otimes^*(\varepsilon_4) = 0.$$

Proof. The proof relies on the simple fact that switching the two factors changes the sign of the orientation of the tensor product, and hence of the Euler class, while it does not change the sign of the generator of $H_c^4(\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})) \cong \mathbb{R}$. We give the detailed proof for the convenience of the reader.

Let $\tau : \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ denote the homomorphism permuting the two factors: $\tau(A, B) = (B, A)$, for $(A, B) \in \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$. Let E_{23} denote the odd permutation

$$E_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and let $\rho_{E_{23}} : \mathrm{SL}(4, \mathbb{R}) \to \mathrm{SL}(4, \mathbb{R})$ denote the corresponding conjugation by E_{23},

$$\rho_{E_{23}}(A) = E_{23} A E_{23}^{-1},$$

for $A \in \mathrm{SL}(4, \mathbb{R})$. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) & \xrightarrow{\ \tau\ } & \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \\
\Big\downarrow{\scriptstyle \rho_\otimes} & & \Big\downarrow{\scriptstyle \rho_\otimes} \\
\mathrm{SL}(4, \mathbb{R}) & \xrightarrow{\ \ \rho_{E_{23}}\ \ } & \mathrm{SL}(4, \mathbb{R}),
\end{array}
$$

so that in particular

$$\tau^*(\rho_\otimes^*(\varepsilon_4)) = \rho_\otimes^*(\rho_{E_{23}}^*(\varepsilon_4)).$$

But since E_{23} has negative determinant, by Lemma 2.1, $\rho_{E_{23}}^*(\varepsilon_4) = -\varepsilon_4$. Thus we get, on the one hand, that

(2) $$\tau^*(\rho_\otimes^*(\varepsilon_4)) = -\rho_\otimes^*(\varepsilon_4).$$

On the other hand, letting $E_{(13)(24)}$ denote the even permutation

$$E_{(13)(24)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and $\rho_{E_{(13)(24)}}$ the corresponding conjugation by $E_{(13)(24)}$, we also have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) & \xrightarrow{\ \tau\ } & \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \\
\downarrow{\scriptstyle \rho_\Delta} & & \downarrow{\scriptstyle \rho_\Delta} \\
\mathrm{SL}(4,\mathbb{R}) & \xrightarrow{\ \rho_{E_{(13)(24)}}\ } & \mathrm{SL}(4,\mathbb{R}),
\end{array}
$$

where $\rho_\Delta : \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(4,\mathbb{R})$ is the diagonal embedding. By Lemma 2.1, we have

$$(3) \qquad \tau^*(\rho_\Delta^*(\varepsilon_4)) = \rho_\Delta^*(\rho_{E_{(13)(24)}}^*(\varepsilon_4)) = \rho_\Delta^*(\varepsilon_4),$$

since $E_{(13)(24)}$ has positive determinant. The Whitney product formula for the Euler class implies $\rho_\Delta^*(\varepsilon_4) = \varepsilon_2 \cup \varepsilon_2$, so we can rewrite Equation (3) as

$$(4) \qquad \tau^*(\varepsilon_2 \cup \varepsilon_2) = \varepsilon_2 \cup \varepsilon_2.$$

Finally, since $H^4(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))$ is one dimensional, generated by the cup product of Euler classes $\varepsilon_2 \cup \varepsilon_2$, there must exist $\lambda \in \mathbb{R}$ with $\rho_\otimes^*(\varepsilon_4) = \lambda \cdot \varepsilon_2 \cup \varepsilon_2$. It then follows from Equations (2) and (4) that

$$\lambda \cdot \varepsilon_2 \cup \varepsilon_2 = \rho_\otimes^*(\varepsilon_4) = -\tau^*(\rho_\otimes^*(\varepsilon_4)) = -\tau^*(\lambda \cdot \varepsilon_2 \cup \varepsilon_2) = -\lambda \cdot \varepsilon_2 \cup \varepsilon_2.$$

Hence $\lambda = 0$, and consequently $\rho_\otimes^*(\varepsilon_4) = 0$. $\qquad\square$

4.2. Triviality of top cohomology of one factor, implies vanishing of the Euler class.

Lemma 4.2. *Let M_1, M_2 be closed aspherical oriented manifolds of respective dimensions m_1, m_2 and fundamental groups Γ_1, Γ_2. Set $m = m_1 + m_2$ and let*

$$\rho : \Gamma_1 \times \Gamma_2 \longrightarrow GL^+(m,\mathbb{R})$$

be a representation. Let G_1 be the Zariski closure of $\rho(\Gamma_1)$. If $H_c^{m_1}(G_1) = 0$ then $\rho^*(\varepsilon_m) = 0 \in H^m(\Gamma_1 \times \Gamma_2)$.

Proof. Let G_2 be the Zariski closure of $\rho(\Gamma_2)$. Denote by $\varphi : G_1 \times G_2 \to GL^+(m,\mathbb{R})$ the homomorphism induced by the inclusions $G_i < GL^+(m,\mathbb{R})$, for $i = 1, 2$. Note that φ is well defined since G_1 and G_2 are by construction commuting subgroups of $GL^+(m,\mathbb{R})$, but it is not necessarily injective since G_1 and G_2 may have a nontrivial central intersection. The representation ρ naturally induces a homomorphism

$$\overline{\rho} : \Gamma_1 \times \Gamma_2 \longrightarrow G_1 \times G_2,$$

defined on the factors Γ_i, for $i = 1, 2$, as the restriction of ρ to Γ_i. The original representation ρ is now nothing else but the composition

$$\Gamma_1 \times \Gamma_2 \xrightarrow{\ \overline{\rho}\ } G_1 \times G_2 \xrightarrow{\ \varphi\ } \mathrm{GL}^+(m, \mathbb{R}).$$

The induced map $\rho^* : H_c^m(\mathrm{GL}^+(m, \mathbb{R})) \to H^m(\Gamma_1 \times \Gamma_2)$ thus factors through $H_c^m(G_1 \times G_2)$, and the lemma will follow from:

Claim. The induced map $\overline{\rho}^* : H_c^m(G_1 \times G_2) \to H^m(\Gamma_1 \times \Gamma_2)$ is zero (in degree m).

To prove the claim, note that the two latter cohomology groups satisfy a Künneth product formula: For $H_c^m(G_1 \times G_2)$, it follows from the Van Est isomorphism [Ve55] and the validity of the Künneth formula for relative Lie algebra cohomology [BoWa00, 1.3]. For $H^m(\Gamma_1 \times \Gamma_2)$, it holds since by asphericality, the cohomology groups of $\Gamma_1 \times \Gamma_2$, Γ_1 and Γ_2 are isomorphic to the cohomology groups of $M_1 \times M_2$, M_1 and M_2 respectively, and the latter cohomology groups satisfy the Künneth formula (see for example [MiSt79, Theorem A.6]). Furthermore, since all isomorphisms are natural, the map ρ^* now becomes

$$\overline{\rho}^* = \rho_1^* \otimes \rho_2^* : \bigoplus_{p+q=m} H_c^p(G_1) \otimes H_c^q(G_2) \longrightarrow \bigoplus_{p+q=m} H^p(\Gamma_1) \otimes H^q(\Gamma_2),$$

where $\rho_i : \Gamma_i \to G_i$, for $i = 1, 2$, denotes the restriction of ρ to Γ_i. Since $m = m_1 + m_2$, the only nonzero summand in the latter direct sum is the one corresponding to $p = m_1, q = m_2$. Thus, the restriction of $\overline{\rho}^*$ to all the summands other then $H_c^{m_1}(G_1) \otimes H_c^{m_2}(G_2)$ is trivial. Hence, the assumption $H_c^{m_1}(G_1) = 0$ gives $\overline{\rho}^* = 0$ as claimed. $\square$

4.3. **Euler class and amenable factors.** One advantage of continuous bounded cohomology is its blindness to amenable factors:

Lemma 4.3. *Let M_1, M_2 be closed aspherical oriented manifolds of respective dimension $\dim(M_1), \dim(M_2) \geq 1$ and fundamental groups Γ_1, Γ_2. Set $m = \dim(M_1) + \dim(M_2)$ and let*

$$\rho : \Gamma_1 \times \Gamma_2 \longrightarrow GL^+(m, \mathbb{R})$$

be a representation. If either $\rho(\Gamma_1)$ or $\rho(\Gamma_2)$ is amenable, then

$$\rho^*(\varepsilon_m) = 0 \in H^m(\Gamma_1 \times \Gamma_2).$$

Proof. Suppose that $\rho(\Gamma_1)$ is amenable. Denote by G_i, for $i = 1, 2$, the Zariski closure of $\rho(\Gamma_i)$. By the Tits alternative, $\rho(\Gamma_1)$ being amenable, is virtually solvable, thus its Zariski closure G_1 is also virtually solvable hence

amenable. Therefore the projection $G_1 \times G_2 \to G_2$ induces, in virtue of [Mo01, Lemma 7.5.10], an isometric isomorphism $H^*_{c,b}(G_2) \cong H^*_{c,b}(G_1 \times G_2)$ on the respective continuous bounded cohomology groups.

As in the proof of Lemma 4.2, let $\varphi : G_1 \times G_2 \to \mathrm{GL}^+(m, \mathbb{R})$ be the homomorphism induced by the inclusion of the factors. Again, observe that $\rho = \varphi \circ \overline{\rho}$, where $\overline{\rho} : \Gamma_1 \times \Gamma_2 \to G_1 \times G_2$ is as in the proof of Lemma 4.2. Consider the commutative diagram

$$
\begin{array}{c}
\end{array}
$$

Here, all vertical arrows are the natural comparison maps between continuous bounded and continuous cohomology groups. The horizontal maps are induced from $\varphi, \rho, \overline{\rho}$ and the restriction of ρ to Γ_2. The upwards diagonal maps are induced by the canonical projection on the second factor.

Since the Euler class $\varepsilon_m \in H^m_c(\mathrm{GL}^+(m, \mathbb{R}))$ is bounded [IvTu82], it follows from the commutativity of the above diagram and the isomorphism $H^m_{c,b}(G_1 \times G_2) \cong H^m_{c,b}(G_2)$ that its image in $H^m_c(G_1 \times G_2)$ is in the image of the map $H^m_c(G_2) \hookrightarrow H^m_c(G_1 \times G_2)$, and furthermore $\rho^*(\varepsilon_m)$ is in the image of $H^m(\Gamma_2) \hookrightarrow H^m(\Gamma_1 \times \Gamma_2)$. But since M_2 is aspherical and $m > \dim(M_2)$, we have $H^m(\Gamma_2) \cong H^m(M_2) = 0$, and hence also $\rho^*(\varepsilon_m) = 0$. $\qquad \square$

We will also make use of the following:

Lemma 4.4. *Let G be a closed subgroup of $\mathrm{GL}^+(m, \mathbb{R})$, and suppose that $G = S \ltimes A$ is a decomposition of G into a semidirect product where A is a closed amenable normal subgroup. Denote by p the projection $p : S \ltimes A \to S$. Let Γ be a discrete group and*

$$
\rho : \Gamma \longrightarrow S \ltimes A < GL^+(m, \mathbb{R})
$$

a representation. Then

$$
\rho^*(\varepsilon_m) = (p \circ \rho)^*(\varepsilon_m) \in H^m(\Gamma, \mathbb{R}).
$$

Proof. Let $i : S \hookrightarrow S \ltimes A$ denote the embedding of S in the semidirect product of S and A. Since $p \circ i$ is the identity on S, the induced map $(p \circ i)^* = i^* \circ p^*$ is the identity both on $H_c^*(S)$ and on $H_{c,b}^*(S)$.

On the bounded continuous cohomology groups, it follows from [Mo01, Lemma 7.5.10] that the projection p furthermore induces an isometric isomorphism

$$p^* : H_{c,b}^m(S) \xrightarrow{\cong} H_{c,b}^m(S \ltimes A).$$

Consider the following commutative diagram:

$$
\begin{array}{ccccc}
 & & & & H^m(\Gamma) \\
 & & & \nearrow{\scriptstyle \rho^*} & \uparrow{\scriptstyle \rho^* \circ p^*} \\
H_c^m(\mathrm{GL}^+(m,\mathbb{R})) & \longrightarrow & H_c^m(S \ltimes A) & \xleftarrow{p^*} & H_c^m(S) \\
\uparrow{\scriptstyle c} & & \uparrow{\scriptstyle c} & & \uparrow{\scriptstyle c} \\
H_{c,b}^m(\mathrm{GL}^+(m,\mathbb{R})) & \longrightarrow & H_{c,b}^m(S \ltimes A) & \xleftarrow{\cong} & H_{c,b}^m(S).
\end{array}
$$

On the continuous cohomology groups, p^* is not necessarily the inverse of $i^* : H_c^m(S \ltimes A) \to H_c^m(S)$. However, we get from the above diagram that $p^* \circ i^* : H_c^m(S \ltimes A) \to H_c^m(S \ltimes A)$ is the identity when restricted to the image of $H_{c,b}^m(S \ltimes A) \to H_c^m(S \ltimes A)$. In particular, since the Euler class is bounded, denoting by ε_m the restriction of the Euler class both to $S \ltimes A$ and to S, we obtain $\varepsilon_m = p^*(\varepsilon_m)$ as elements of $H_c^m(S \ltimes A)$, and hence also $\rho^*(\varepsilon_m) = (p \circ \rho)^*(\varepsilon_m)$. $\qquad\square$

5. Euler class norm as a proportionality constant

The main purpose of this section is to prove the following:

Proposition 5.1. *Let M be a closed oriented $2n$-dimensional Riemannian manifold whose universal cover is isometric to the Cartesian product of n copies of the hyperbolic plane $\mathcal{H}$. Then*

$$\|M\| = \frac{\chi(M)}{(-2)^n \|\varepsilon_2 \cup \ldots \cup \varepsilon_2\|_\infty}.$$

For a general closed oriented Riemannian manifold M, the simplicial volume and the volume are related by the Gromov–Thurston fundamental proportionality principle (see [Gr82, Th78])

$$\|M\| = \frac{\mathrm{Vol}(M)}{c(\widetilde{M})},$$

where $c(\widetilde{M}) \in \mathbb{R}_+ \cup \{+\infty\}$ is a constant depending only on the Riemannian universal cover $\widetilde{M}$ of M.

For locally symmetric spaces of noncompact type, it is shown in [Bu06] that the proportionality constant satisfies

$$c(\widetilde{M}) = \left\|\omega_{\widetilde{M}}\right\|_\infty,$$

where $\omega_{\widetilde{M}} \in H_c^n(\text{Isom}(\widetilde{M})^\circ)$ is the image under the Van Est isomorphism of the $\text{Isom}(\widetilde{M})^\circ$–invariant volume form $\omega_{\widetilde{M}}$ on the symmetric space $\widetilde{M}$.

When $\widetilde{M} = \mathcal{H}^n$, the Euler characteristic and the volume of M are related by Hirzebruch's proportionality principle [Hi58]

$$\text{Vol}(M) = (-2\pi)^n \chi(M).$$

Thus, for such manifolds we have

$$\|M\| = \frac{\text{Vol}(M)}{\|\omega_{\mathcal{H}^n}\|_\infty} = \frac{(-2\pi)^n \chi(M)}{\|\omega_{\mathcal{H}^n}\|_\infty}.$$

In order to prove the proposition, we need to understand the relation between the norms of the volume form and of the Euler class. On the one hand, since the projection $\prod_{i=1}^n \text{SL}(2, \mathbb{R}) \to \prod_{i=1}^n \text{PSL}(2, \mathbb{R})$ is finite-to-one, the volume form $\omega_{\mathcal{H}^n}$ defines continuous cohomology classes with the same sup norm in $H_c^{2n}(\prod_{i=1}^n \text{PSL}(2, \mathbb{R}))$ and in $H_c^{2n}(\prod_{i=1}^n \text{SL}(2, \mathbb{R}))$. On the other hand, since $\text{GL}^+(2, \mathbb{R}) \cong \text{SL}(2, \mathbb{R}) \times \mathbb{R}^{>0}$, by Lemma 4.4 the sup norm of the Euler class is unchanged by replacing it with its restriction to $\prod_{i=1}^n \text{SL}(2, \mathbb{R})$ which we still denote by $\varepsilon_2 \cup \ldots \cup \varepsilon_2$. The proposition will follow from:

Lemma 5.2. *The volume form and the Euler class are related by*

$$\omega_{\mathcal{H}^n} = (-4\pi)^n \varepsilon_2 \cup \ldots \cup \varepsilon_2$$

as elements in $H_c^{2n}(\prod_{i=1}^n \text{SL}(2, \mathbb{R}))$.

Proof of Lemma 5.2. As the Van Est isomorphism is multiplicative, we have

$$\omega_{\mathcal{H} \times \ldots \times \mathcal{H}} = \omega_{\mathcal{H}} \cup \ldots \cup \omega_{\mathcal{H}} \in H_c^{2n}(\prod_{i=1}^n \text{SL}(2, \mathbb{R})),$$

so it is enough to show that

(5) $\omega_{\mathcal{H}} = -4\pi\varepsilon_2$

as elements of $H_c^2(\text{SL}(2, \mathbb{R}))$.

Let Σ_g be a closed surface of genus $g \geq 2$ and let $\rho : \pi_1(\Sigma_g) \to \text{PSL}(2, \mathbb{R})$ be the embedding of $\pi_1(\Sigma_g)$ corresponding to an arbitrary complete hyperbolic structure of Σ_g. It is shown in [Mi58] (see also [MiSt79, pages 312-314] for an alternative topological proof) that ρ lifts to a representation $\overline{\rho} : \pi_1(\Sigma_g) \to \text{SL}(2, \mathbb{R})$ and that

$$\langle \overline{\rho}^*(\varepsilon_2), [\Sigma_g] \rangle = 1 - g = \frac{\chi(\Sigma_g)}{2}.$$

Since $H^2(\pi_1(\Sigma_g))$ is one dimensional, and

$$\langle \overline{\rho}^*(\omega_{\mathcal{H}}), [\Sigma_g]\rangle = \langle \rho^*(\omega_{\mathcal{H}}), [\Sigma_g]\rangle = \mathrm{Vol}(\Sigma_g) = -2\pi\chi(\Sigma_g),$$

we have $\overline{\rho}^*(\omega_{\mathcal{H}}) = -4\pi\overline{\rho}^*(\varepsilon_2)$. Moreover, as $\overline{\rho}(\Sigma_g)$ is a cocompact lattice of $\mathrm{SL}(2,\mathbb{R})$, $\overline{\rho}^*$ induces an isomorphism between $H^2_c(\mathrm{SL}(2,\mathbb{R}))$ and $H^2(\pi_1(\Sigma_g))$, and hence $\omega_{\mathcal{H}} = -4\pi\varepsilon_2$. This completes the proof of Lemma 5.2, and hence of Proposition 5.1. $\qquad\square$

Remark 5.3. By Proposition 5.1, computing the simplicial volume of a closed $\mathcal{H}^n$-manifold is equivalent to computing the sup norm of $\varepsilon_2 \cup \ldots \cup \varepsilon_2 \in H^{2n}_c(\Pi^n_{i=1}\mathrm{GL}^+(2,\mathbb{R}))$. Milnor's original inequality [Mi58] amounts to showing $\|\varepsilon_2\|_\infty = 1/4$, and in [Bu07], the first author proved $\|\varepsilon_2 \cup \varepsilon_2\|_\infty = 1/24$. For more than two factors, only rough lower and upper bounds are currently known.

6. Representations of product groups

In this section we prove the following general proposition[¶]

Proposition 6.1. *Let* $H = \prod^n_{i=1} H_i$ *be a direct product of* n *groups and let* $\rho : H \to GL^+(m,\mathbb{R})$ *be an oriented representation such that* $\rho(H_i)$ *is nonamenable for each* i. *Then* $m \geq 2n$. *When* $m = 2n$, *the identity component of the Zariski closure of* $\rho(H)$ *is reductive, and* ρ *factors through a map to* $\prod^n_{i=1} GL(2,\mathbb{R})$

$$\prod^n_{i=1} H_i \xrightarrow{\rho_1} \prod^n_{i=1} GL(2,\mathbb{R}) \xrightarrow{\rho_2} GL^+(2n,\mathbb{R}),$$

(with ρ spanning the full composition)

and there is an even integer $0 \leq t \leq n$ *such that, up to reordering the factors* H_i, $\rho_1 = \prod^n_{i=1} \rho_{1,i}$ *where* $\rho_{1,i}$ *is a representation of* H_i *and the restriction of its image in* $\prod^n_{i=1} GL_2(\mathbb{R})$ *to the* j*'th factor is irreducible when* $j = i$ *and scalar otherwise, and* $V = \mathbb{R}^{2n}$ *decomposes as an invariant direct sum*

$$V = \bigoplus^{t/2}_{i=1} W_i \oplus \bigoplus^n_{i=t+1} V_i,$$

where the W_i*'s are* 4*–dimensional and the* V_i*'s are* 2*–dimensional. The restriction of* ρ_2 *to any pair of factors corresponding to indices* $(2i-1, 2i)$ *with* $i \leq t/2$ *on the invariant subspace* W_i *is isomorphic to the tensor of two standard representations of* $GL_2(\mathbb{R})$ *on* $\mathbb{R}^2$, *and is trivial on any of the other* W_j *and* V_j. *The restriction of* ρ_2 *to a factor corresponding to* $i > t$ *and to* V_i *is isomorphic to the standard representation of* $GL(2,\mathbb{R})$ *on* $\mathbb{R}^2$, *while it is trivial on any other* V_j *and* W_j. *The decomposition is uniquely determined,*

[¶]Unfortunately the variant of this proposition that we included in the announcement [BuGe08, Lemma 3.1] is erroneous.

and in particular the number $t = t(\rho)$ is well defined. When $t = 0$, $\rho(H)$ is conjugated to a subgroup of $Bl_{2,n}^+$.

The following classical generalization of Schur's lemma is implicit in the proof of Theorem 3 in [Cl37] when restricting to the case of product groups:

Lemma 6.2. *Let $r : H_1 \times H_2 \to GL(W)$ be an irreducible representation of a direct product over an algebraically closed field. Then r is isomorphic to the tensor of some (irreducible) representations $r_i : H_i \to GL(\tilde{W}_i)$, $i = 1, 2$.*

Moreover, any linear automorphism $T \in GL(W)$ which commutes with $r(H_2)$ is of the form $\tilde{T} \otimes 1$ for some $\tilde{T} \in GL(\tilde{W}_1)$.

Proof. Let $U \subset W$ be an $r(H_1)$ nontrivial irreducible subspace. If $U = W$, the lemma follows directly from Schur's lemma with $\tilde{W}_1 = W$ and $\dim(\tilde{W}_2) = 1$. Suppose that $U \neq W$. Since W is irreducible, we can find $h_1, \ldots, h_k \in H_2$ with $h_1 = 1$ such that $W = \sum_{i=1}^{k} r(h_i)U$ and $r(h_j)U$ is not contained in $\sum_{i=1}^{j-1} r(h_i)U$ for every $j \leq k$. Since the h_i commute with H_1, $r(h_j)U$ are H_1 isomorphic invariant subspaces, hence H_1–irreducible, so we conclude that $W = \bigoplus_{i=1}^{k} r(h_i)U$ is a direct sum. Choose an ordered basis B_1 of U and extend it to an ordered basis B of W by adding $r(h_i)B_1$, $i = 2, \ldots, k$, to it. With respect to the basis B the elements of $r(H_1)$ are block diagonal with k identical blocks. To prove the lemma, we have to show that if T is a linear transformation of W commuting with $r(H_1)$ then each of the $k \times k$ blocks of T determined by the basis $B = B_1 \cup \ldots \cup B_k$ is a scalar matrix. In other words, denoting by $P_j : W \to r(h_j)U$, $j = 1, \ldots, k$, the associated projections, we have to show that $r(h_j^{-1}) \circ P_j \circ T \circ r(h_i)$ is a scalar transformation of U for every i, j. Since all four transformations involved commute with $r(H_1)$, and U is H_1 irreducible, this is a consequence of Schur's lemma. $\square$

The following simple observation will be used in the proof below.

Lemma 6.3. *Suppose that Δ is a subgroup of $GL(m, \mathbb{C})$ and $\mathbb{C}^m$ decomposes as a direct sum of one Δ–invariant subspace U of dimension > 1 and $(m - \dim U)$ Δ–invariant lines. Then every Δ–irreducible subspace of dimension > 1 is contained in U.*

Proof. Let W be a Δ–invariant subspace of dimension > 1. If W is not contained in U, then the corresponding projection P_L on one of the lines L in the given decomposition is nonzero on W. This implies that W is not irreducible since $\ker(P_L|_W)$ is a Δ–invariant subspace of W of codimension 1. $\square$

Proof of Proposition 6.1. Consider $V_{\mathbb{C}} = \mathbb{C} \otimes V$ as a complex vector spaces. By choosing a basis for V we may identify $GL(m, \mathbb{C})$ with the group of complex linear automorphisms of $V_{\mathbb{C}}$ and $GL(m, \mathbb{R})$ as the subgroup preserving the real subspace V.

Let G_i be the Zariski closure of $\rho(H_i)$ in $\mathrm{GL}(m, \mathbb{C})$. Let G_i° be the identity connected component of G_i with respect to the Zariski topology, and let $G_i^{\circ} = S_i \ltimes U_i$ be a Levi decomposition of the complex algebraic group G_i°, S_i being reductive and U_i unipotent (see [Ma92], 0.28). The G_i commute with each other, being the Zariski closure of commuting groups. Hence the S_i also commute with each other. Let $S = \prod_{i=1}^{n} S_i$ be their product. Then S is reductive. Our nonamenability assumption implies in particular that each S_i is nonabelian.

We will first prove the analog of the statement when $\prod H_i$ is replaced by $S = \prod S_i$, ρ by the *complex* representation corresponding to the inclusion of S in $\mathrm{GL}(m, \mathbb{C})$, and the nonamenability assumption by the data that each S_i is nonabelian. That is, we will show that $m \geq 2n$ and in the equality case, $V_{\mathbb{C}}$ decomposes as an S–invariant direct sum $\bigoplus_{i=1}^{t/2} W_{\mathbb{C},i} \oplus \bigoplus_{i=t+1}^{n} V_{\mathbb{C},i}$ where the $W_{\mathbb{C},i}$ are 4–dimensional and isomorphic to tensors of pairs of invariant complex planes corresponding to pairs of factors of S, the $V_{\mathbb{C},i}$ are planes, and each S_i acts nonscalarly on exactly one subspace in this decomposition. Moreover we will show that this decomposition is unique.

To prove this claim we argue by induction on n, the case $n = 1$ being trivial. Suppose $n > 1$. Since S is reductive, $V_{\mathbb{C}}$ decomposes as an S–invariant direct sum of S–irreducible subspaces $\bigoplus_{j=1}^{l} \mathcal{V}_{\mathbb{C},j}$.

Suppose first that $l > 1$. Then since each S_i is nonabelian, its restriction to at least one of the $\mathcal{V}_{\mathbb{C},j}$ is nonabelian, and the conclusion that $m \geq 2n$ follows from the induction hypothesis applied to all the $\mathcal{V}_{\mathbb{C},j}$ simultaneously, i.e.

$$m = \sum_{j=1}^{l} \dim \mathcal{V}_{\mathbb{C},j} \geq 2 \sum_{j=1}^{l} \#\{i : \text{the restriction of } S_i \text{ to } \mathcal{V}_{\mathbb{C},j} \text{ is nonabelian}\} \geq 2n.$$

Moreover if $m = 2n$, each S_i is nonabelian on exactly one $\mathcal{V}_{\mathbb{C},j}$, and since the restriction of S_i to every other $\mathcal{V}_{\mathbb{C},j}$ (i.e. one on which it acts commutatively) is central, it follows from Schur's lemma that it acts scalarly there. The uniqueness of the decomposition $V_{\mathbb{C}} = \bigoplus_{i=1}^{t/2} W_{\mathbb{C},i} \oplus \bigoplus_{i=t+1}^{n} V_{\mathbb{C},i}$ follows in this case from the induction hypothesis and Lemma 6.3.

Suppose now that $l = 1$, i.e. S acts irreducibly on $V_{\mathbb{C}}$. Writing $S = S_1 \times \prod_{i=2}^{n} S_i$, we derive from Lemma 6.2 that $V_{\mathbb{C}} \cong \tilde{V}_1 \otimes \tilde{V}_2$ where $\tilde{V}_1$ is a representation of S_1 and $\tilde{V}_2$ is a representation of $\prod_{i=2}^{n} S_i$. The assumption that all S_i are nonabelian and the inductive hypothesis give $\dim \tilde{V}_1 \geq 2$ and $\dim \tilde{V}_2 \geq 2(n-1)$. Thus $m = \dim V_{\mathbb{C}} \geq 4(n-1) \geq 2n$, and equality implies that $n = 2, m = 4$ and our representation is isomorphic to the tensor of two 2–dimensional representations of S_1 and $S_{2=n}$. This finishes the proof of the analogous statement for the complex representation given by the inclusion of S in $\mathrm{GL}(m, \mathbb{C})$.

We shall denote by j° and k° either a pair of indices $(2i-1, 2i)$ or a single index i according to whether $i \leq t/2$ or $i > t$. We let $|j^\circ| = 1, 2$ according to whether j° is a single index or a pair of indices. For each pair $j^\circ = (2i-1, 2i)$, $1 \leq i \leq t/2$, we let

$$G_{j^\circ}^\circ = G_{2i-1}^\circ \times G_{2i}^\circ, \ S_{j^\circ} = (S_{2i-1} \times S_{2i}) \ \text{ and } \ U_{j^\circ} = (U_{2i-1} \times U_{2i}),$$

and let $V_{\mathbb{C},j^\circ}$ be the unique S_{j°–irreducible space of dimension > 1 (Lemma 6.3).

Consider now a unipotent element $u \in U_{k^\circ}$. Since for $j^\circ \neq k^\circ$, u centralizes S_{j°, it preserves $V_{\mathbb{C},j^\circ}$, and by Schur's lemma acts scalarly, hence, being unipotent, trivially on $V_{\mathbb{C},j^\circ}$. To show that $V_{\mathbb{C},k^\circ}$ is u invariant as well, we argue by way of contradiction. Suppose that $u(v) \notin V_{\mathbb{C},k^\circ}$ for some $v \in V_{\mathbb{C},k^\circ}$. Let $j^\circ \neq k^\circ$ be such that the projection of $u(v)$ to $V_{\mathbb{C},j^\circ}$ is nontrivial, and choose $s \in S_{j^\circ}$ for which $su(v)$ is not on the line spanned by $u(v)$. Such s exists since S_{j° acts irreducibly on $V_{\mathbb{C},j^\circ}$. This however gives the impossible

$$\begin{aligned} 2 &= \operatorname{rank}\{su(v), u(v)\} = \operatorname{rank}\{u^{-1}su(v), v\} \\ &= \operatorname{rank}\{s^{-1}u^{-1}su(v), s^{-1}(v)\} = 1, \end{aligned}$$

where the last equality holds since s and u commute and s acts scalarly on $\mathbb{C}v$. This holds for any $u \in U_{k^\circ}$, so all the $V_{\mathbb{C},j^\circ}$ are $G_{k^\circ} = S_{k^\circ} \ltimes U_{k^\circ}$–invariant, and since k° is arbitrary, it follows that the decomposition $V_{\mathbb{C}} = \bigoplus V_{\mathbb{C},j^\circ}$ is $G = S \ltimes U$ invariant.

Next we show that G° is reductive. Consider first k° with $|k^\circ| = 1$. Since U_{k° is unipotent, the subspace of U_{k°–invariant vectors in $V_{\mathbb{C},k^\circ}$ is non-trivial. As U_{k° is normalized by S_{k°, this subspace is S_{k°–invariant, and as S_{k° is irreducible on $V_{\mathbb{C},k^\circ}$, it must be the full space, i.e. U_{k° acts trivially on $V_{\mathbb{C},k^\circ}$. We already saw in the previous paragraph that U_{k° acts trivially on $V_{\mathbb{C},j^\circ}$ for any $j^\circ \neq k^\circ$, so U_{k° is trivial. Similarly, if $|k^\circ| = 2$, we write $G_{k^\circ}^\circ = G_{2i-1}^\circ \times G_{2i}^\circ = (S_{2i-1} \times S_{2i}) \ltimes (U_{2i-1} \times U_{2i})$ and $V_{\mathbb{C},k^\circ} = \tilde{V}_{2i-1} \otimes \tilde{V}_{2i}$ where the $\tilde{V}_j$ are irreducible G_j spaces, $j \in \{2i-1, 2i\}$. If u is a unipotent belonging to one of the factors, say $u \in U_{2i-1}$, then, being in the centralizer of the other factor, u preserves the tensor structure of $V_{\mathbb{C},k^\circ}$ and acts trivially on its second factor (Lemma 6.2). As above we deduce that the space of U_{2i-1} invariants in $\tilde{V}_{2i-1}$ is the full space, and hence that U_{2i-1} acts trivially on $V_{\mathbb{C},k^\circ}$ as well as on $V_{\mathbb{C},j^\circ}$ for any j°. Thus U is trivial, i.e. $G^\circ = S$ is reductive.

Finally, since the decomposition established above is unique, and the $G_{k^\circ}^\circ$ are defined over $\mathbb{R}$, each $V_{\mathbb{C},k^\circ}$ is invariant under complex conjugation, hence is defined over $\mathbb{R}$. In other words $V_{\mathbb{C},k^\circ} = \mathbb{C} \otimes V_{k^\circ}$ – the complexification of the real space $V_{k^\circ} = V_{\mathbb{C},k^\circ} \cap V$. The uniqueness of the decomposition also implies that the result holds for $G = \prod G_i$ rather than its Zariski identity component $\prod G_i^\circ$.

Moreover, in the case $t = 0$, since the representation ρ is oriented and the elements of $\rho(H_i)$ act scalarly and hence preserve orientation on the real 2-dimensional spaces V_j, for $j \neq i$, they also preserve the orientation on V_i. This proves the last statement of the proposition. $\qquad\square$

Remark 6.4. (i) In later sections we shall need to apply Proposition 6.1 in a slightly more general setup where H is an almost direct product of n factors, i.e $H = (\prod_{i=1}^{n} H_i)/C$ where $C \lhd \prod_{i=1}^{n} H_i$ is a (finite) central subgroup. In this case, given $\rho : H \to \mathrm{GL}^+(2n, \mathbb{R})$ and letting $\tilde{\rho} : \prod_{i=1}^{n} H_i \to \mathrm{GL}^+(2n, \mathbb{R})$ be the induced representation, we can decompose $\tilde{\rho} = \tilde{\rho}_2 \circ \tilde{\rho}_1$, as in Proposition 6.1. Then $\tilde{\rho}_1(C)$ lies both in the center of $\prod_{i=1}^{n} \mathrm{GL}(2, \mathbb{R})$ and in the kernel of $\tilde{\rho}_2$, and ρ decomposes as $\rho = \rho_2 \circ \rho_1$ through a map to $\prod_{i=1}^{n} \mathrm{GL}(2, \mathbb{R})/\rho_1(C)$ which is an almost direct product. If the restriction of $\tilde{\rho}_2$ to a factor or a pair of factors of $\prod_{i=1}^{n} \mathrm{GL}(2, \mathbb{R})$ is irreducible or tensor of irreducible or scalar on some subspace of $V = \mathbb{R}^{2n}$ the same holds for the restriction of ρ_2 to the image of that factor (or pair of factors) in $\prod_{i=1}^{n} \mathrm{GL}(2, \mathbb{R})/\rho_2(C)$. We shall denote $t(\rho) = t(\tilde{\rho})$.

(ii) In the case when $S \leq \mathrm{GL}^+(2n, \mathbb{R})$ is an almost direct product of $2n$ nonamenable subgroups, we shall simply write $t(S) = t(i)$, where $i : S \hookrightarrow \mathrm{GL}^+(2n, \mathbb{R})$ is the inclusion.

We end this section with an extension of Lemma 4.1 that will be applied repeatedly in the sequel combined with Proposition 6.1.

Lemma 6.5. *Let* $\rho : \prod_{i=1}^{n} H_i \to GL^+(2n, \mathbb{R})$ *be a representation such that* $\rho(H_i)$ *is nonamenable for every* i. *If* $t(\rho) > 0$ *then*

$$\rho^*(\varepsilon_{2n}) = 0.$$

Proof. By Proposition 6.1, ρ factors through a map to $\prod_{i=1}^{n} \mathrm{GL}_2(\mathbb{R})$:

$$\prod_{i=1}^{n} H_i \xrightarrow{\ \rho_1\ } \prod_{i=1}^{n} \mathrm{GL}_2(\mathbb{R}) \xrightarrow{\ \rho_2\ } \mathrm{GL}^+(2n, \mathbb{R}).$$

Up to replacing $\prod_{i=1}^{n} H_i$ by a finite index subgroup, which has no influence on the vanishing of the Euler class, we can assume that the image of ρ_1 is contained in the product $\prod_{i=1}^{n} \mathrm{GL}_2^+(\mathbb{R})$. If $t(\rho) > 0$, then up to permuting the factors, Proposition 6.1 allows us further to describe ρ_2 as follows: Let ι denote the canonical inclusion of $\prod_{i=1}^{n} \mathrm{SL}_2(\mathbb{R})$ in $\prod_{i=1}^{n} \mathrm{GL}_2^+(\mathbb{R})$. Precomposed with ι, the representation ρ_2 has the form

$$\rho_2 \circ \iota(A, B, C) = \begin{pmatrix} A \otimes B & 0 \\ 0 & \rho_2'(C) \end{pmatrix},$$

for $A, B \in \mathrm{SL}_2(\mathbb{R})$ and $C \in \Pi_{i=3}^n \mathrm{SL}_2(\mathbb{R})$, where $\rho_2' : \Pi_{i=3}^n \mathrm{SL}_2(\mathbb{R}) \to \mathrm{GL}_{2n-4}^+(\mathbb{R})$ is some representation. Denote by $p : \Pi_{i=1}^n \mathrm{GL}_2^+(\mathbb{R}) \to \Pi_{i=1}^n \mathrm{SL}_2(\mathbb{R})$ the natural projection. Note that ρ_2 and $\rho_2 \circ \iota \circ p$ lie in the same path connected component in the space of representations, thus $\rho_2^*(\varepsilon_m) = (\rho_2 \circ \iota \circ p)^*(\varepsilon_m)$. Finally, since $\rho_2 \circ \iota$ is the inclusion of the direct sum of the tensor representation $\rho_\otimes$ of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ and the representation ρ_2' on $\Pi_{i=3}^n \mathrm{SL}_2(\mathbb{R})$, we have

$$(\rho_2 \circ \iota)^*(\varepsilon_m) = \rho_\otimes^*(\varepsilon_4) \cup (\rho_2')^*(\varepsilon_{m-4}) = 0,$$

since $\rho_\otimes^*(\varepsilon_4) = 0$ by Lemma 4.1. $\qquad\square$

7. Lattices in $\mathrm{PSL}(2, \mathbb{R})^n$ and their representations

Recall that a lattice in a semisimple Lie group G is said to be *irreducible* if it projects densely to every proper quotient of G. Let $G = \prod_{i=1}^n \mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{Isom}(\mathcal{H}^n)^\circ$, and let Γ be a lattice[||] in G. By [R72, Theorem 5.22], up to replacing Γ by a finite index subgroup $\Gamma' \le \Gamma$, G decomposes as a direct product $G = \prod_{j=1}^m G_j$ such that:

- each G_j is a direct product of $\mathrm{PSL}(2, \mathbb{R})$'s,
- $\Gamma_j' := \Gamma' \cap G_j$ is an irreducible lattice in G_j, and
- Γ' is the direct product of the Γ_j''s.

For the rest of this section let us assume that $\Gamma = \Gamma'$, i.e. that Γ itself decomposes as a direct product of irreducible factors $\Gamma_j = \Gamma \cap G_j$. Let $\rho : \Gamma \to \mathrm{GL}^+(2n, \mathbb{R})$ be an orientable $2n$–dimensional representation of Γ. In order to analyze the possible images of ρ, we shall distinguish between 3 cases:

Case 1: Γ is completely reducible, i.e. $\Gamma = \prod_{i=1}^n \Gamma_i$, where each Γ_i is a lattice in $\mathrm{PSL}(2, \mathbb{R})$. It is an immediate consequence of Proposition 6.1 that:

Proposition 7.1. *In this case either*

- *$\rho(\Gamma_i)$ is amenable for some $i = 1, \ldots, n$,*
- *$t(\rho) > 0$, or*
- *$\rho(\Gamma)$ is conjugate to a subgroup of $Bl_{2,n}^+$.*

Case 2: Γ is rigid, i.e. each G_j has real rank at least 2, or equivalently no G_j is $\mathrm{PSL}(2, \mathbb{R})$. By Margulis arithmeticity theorem, every Γ_i is a Hilbert–Blumenthal modular group.

Proposition 7.2. *In this case, up to replacing Γ by a finite index subgroup, $\rho(\Gamma)$ is contained in a connected semisimple Lie group S for which all the noncompact simple factors are locally isomorphic to $\mathrm{PSL}(2, \mathbb{R})$. The number of the simple factors of S is at most n, and in the case it is exactly n, either*

[||]In this section we do not need to assume that Γ is cocompact.

(a) $t(S) > 0$, or

(b) S has no compact factors, and

(b_1) it is conjugate to $SBl_{2,n}$, the diagonal product of $SL_2(\mathbb{R})'s$,

(b_2) ρ is faithful and $\rho(\Gamma) \cap Z(S) = \emptyset$, and

(b_3) the image of $\rho(\Gamma)$ in S is a lattice and is cocompact in the case $\Gamma \leq G$ is cocompact.

Proof. By Whitney's theorem (see [PR94, Theorem 3.6]), the Zariski closure of $\rho(\Gamma_j)$ in $\mathrm{GL}^+(2n, \mathbb{R})$, which we shall denote below by $\overline{\rho(\Gamma_j)}^z$, has finitely many connected components with respect to the Hausdorff topology induced from $\mathrm{GL}^+(2n, \mathbb{R})$. Hence, up to replacing Γ by a finite index subgroup, we may assume that $\overline{\rho(\Gamma_j)}^z$ is connected (as a real group) for each $j \leq m$. Moreover, as $\overline{\rho(\Gamma_j)}^z$ is closed in $\mathrm{GL}^+(2n, \mathbb{R})$, it is a Lie group (see [Kn96, Proposition 1.75]).

Since for each j, $\mathrm{rank}_{\mathbb{R}}(G_j) \geq 2$, it follows from Margulis' Theorem (see [Ma92, Ch. IX, Theorem 5.8]) that $\overline{\rho(\Gamma_j)}^z$ is semisimple, and hence $\overline{\rho(\Gamma)}^z$ is also semisimple. Lets denote $S = \overline{\rho(\Gamma)}^z$, and let S_{nc}, S_c be the product of noncompact, resp. compact, simple factors of S. Then S is an almost direct product $S = S_{nc} \times S_c$ (i.e. $S_{nc} \cap S_c$ is finite and central in S).

Let $\tilde{\rho} : \Gamma_j \to \mathrm{Ad}(S_{nc})$ be the representation induced from ρ by dividing out S_c and composing with the Adjoint representation of S_{nc}. Since each simple factor S' of S_{nc} is noncompact, the projection of the image of each $\tilde{\rho}(\Gamma_j)$ to S', being normal, is either trivial or Zariski dense, and $\prod_{j=1}^m \Gamma_j$ is superrigid in $G = \prod_{j=1}^m G_j = \prod_{i=1}^n \mathrm{PSL}(2, \mathbb{R})$ (as $\mathrm{rank}_{\mathbb{R}}(G_j) \geq 2$, $\forall j$), it follows from Margulis' Superrigidity Theorem [Ma92, Ch. VII, Sec. 5] that $\tilde{\rho}$ extends to a representation of $G = \prod_{i=1}^n \mathrm{PSL}(2, \mathbb{R})$. Thus $\mathrm{Ad}(S_{nc})$ is a homomorphic image of $\prod_{i=1}^n \mathrm{PSL}(2, \mathbb{R})$. This proves the first statement of the proposition.

In order to see that the number of simple factors of S is at most n, we apply Proposition 6.1 to the product of the simple factors of $\tilde{S}$, the universal covering Lie group of S. Note that all the simple factors of S (including the compact ones) considered as abstract groups with no topology are nonamenable. Suppose now that S has exactly n simple factors. Then if $t(S) > 0$ we are in case (a). Assuming $t(S) = 0$, we get from Proposition 6.1 that S is conjugate to a subgroup of $\mathrm{Bl}_{2,n}^+$. Then all the simple factors of S are locally isomorphic to $\mathrm{PSL}_2(\mathbb{R})$ (i.e. $S = S_{nc}$), and hence $\dim(S) = 3n$. Furthermore, since S is connected and semisimple, its conjugate in $\mathrm{Bl}_{2,n}^+$ is contained in, and hence, by dimension equality, coincides with the commutator group of $\mathrm{Bl}_{2,n}^+$ which is $SBl(2, \mathbb{R})$. This proves (b_1).

(b_2) follows from Margulis' Superrigidity Theorem as the map $G \to \mathrm{Ad}(S)$ extending the map $\Gamma \ni \gamma \mapsto \mathrm{Ad}_S(\rho(\gamma))$ is an isomorphism.

The fact (b_3), that the image of Γ in S is a (cocompact) lattice, holds because G/Γ is (equivariantly) homeomorphic to $\mathrm{Ad}(S)/\mathrm{Ad}_S(\rho(\Gamma))$ and S has a finite center by (b_1). □

Case 3: The mixed case. The mixed case is when some, but not all, of the G_j are isomorphic to $\mathrm{PSL}(2,\mathbb{R})$. Denote by $\mathcal{R} \subset \{1,\ldots,m\}$ the set of indices corresponding to rigid factors of $\Gamma = \prod_{j=1}^m \Gamma_j$. In this case we have:

Proposition 7.3. *Up to replacing each rigid factor by a finite index subgroup (without changing the nonrigid factors), we have that either*
(i) for some $j \notin \mathcal{R}$, $\rho(\Gamma_j)$ is amenable,
(ii) for some $j \in \mathcal{R}$, $\rho(\Gamma_j)$ is contained in a group locally isomorphic to $\prod_{i=1}^k \mathrm{PSL}(2,\mathbb{R})$ where k is strictly smaller than the number of factors of G_j, or
(iii) there are n groups H_i, $i = 1,\ldots,n$ and an almost direct product $H = (\prod_{i=1}^n H_i)/C$ such that ρ factors through a map

$$\Gamma \xrightarrow{\ \rho_1\ } H \xrightarrow{\ \rho_2\ } GL^+(2n,\mathbb{R}),$$

and we have that either
  (iii′) $t(\rho_2) > 0$, or
  (iii″) $\rho(\Gamma)$ is conjugated to a subgroup of $Bl_{2,n}^+$.

Proof. Fix a labeling of the simple factors of $\mathrm{PSL}(2,\mathbb{R})^n$ by $\{1,\ldots,n\}$, and for $j \in \mathcal{R}$ denote by K_j the set of indices corresponding to the factors of G_j under this labeling. By replacing every Γ_j, $j \in \mathcal{R}$, by a finite index subgroup, we may assume that the Zariski closure $F_j = \overline{\rho(\Gamma_j)}^z$ of $\rho(\Gamma_j)$ in $GL(2n,\mathbb{R})$ is connected in the Hausdorff topology. Then, as in Case (2), for each $j \in \mathcal{R}$, F_j is locally isomorphic to a product of $\mathrm{PSL}(2,\mathbb{R})$'s, and the number of factors is at most $\dim(G_j)/3$. Assuming that (ii) above is not satisfied, for every rigid component j, F_j is an almost direct product of exactly $|K_j|$ such factors, so let us name its simple factors by H_i, $i \in K_j$, and write $F_j = \prod_{i \in K_j} H_i/C_j$ where C_j is a finite central group in $\prod_{i \in K_j} H_i$. Set $C = \prod_{j \in \mathcal{R}} C_j$, for $j \notin \mathcal{R}$, set $H_i = F_j = \rho(\Gamma_j)$, where $i = i(j) \in \{1,\ldots,n\}$ is the labeling index of G_j, and define $H = \prod_{i=1}^n H_i/C = \prod_{j=1}^m F_j$. Since the Γ_j's commute which one another, the F_j's commute and the product of the inclusion maps $F_j \to GL^+(2n,\mathbb{R})$ gives a well defined representation, which we denote by $\rho_2 : H \to GL^+(2n,\mathbb{R})$. Obviously we set $\rho_1 : \Gamma \to \prod_{j=1}^m F_j = H$ to be the product of the restrictions $\rho|_{\Gamma_j} : \Gamma_j \to F_j$. Finally if we assume in addition that (i) is not satisfied, we may apply Proposition 6.1 and conclude that either $(iii′)$ or $(iii″)$ must hold. □

8. An inequality of Euler classes

In this section we prove the following:

Theorem 8.1. *Let Γ be a cocompact lattice in $\mathrm{Isom}(\mathcal{H}^n)^+$ and*

$$\rho : \Gamma \longrightarrow \mathrm{GL}^+(2n, \mathbb{R})$$

a representation. Then

$$\|\rho^*(\varepsilon_{2n})\|_\infty \leq \|\underbrace{\varepsilon_2 \cup \ldots \cup \varepsilon_2}_{n\ times}\|_\infty.$$

Proof. The inclusion of a finite index subgroup Δ in Γ induces isometric embeddings

$$H^*(\Gamma) \longrightarrow H^*(\Delta) \quad \text{and} \quad H_b^*(\Gamma) \longrightarrow H_b^*(\Delta)$$

both on the standard and bounded cohomology groups. The statement for bounded cohomology groups is a particular case of [Mo01, Proposition 8.6.2] and the standard case is proven identically. As a result, we may replace Γ by a finite index subgroup which is contained in the identity connected component $\mathrm{Isom}(\mathcal{H}^n)^\circ \cong \prod_{i=1}^n \mathrm{PSL}(2, \mathbb{R})$ and decomposes as in Section 7. We will argue case by case, showing that if $\rho^*(\varepsilon_{2n}) \neq 0$ then, up to replacing Γ again by a finite index subgroup, $\rho(\Gamma)$ is conjugate to a subgroup of $\mathrm{Bl}_{2,n}^+$. By Lemma 4.3, if for some j, $\rho(\Gamma_j)$ is amenable, then $\rho^*(\varepsilon_{2n}) = 0$. We shall assume below that this is not the case.

Case 1: Γ is completely reducible.

In this case, assuming $\rho^*(\varepsilon_{2n}) \neq 0$ we have by Lemma 6.5 that $t(\rho) = 0$, hence Proposition 7.1 gives that $\rho(\Gamma)$ is conjugate to a subgroup of $\mathrm{Bl}_{2,n}^+$.

Case 2: Γ is rigid.

Replacing Γ by a further finite index subgroup, if necessary, we get from Proposition 7.2 that $\rho(\Gamma)$ is contained in a connected Lie group S locally isomorphic to $\left(\mathrm{PSL}(2, \mathbb{R})\right)^k$ with $k \leq n$. Note that ρ^* factors through

$$H_c^{2n}(\mathrm{GL}^+(2n, \mathbb{R})) \longrightarrow H_c^{2n}(S) \cong H_c^{2n}\left(\Pi_{i=1}^k \mathrm{PSL}(2, \mathbb{R})\right) \longrightarrow H^{2n}(\Gamma).$$

If $k < n$, then the middle cohomology group is zero and hence $\rho^*(\varepsilon_{2n})$ vanishes. If $k = n$, then Proposition 7.2 gives us further that either $t(S) > 0$, in which case $\rho^*(\varepsilon_{2n}) = 0$ (Lemma 6.5), or S is conjugate to a subgroup of $\mathrm{Bl}_{2,n}^+$.

Case 3: The mixed case.

Applying Proposition 7.3 and arguing as above, we derive that if either $(i), (ii)$ or (iii') of 7.3 holds then $\rho^*(\varepsilon_{2n}) = 0$. Thus we may assume in this case as well that $\rho(\Gamma)$ is conjugate to a subgroup of $\mathrm{Bl}_{2,n}^+$.

We have therefore reduced to the situation where, up to conjugating the image, ρ factors through a map ρ_0 into $\mathrm{Bl}_{2,n}^+$:

$$\Gamma \xrightarrow{\ \rho\ } \mathrm{GL}^+(2n, \mathbb{R})$$

$$\rho_0 \searrow \quad \uparrow i$$

$$\mathrm{Bl}_{2,n}^+.$$

The Whitney product formula for the Euler class gives

$$i^*(\varepsilon_{2n}) = \varepsilon_2 \cup \ldots \cup \varepsilon_2 \in H_c^{2n}(\mathrm{Bl}_{2,n}^+),$$

and hence

$$\left\| \rho^*(\varepsilon_{2n}) \right\|_\infty = \left\| \rho_0^* \circ i^*(\varepsilon_{2n}) \right\|_\infty = \left\| \rho_0^*(\varepsilon_2 \cup \ldots \cup \varepsilon_2) \right\|_\infty \leq \left\| \varepsilon_2 \cup \ldots \cup \varepsilon_2 \right\|_\infty.$$

$\square$

9. The proofs of Theorems 1.1, 1.6, 1.7

Proof of Theorem 1.1. Let M be a closed oriented Riemannian manifold with universal cover $\mathcal{H}^n = \prod_{i=1}^n \mathcal{H}$. Let Γ be the fundamental group of M embedded as a cocompact lattice in $\mathrm{Isom}(\mathcal{H}^n)^+$ acting on $\mathcal{H}^n$ by deck transformations. Let ξ be a $\mathrm{GL}^+(2n, \mathbb{R})$–bundle over M. Suppose that ξ admits a flat structure, and let $\rho : \Gamma \to \mathrm{GL}^+(2n, \mathbb{R})$ be the corresponding representation. Identifying $H^{2n}(M)$ with $H^{2n}(\Gamma)$, $\varepsilon(\xi)$ considered as an element of $H^{2n}(\Gamma)$ is equal to $\rho^*(\varepsilon_{2n})$. Since this identification is an isometry, Theorem 8.1 gives

$$\left\| \varepsilon_{2n}(\xi) \right\|_\infty = \left\| \rho^*(\varepsilon_{2n}) \right\|_\infty \leq \left\| \varepsilon_2 \cup \ldots \cup \varepsilon_2 \right\|_\infty.$$

Combining Equation (1), the proportionality principle established in Proposition 5.1 and the last inequality, we conclude

$$\left| \langle \varepsilon(\xi), [M] \rangle \right| = \left\| \varepsilon(\xi) \right\|_\infty \left\| M \right\| = \frac{\left\| \varepsilon(\xi) \right\|_\infty}{\left\| \varepsilon_2 \cup \ldots \cup \varepsilon_2 \right\|_\infty} \frac{\chi(M)}{(-2)^n} \leq \frac{1}{(-2)^n} \chi(M),$$

which completes the proof of the theorem. $\square$

Proof of Theorem 1.6. Let $\Gamma < \mathrm{Isom}(\mathcal{H}^n)^+$ be the fundamental group of M with $\tilde{\Gamma}$ a cofaithful lift of Γ in G_n^+ and $\rho : \Gamma \to \tilde{\Gamma} < G_n^+$ the cofaitfhul map. Up to replacing M by a finite cover, which amounts to multiplying each side of the equality we are proving by the same number (the degree of the cover), we may assume that $\Gamma \leq \prod_{i=1}^n \mathrm{PSL}(2, \mathbb{R})$ and $\tilde{\Gamma} \leq \prod_{i=1}^n \mathrm{SL}(2, \mathbb{R})$.

We thus have a commutative diagram

$$
\begin{array}{ccc}
\Gamma & \hookrightarrow & \Pi_{i=1}^n \mathrm{PSL}(2,\mathbb{R}) \\
& \searrow{\scriptstyle\rho} & \uparrow \\
& & \Pi_{i=1}^n \mathrm{SL}(2,\mathbb{R}).
\end{array}
$$

Since ρ is injective and $\rho(\Gamma)$ is a cocompact lattice in $\Pi_{i=1}^n \mathrm{SL}(2,\mathbb{R})$, the induced map $\rho^* : H_c^*(\Pi_{i=1}^n \mathrm{SL}(2,\mathbb{R})) \to H^*(\Gamma)$ is an isometric embedding (cf. [Mo01, Proposition 8.6.2] or [Bu06, Theorem 3]). Thus

$$
\|\rho^*(\varepsilon_2 \cup \ldots \cup \varepsilon_2)\|_\infty = \|\varepsilon_2 \cup \ldots \cup \varepsilon_2\|_\infty.
$$

Let ξ_ρ be the flat $\mathrm{GL}^+(2n,\mathbb{R})$–bundle over M corresponding to the representation $\rho : \Gamma \to \Pi_{i=1}^n \mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SBl}_{2,n}(\mathbb{R}) < \mathrm{GL}^+(2n,\mathbb{R})$. Then

$$
|\chi(\xi_\rho)| = |\langle \rho^*(\varepsilon_2 \cup \ldots \cup \varepsilon_2), [M] \rangle| = \|\varepsilon_2 \cup \ldots \cup \varepsilon_2\|_\infty \|M\| = \frac{1}{2^n} |\chi(M)|,
$$

where the last equality follows from Proposition 5.1. $\qquad\square$

Proof of Theorem 1.7. Let N be a closed $\mathcal{H}^n$–manifold of the form

$$
N = \Sigma_{g_1} \times \ldots \times \Sigma_{g_k} \times N',
$$

where N' is rigid, $k \geq 0$, and the Σ_{g_i}'s are surfaces of genus $g_i \geq 2$, and let ξ_N be a flat $\mathrm{GL}^+(2n,\mathbb{R})$-bundle over N. In order to show that

$$
\chi(\xi_N) \in \left\{ \pm \frac{\chi(N')}{2^{\mathrm{Dim}(N')/2}} \Pi_{i=1}^k \ell_i : |\ell_i| \leq g_i - 1 \right\} \cup \{0\},
$$

we split the proof into three different cases, as before:

Case 1: $N = \Sigma_{g_1} \times \cdots \times \Sigma_{g_n}$ is a product of surfaces and N' is trivial. The fundamental group Γ of N is a product of n surface groups Γ_i. Let $\rho : \Gamma \to \mathrm{GL}^+(2n,\mathbb{R})$ be the oriented representation corresponding to ξ_N. If the restriction of ρ to any of the surface groups is amenable, then $\rho^*(\varepsilon_{2n}) = 0$ by Lemma 4.3, and if $t(\rho) > 0$ then again $\rho^*(\varepsilon_{2n}) = 0$ by Lemma 6.5, so let us assume that this is not the case. It now follows from Proposition 7.1, that $\rho(\Gamma)$ is conjugate to a subgroup of $\mathrm{Bl}_{2,n}^+$. Since conjugation induces isomorphisms of cohomology groups, we can without loss of generality assume that the image of ρ is in fact contained in $\mathrm{Bl}_{2,n}^+$. Moreover, it follows from Proposition 7.1 that up to reordering the factors if necessary, the i-th factor Γ_i is mapped irreducibly to the i-th factor of $\mathrm{Bl}_{2,n}^+ \cong \Pi_{i=1}^n \mathrm{GL}^+(2,\mathbb{R})$ and scalarly to every other factor. Denote by $r_i : \Gamma_i \to \mathrm{GL}^+(2,\mathbb{R})$ the restriction of ρ to Γ_i composed with the projection onto the i-th factor of $\mathrm{Bl}_{2,n}^+$. Then ρ and $(r_1, \ldots, r_n)$ have the same projection to $\Pi_{i=1}^n \mathrm{PSL}(2,\mathbb{R})$. Thus by Lemma 2.2,

$$
\rho^*(\varepsilon_{2n}) = r_1^*(\varepsilon_2) \cup \ldots \cup r_n^*(\varepsilon_2).
$$

We hence obtain that

$$
\begin{aligned}
\chi(\xi_N) = \langle \varepsilon_{2n}, [N] \rangle &= \langle r_1^*(\varepsilon_2) \cup \ldots \cup r_n^*(\varepsilon_2), [\Sigma_{g_1} \times \ldots \times \Sigma_{g_n}] \rangle \\
&= \langle r_1^*(\varepsilon_2), [\Sigma_{g_1}] \rangle \cdot \ldots \cdot \langle r_n^*(\varepsilon_2), [\Sigma_{g_n}] \rangle.
\end{aligned}
$$

Finally, for every i, $\langle r_i^*(\varepsilon_2), [\Sigma_{g_i}] \rangle$ is the Euler number of a flat bundle over the surface Σ_{g_i}, and hence, by Equation (1) and Proposition 5.1, satisfies

$$
|\langle r_i^*(\varepsilon_2), [\Sigma_{g_i}] \rangle| = \|r_i^*(\varepsilon_2)\|_\infty \|\Sigma_{g_i}\| \leq \frac{\|r_i^*(\varepsilon_2)\|_\infty}{\|\varepsilon_2\|_\infty} \frac{\chi(\Sigma_{g_i})}{-2} \leq g_i - 1,
$$

which is Milnor's classical inequality [Mi58].

Case 2: $N = N'$ is rigid. Let Γ denote the fundamental group of N and let $\rho : \Gamma \to \mathrm{GL}^+(2n, \mathbb{R})$ be a representation inducing the flat bundle ξ_N. Proposition 5.1 gives

$$
|\chi(\xi_N)| = \|\rho^*(\varepsilon_{2n})\|_\infty \|M\| = \frac{\|\rho^*(\varepsilon_{2n})\|_\infty}{\|\varepsilon_2 \cup \ldots \cup \varepsilon_2\|_\infty} \frac{1}{2^n} |\chi(M)|.
$$

Hence the result will follow from the next claim:

Claim. Either $\rho^*(\varepsilon_{2n}) = 0$ or $\|\rho^*(\varepsilon_{2n})\|_\infty = \|\varepsilon_2 \cup \ldots \cup \varepsilon_2\|_\infty$.

Since the inclusion of a finite index subgroup in Γ induces isometric embeddings on the standard and bounded cohomology groups, we can without loss of generality replace Γ by a finite index subgroup. Thus, as in the proof of Theorem 8.1, applying Proposition 7.2, we can reduce to the situation where either $\rho^*(\varepsilon_{2n}) = 0$, or ρ is injective, and its image $\rho(\Gamma)$ is, up to conjugation, contained, discrete and cocompact in $S = \mathrm{SBl}_{2,n}$. In the latter case, we again invoke [Mo01, Proposition 8.6.2] (or [Bu06, Theorem 3]) to conclude that the induced map $\rho^* : H_c^*(\mathrm{SBl}_{2,n}) \to H^*(\Gamma)$ is an isometric embedding and

$$
\|\rho^*(\varepsilon_{2n})\|_\infty = \|\varepsilon_2 \cup \ldots \cup \varepsilon_2\|_\infty,
$$

which finishes the proof of the claim.

Case 3: The mixed case: $N = \Sigma_{g_1} \times \ldots \times \Sigma_{g_k} \times N'$ with $k > 0$ and N' nontrivial. The fundamental group Γ of N decomposes as $\Gamma = \Gamma_1 \times \ldots \times \Gamma_k \times \Gamma'$, where the Γ_i's are surface groups and Γ' is the fundamental group of the rigid factor N'. Let $\rho : \Gamma \to \mathrm{GL}^+(2n, \mathbb{R})$ be a representation inducing the flat bundle ξ_N.

Replacing the rigid factor Γ' by a finite index subgroup has no effect on the equality we are proving (both sides are multiplied by the index of the subgroup). Hence, assuming that $\rho^*(\varepsilon_n) \neq 0$, arguing as above, we may reduce the situation to case (iii') of Proposition 7.3.

Furthermore, by Proposition 6.1, up to conjugation we can assume that the representation

$$\rho : \Gamma = \Gamma_1 \times \ldots \times \Gamma_k \times \Gamma' \to \mathrm{Bl}_{2,n}^+ \cong \mathrm{GL}^+(2,\mathbb{R}) \times \ldots \times \mathrm{GL}^+(2,\mathbb{R})$$

maps surface group factor Γ_i irreducibly to the i–th factor of $\mathrm{Bl}_{2,n}^+$ and scalarly to the others and, the rigid factor Γ' is mapped scalarly to each of the k first factors of $\mathrm{Bl}_{2,n}^+$. Define $r_i : \Gamma_i \to \mathrm{GL}^+(2,\mathbb{R})$ as the restriction of ρ to Γ_i composed with the projection onto the i–th factor of $\mathrm{Bl}_{2,n}^+$ and $r' : \Gamma' \to \Pi_{i=k+1}^n \mathrm{GL}^+(2,\mathbb{R})$ as the restriction of ρ to Γ' composed with the projection onto the last $n - k$ factors of $\mathrm{Bl}_{2,n}^+$. Then ρ and

$$(r_1, \ldots, r_k, r') : \Gamma_1 \times \ldots \times \Gamma_k \times \Gamma' \to \mathrm{Bl}_{2,n}^+$$

have the same projection to $\Pi_{i=1}^n \mathrm{PSL}(2,\mathbb{R})$, hence by Lemma 2.2 the corresponding pullbacks of the Euler class coincide. Thus the first part of the theorem follows in this case from its validity in cases 1 and 2.

Let us now explain the second statement of the theorem, namely that if N is cofaithful, then all the integers satisfying the given rule are actually attained as Euler numbers of flat bundles. The assertion is true for surfaces by [Mi58]. It is further true for any rigid cofaithful manifold N', since the Euler number is then equal to either 0 or $\pm\chi(N')/2^{\dim(N')/2}$, where 0 is realized for example by the trivial bundle, and the latter, up to orientation, by the cofaithful map (Theorem 1.6). The general case follows by considering direct products of flat bundles over the factors of N with the appropriate Euler numbers. $\qquad\square$

10. The proof of Theorem 1.10

For a group G we denote by $S_n \ltimes G^n$ the semidirect product where S_n permutes the factors of $G^n = \prod_{i=1}^n G$. When G is a subgroup of $\mathrm{GL}(2,\mathbb{R})$, $S_n \ltimes G^n$ admits a natural embedding in $\mathrm{GL}(2n,\mathbb{R})$ where G^n embeds diagonally in $\mathrm{Bl}_{2,n}$. We will not distinguish between $S_n \ltimes G^n$ and this representation of it, and denote by $(S_n \ltimes G^n)^+$ its intersection with $\mathrm{GL}^+(2n,\mathbb{R})$. In particular $G_n^+ = (S_n \ltimes \mathrm{GL}^1(2,\mathbb{R}))^+$, the 2^n sheeted cover of $\mathrm{Isom}(\mathcal{H}^n)^+$, is imbedded in this form.

Recall that s denotes the quotient map

$$s : \mathrm{Isom}(\mathcal{H}^n) \cong S_n \ltimes \mathrm{PGL}(2,\mathbb{R})^n \to S_n.$$

For a subgroup $\Gamma \le \mathrm{Isom}(\mathcal{H}^n)$ and an abelian group A we denote by $H^1_{\mathrm{Sym}}(\Gamma, A^n)$ the first cohomology group of Γ in A^n with respect to the action of Γ on A^n that permutes the factors according to the map s.

Lemma 10.1. *Suppose that $\Gamma \le \mathrm{Isom}(\mathcal{H}^n)$ is a rigid lattice. Then:*
    (1) $H^1_{Sym}(\Gamma, \mathbb{R}^n) = \langle 0 \rangle.$

(2) *The inclusion of $\langle \pm 1 \rangle$ in $\mathbb{R}^*$ induces an isomorphism $H^1_{Sym}(\Gamma, (\langle \pm 1 \rangle)^n) \cong$*
    $H^1_{Sym}(\Gamma, (\mathbb{R}^*)^n)$, and this cohomology group is finite.

Proof. Let Γ_0 be the intersection of Γ with the identity connected component of $\mathrm{Isom}(\mathcal{H}^n)$. Then Γ_0 is a rigid lattice in $\mathrm{PSL}(2,\mathbb{R})^n$, hence has finite abelianization and in particular, no nontrivial homomorphisms to $\mathbb{R}$, as follows for instance by Margulis' normal subgroups theorem. Moreover, as $s|_{\Gamma_0}$ is trivial, any element of $H^1_{\mathrm{Sym}}(\Gamma, \mathbb{R}^n)$ restricts to a homomorphism on Γ_0, and is hence identically 0 on Γ_0. It follows from the cocycle equation that any element $\alpha \in H^1_{\mathrm{Sym}}(\Gamma, \mathbb{R}^n)$ is constant on cosets of Γ_0:

$$\alpha(\gamma_0 \gamma) = \alpha(\gamma_0) + s(\gamma_0) \cdot \alpha(\gamma) = \alpha(\gamma), \ \forall \gamma_0 \in \Gamma, \gamma \in \Gamma.$$

Thus $H^1_{\mathrm{Sym}}(\Gamma, \mathbb{R}^n) \cong H^1_{\mathrm{Sym}}(\Gamma/\Gamma_0, \mathbb{R}^n)$, which is trivial since Γ/Γ_0 is finite and hence has property (T). This proves (1).

Since $\mathbb{R}^* \cong \mathbb{R} \oplus \mathbb{Z}/(2)$ we have

$$H^1_{\mathrm{Sym}}(\Gamma, (\mathbb{R}^*)^n) \cong H^1_{\mathrm{Sym}}(\Gamma, \mathbb{R}^n) \oplus H^1_{\mathrm{Sym}}(\Gamma, (\mathbb{Z}/(2))^n),$$

and the first statement of (2) follows from (1).

Moreover, since Γ_0 is a lattice in $\prod_{i=1}^n \mathrm{PSL}(2,\mathbb{R})$, it is finitely generated, and hence $\mathrm{Hom}(\Gamma_0, (\mathbb{Z}/(2))^n)$ is finite. By the cocycle equation, an element $\alpha \in H^1_{\mathrm{Sym}}(\Gamma, (\mathbb{Z}/(2))^n)$ is determined by its restriction to Γ_0 and by its values on a finite set of cosets representatives for Γ_0 in Γ. Thus $H^1_{\mathrm{Sym}}(\Gamma, (\mathbb{Z}/2)^n)$ is finite. $\qquad \square$

We now give the proof of Theorem 1.10: Let M be a closed locally $\mathcal{H}^n$ rigid manifold with fundamental group Γ.

Nonzero Euler number implies cofaithfulness: Suppose that M admits a $\dim(M)$–dimensional flat vector bundle with nonzero Euler number. Up to reversing the orientation we may assume that the Euler number is positive, hence by Corollary 1.9, equals $|\frac{1}{2^n} \chi(M)|$. Let $\rho : \Gamma \to \mathrm{GL}^+(2n, \mathbb{R})$ be the linear representation inducing this structure. By Proposition 7.2, Γ admits a normal finite index subgroup Δ such that $\rho(\Delta)$ is contained in a semisimple Lie group S with at most n simple factors, for which every noncompact simple factor is locally isomorphic to $\mathrm{PSL}(2,\mathbb{R})$. If S admits less than n factors or a compact factor, Lemma 4.4 combined with the fact that S_{nc} (the product of noncompact factors of S) is locally isomorphic in that case to the isometry group of a symmetric space of dimension strictly smaller than $2n$, yields that $\rho^*(\varepsilon_{2n}) = 0$, contrary to our assumption. Therefore, S is locally isomorphic to $\mathrm{PSL}(2,\mathbb{R})^n$. By Lemma 6.5, $t(S) = 0$, hence we may assume that $(b_1), (b_2), (b_3)$ of Proposition 7.2 hold for Δ and ρ, and in particular, up to replacing ρ by some conjugate representation, that $S = \mathrm{SBl}_{2,n}$ and $\rho(\Delta)$ is a uniform lattice there. In particular, since $\rho(\Delta)$ is normal in $\rho(\Gamma)$ and, by $(b3)$ and Borel's density theorem, Zariski dense in S, it follows that $\rho(\Gamma)$ is

contained in the normalizer of $S = \text{SBl}_{2,n}$, namely, in $(S_n \ltimes \text{GL}(2, \mathbb{R})^n)^+$. By replacing Δ further by some characteristic finite index subgroup if necessary, we may assume additionally that Δ itself lies in $\text{PSL}(2, \mathbb{R})^n$, the identity connected component of $\text{Isom}^+(\mathcal{H}^n) \cong (S_n \ltimes \text{PGL}(2, \mathbb{R})^n)^+$.

Let $f : \Delta \to \prod_{i=1}^n \text{PSL}_2(\mathbb{R})$ be the composition of $\rho|_\Delta$ followed by $\text{Ad} :$ $\text{SBl}_{2,n} \to \text{Ad}(\text{SBl}_{2,n}) \cong \prod_{i=1}^n \text{PSL}_2(\mathbb{R})$. By Mostow's rigidity theorem f extends to an isomorphism $\tilde{f} : \prod_{i=1}^n \text{PSL}_2(\mathbb{R}) \to \prod_{i=1}^n \text{PSL}_2(\mathbb{R})$ which, by reordering the factors of the target group, preserves the order of the factors. For every $\gamma \in \Gamma$ and $\delta \in \Delta$ we have

$$f(\gamma \delta \gamma^{-1}) = \rho(\gamma) f(\delta) \rho(\gamma)^{-1}$$

where the term on the right hand side is understood as $\text{Ad}(\rho(\gamma) \rho(\delta) \rho(\gamma)^{-1})$. Hence by Borel's density theorem we have

(6) $$\tilde{f}(\gamma g \gamma^{-1}) = \rho(\gamma) \tilde{f}(g) \rho(\gamma)^{-1}$$

for any $g \in \text{PSL}(2, \mathbb{R})^n$. In particular, $s(\rho(\gamma)) = s(\gamma)$, i.e. $\rho(\gamma)$ and γ induce the same permutation on the factors of $\text{SL}(2, \mathbb{R})^n$ and $\text{PSL}(2, \mathbb{R})^n$ respectively. Thus we can write $\rho(\gamma) \in (S_n \ltimes \text{GL}(2, \mathbb{R})^n)^+$ as

$$(s(\gamma), \rho_1(\gamma), \ldots, \rho_n(\gamma)),$$

where $\rho_i(\gamma)$ is the component of $\rho(\gamma)$ in the i'th factor of $\text{GL}(2, \mathbb{R})^n$. One can easily verify that the map $\alpha : \Gamma \to \mathbb{R}^n$ given by

$$\alpha(\gamma) := (\log \det(\rho_1(\gamma)^2), \ldots, \log \det(\rho_n(\gamma)^2))$$

is a cocycle, and hence by Lemma 10.1 cohomologous to the trivial map. Thus there exists a diagonal matrix of the form $\Lambda = \text{diag}(\lambda_1, \lambda_1, \ldots, \lambda_n, \lambda_n)$ such that the image of ρ conjugated by Λ is contained in $G_n^+ = (S_n \ltimes \text{GL}^1(2, \mathbb{R})^n)^+$.

Finally let $\psi : (S_n \ltimes \text{GL}^1(2, \mathbb{R})^n)^+ \to (S_n \ltimes \text{PGL}(2, \mathbb{R})^n)^+$ be the canonical projection determined by modding out the center of $\text{GL}^1(2, \mathbb{R})^n$. Since $(S_n \ltimes \text{PGL}(2, \mathbb{R})^n)^+$ is isomorphic to $\text{Aut}^+(\text{PSL}(2, \mathbb{R})^n)$, where the faithful action of the first on $\text{PSL}(2, \mathbb{R})^n$ is by conjugation on its identity connected component, we deduce from Equation 6 that $\psi \circ \rho : \Gamma \to (S_n \ltimes \text{PGL}(2, \mathbb{R})^n)^+$ is injective. Hence

$$\rho \circ (\psi \circ \rho)^{-1} : \psi(\rho(\Gamma)) \to (S_n \ltimes \text{GL}^1(2, \mathbb{R})^n)^+$$

induces a faithful lift, showing that M is cofaithful.

Characterisation of flat structures with positive Euler number as elements in $H^1_{\mathbf{Sym}}(\Gamma, \mathbb{Z}/(2)^n)$: Suppose now that M is cofaithful, and let $\rho_1 : \Gamma \to G_n^+ \le \text{GL}^+(2n, \mathbb{R})$ be the representation induced by the cofaithful lift of Γ from $S_n \ltimes (\text{PGL}(2, \mathbb{R})^n)^+$ to $(S_n \ltimes (\text{GL}_2^1)^n)^+ \cong G_n^+$ where the latter

is realized as the subgroup $\mathrm{GL}(2n, \mathbb{R})^+$ described at the beginning of this section. Let $\rho : \Gamma \to \mathrm{GL}^+(2n, \mathbb{R})$ be another representation, and suppose that $\rho^*(\varepsilon_{2n}) \neq 0$. As above, we derive from Proposition 7.2 that there is a finite index normal subgroup $\Delta \leq \Gamma$ such that the restriction of ρ to Δ is faithful and $\rho(\Delta)$ is conjugated to some cocompact lattice in $\mathrm{SBL}_{2,n}(\mathbb{R})$. Thus, it follows from Mostow's rigidity theorem that $\rho^g|_\Delta = \rho_1|_\Delta$ for an appropriate element $g \in \mathrm{GL}(2n, \mathbb{R})$ (by ρ^g we mean ρ composed with the conjugation by g). Now let γ be an arbitrary element of Γ. Since Δ is normal in Γ, we have

$$
\begin{aligned}
\rho_1(\gamma^{-1}) \rho_1(\delta) \rho_1(\gamma) &= \rho_1(\gamma^{-1} \delta \gamma) = \rho^g(\gamma^{-1} \delta \gamma) = \rho^g(\gamma^{-1}) \rho^g(\delta) \rho^g(\gamma) \\
&= \rho^g(\gamma^{-1}) \rho_1(\delta) \rho^g(\gamma),
\end{aligned}
$$

for every $\delta \in \Delta$. Thus $\rho^g(\gamma) \rho_1(\gamma)^{-1}$ lies in the centralizer of $\rho_1(\Delta)$, and by Borel's density theorem, in the centralizer of $\mathrm{SBL}_{2,n}(\mathbb{R})$. Hence $\rho^g(\gamma)$ is equal to $\rho_1(\gamma)$ multiplied from the left by a matrix of the form

$$
\mathrm{diag}(\chi_1(\gamma), \chi_1(\gamma), \ldots, \chi_n(\gamma), \chi_n(\gamma))
$$

where the $\chi_1, \ldots, \chi_n$ are functions on Γ taking values in $\mathbb{R}^*$. It also follows that the permutation representation determined by conjugating by $\rho^g(\gamma)$ on the factors of $\mathrm{SBl}_{2,n}$ is the same as the one coming from conjugation by $\rho_1(\gamma)$, namely that $s(\rho(\gamma)) = s(\gamma)$. A simple calculation shows that the map $\alpha : \Gamma \to (\mathbb{R}^*)^n$ given by $\alpha(\gamma) = (\chi_1(\gamma), \ldots, \chi_n(\gamma))$ is a cocycle. Hence by Lemma 10.1, after conjugating ρ^g further by some diagonal matrix of the form $\mathrm{diag}(\lambda_1, \lambda_1, \ldots, \lambda_n, \lambda_n)$ if necessary, we may assume that α takes its values in $\langle \pm 1 \rangle^n$. It follows that the conjugacy class of ρ in $\mathrm{GL}(2n, \mathbb{R})^+$ is completely determined by the cocycle α and the sign of the determinant of g, where $\det(g) < 0$ corresponds to negative Euler number.

Let Γ_0 be as in the proof of Lemma 10.1. Then the restriction of every $[\alpha] \in H^1_s(\Gamma, \langle \pm 1 \rangle^n)$ to Γ_0 is a homomorphism. Set

$$
\Gamma^0 = \Gamma_0 \cap \{\ker[\alpha] : [\alpha] \in H^1_s(\Gamma, \langle \pm 1 \rangle^n).
$$

Then Γ^0 is a characteristic subgroup of finite index in Γ and the restrictions to Γ^0 of all the representations of Γ which induce flat vector bundles with nonzero (resp. positive) Euler number are conjugate in $\mathrm{GL}(2n, \mathbb{R})$ (resp. in $\mathrm{GL}^+(2n, \mathbb{R})$).

For the other direction, any cocycle $\alpha : \Gamma \to \langle \pm 1 \rangle^n$ produces such a representation by setting

$$
\rho_\alpha(\gamma) := \mathrm{diag}\big(\alpha(\gamma)_1, \alpha(\gamma)_1, \ldots, \alpha(\gamma)_n, \alpha(\gamma)_n\big) \rho_1(\gamma)
$$

which gives rise to a flat vector bundle with positive Euler number. Moreover if α and β are two such cocycles and the corresponding representations ρ_α, ρ_β are conjugate in $\mathrm{GL}^+(2n, \mathbb{R})$, the conjugating element must centralize

$\rho_\alpha(\Gamma^0)$ and hence, by the Borel density theorem, belongs to the centralizer of $\mathrm{SBl}_{2,n}(\mathbb{R})$, i.e. it is of the form $\mathrm{diag}(\lambda_1, \lambda_1, \ldots, \lambda_n, \lambda_n)$. This shows that α and β are cohomologus as cocycles with coefficients in $(\mathbb{R}^*)^n$, namely, they represent the same element in $H^1_{\mathrm{Sym}}(\Gamma, (\mathbb{R}^*)^n)$. By Lemma 10.1 (2), they represent the same element in $H^1(\Gamma, \langle \pm 1 \rangle)$. This completes the proof of the theorem. $\qquad\square$

Remark 10.2. The finiteness of the number of flat vector bundles of any given dimension over any rigid manifold is true in general. Indeed, one can deduce from the proof of Margulis superrigidity theorem, the fact that any rigid lattice has finite abelianization, and Jordan's theorem, that $\pi_1(M)$ admits only finitely many nonequivalent representations in any fixed dimension.

References

[BePe92]   R. Benedetti, C. Petronio. *Lectures on hyperbolic geometry.* Universitext. Springer-Verlag, Berlin, 1992.

[BCG07]   G. Besson, G. Courtois, S. Gallot, *Inégalités de Milnor-Wood géométriques.* (French) [Geometric Milnor-Wood inequalities] Comment. Math. Helv. 82 (2007), no. 4, 753–803.

[Bor63]   A. Borel, *Compact Clifford-Klein forms of symmetric spaces.* Topology 2, 111-122 (1963).

[BoWa00]   A. Borel, N. Wallach. *Continuous cohomology discrete subgroups, and representations of reductive groups.* Second edition Mathematical Surveys and Monographs, 67. American Mathematical Society, Providence, RI, 2000.

[Bott73]   R. Bott. *On the Chern-Weil homomorphism and the continuous cohomology of Lie-groups.* Advances in Math. 11 (1973), 289-303.

[Bu06]   M. Bucher-Karlsson. *The proportionality constant for the simplicial volume of locally symmetric spaces.* Colloq. Math. 111 No 2 (2008), 183–198.

[Bu07]   M. Bucher-Karlsson, *Finiteness properties of characteristic classes of flat bundles.* L'Ens. Math. 53 (2007), 33-66.

[Bu08]   M. Bucher-Karlsson. *The simplicial volume of closed manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$*, J. of Topology 1 (2008) 584–602.

[BuGe08]   M. Bucher, T. Gelander, *Milnor–Wood inequalities for manifolds locally isometric to a product of hyperbolic planes.* C. R. Acad. Sci. Paris, Ser. I 346 (2008) 661–666.

[BuIo07]   M. Burger, A. Iozzi, *Bounded differential forms, generalized Milnor–Wood inequality and applications to rigidity theory* Geom. Dedicata 125, No 1 (2007) 1–23.

[BILW05]   M. Burger, A. Iozzi, F. Labourie, A. Wienhard, *Maximal representations of surface groups: symplectic Anosov structures*, Quarterly Journal of Pure and Appl. Math., Special Volume in memory of Armand Borel, 1 (2005), no.3, 543–590.

[BIW08]   M. Burger, A. Iozzi, A. Wienhard, *Surface group representations with maximal Toledo invariant*, (2008), to appear in Annals of Math.

[ClOr03]   J.-L. Clerc, B. Ørsted. *The Gromov norm of the Kaehler class and the Maslov index.* Asian J. Math. 7 (2003), no. 2, 269–295.

[Cl37]   A. H. Clifford, *Representations induced in an invariant subgroup.* Ann. of Math. (2) 38 (1937), no. 3, 533–550.

[DoTo87]   A. Domic, D. Toledo. *The Gromov norm of the Kaehler class of symmetric domains.* Math. Ann. 276 (1987), no. 3, 425–432.

[Du76]   J.L. Dupont, *Simplicial de Rham cohomology and characteristic classes of flat bundles.* Topology 15 (1976), no. 3, 233–245.

[Gr82]   M. Gromov. *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. No. 56, (1982), 5–99 (1983).

[Gui80]   A. Guichardet. *Cohomologie des groupe topologiques et des algèbres de Lie.* Textes Mathématiques, 2. CEDIC, Paris, 1980.

[Hi58]   F. Hirzebruch. *Automorphe Formen und der Satz von Riemann-Roch*, in Symposium Internacional de topologia alebraics, Universidad Nacional Autonoma de Mexico y UNESCO 1958.

[Iv85]   N.V. Ivanov, *Foundations of the theory of bounded cohomology*, Zap. Nauchn. Sem. Leningr. Otd. Mat. Inst. 143, 1985, 69–109.

[IvTu82]   N.V. Ivanov, V.G. Turaev, *A canonical cocycle for the Euler class of a flat vector bundle.* Soviet Math. Dokl. Vol. 26 (1982), No. 1, 78-81.

[KaTo68]   F. Kamber, P. Tondeur, *Flat Manifolds.* Lecture Notes in Mathematics, Vol. 67.

[Kn96]   A. W. Knapp, Lie groups beyond an introduction. Progress in Mathematics, 140. Birkhuser Boston, Inc., Boston, MA, 1996.

[KoNo69]   S. Kobayashi, K. Nomizu, *Foundations of differential geometry*, Vol. II. Reprint of the 1969 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley and Sons, Inc., New York, 1996.

[KoSu75]   B. Kostant, D. Sullivan, *The Euler characteristic of a compact affine space form is zero*, Bull. Amer. Math. Soc. 81, (1974) 937-938.

[Lu87]   A. Lubotzky, *On finite index subgroups of linear groups* Bull. London Math. Soc. 19 (1987), no. 4, 325–328.

[McL63]   S. MacLane, *Homology*, Grundlehren Math. Wiss. 114, Springer-Verlag, Berlin-Göttingen-Heidelberg (1963).

[Ma92]   G.A. Margulis, Discrete subgroups of semisimple Lie groups. Springer-Verlag, Berlin, 1991.

[Mi58]   J. Milnor, *On the existence of a connection with curvature zero.* Comment. Math. Helv. 32, 1958, 215-223.

[MiSt79]   J. Milnor, J. Stasheff, *Characteristic classes* Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974.

[Mo01]   N. Monod. *Continuous bounded cohomology of locally compact groups.* Lecture Notes in Mathematics, 1758. Springer-Verlag, Berlin, 2001.

[Ok01]   B. Okun, *Nonzero degree tangential maps between dual symmetric spaces*, Algebraic and Geometric Topology 1 (2001), 709–128.

[PR94]   V. Platonov, A. Rapinchuk, *Algebraic Groups and Number Theory*, Academic Press, 1994.

[R72]   M.S. Raghunathan, *Discrete Subgroups of Lie Groups*, Springer, New York, 1972.

[Sm77a]   J. Smillie, *Affinely flat manifolds*, Ph. D. Thesis, Chicaco, Illionois, 1977, (unpublished).

[Sm77b]   J. Smillie, *Flat manifolds with non-zero Euler characterisitcs*, Comment. Math. Helv. 52 (1977) 453–355.

[Su76]   D. Sullivan, *A generalization of Milnor's inequality concerning affine foliations and affine manifolds*, Comment. Math. Helv. 51 (1976) 183–189,

[Th78]   W. Thurston. *Geometry and topology of 3-manifolds*. Lecture Notes, Princeton (1978).

[Ve55]   W.T. Van Est. *On the algebraic cohomology concepts in Lie groups II*, Nederl. Akad. Wetensch. Proc. Ser. A. 58 (1955), 286-294.

[Wo75]   J. Wood. *Bundles with totally disconnected structure group.* Comment. Math. Helv. 46 (1971), 257–273.