Institutionen för matematik
KTH
Chaotic Dynamical Systems, Fall 2002
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Homework assignment 1

Exercise 1–8 are due October 8, 2002

Exercise 9–13 are due October 15, 2002

1. (Devaney p. 39, 4) Let $T_2(x)$ be the tent map
$$T_2(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}, \\ 2 - 2x, & \frac{1}{2} \le x \le 1. \end{cases}$$
Prove that the set of all period points of $T_2(x)$ are dense in $[0, 1]$.
2. (Devaney p. 39, 8) Show that, at the n^{th} stage of the construction of the Cantor Middle-Thirds set, the sum of the lengths of the remaining intervals is
$$1 - \frac{1}{3}\left(\sum_{i=0}^{n-1}\left(\frac{2}{3}\right)^i\right).$$
3. (Devaney p. 43, 5) Let Σ_N consist of all sequences of natural numbers $1, 2, \ldots, N$. There is a natural shift on Σ_N
   a. How many periodic points does σ have in Σ_N?
   b. Show that σ has a dense orbit in Σ_N.
4. (Devaney p. 43, 6) Let $\mathbf{s} \in \Sigma_2$. Define the stable set of $\mathbf{s}$, $W^s(\mathbf{s})$, as the set of sequences $\mathbf{t}$ such that $d[\sigma^i(\mathbf{s}), \sigma^i(\mathbf{t})] \to 0$ as $i \to \infty$. Identify all of the sequences in $W^s(\mathbf{s})$.
5. (Devaney p. 47, 1) Let $Q_c(x) = x^2 + c$. Prove that if $c < \frac{1}{4}$, there is a unique $\mu > 1$ such that Q_c is topologically conjugate to $F_\mu(x) = \mu x(1 - x)$ via a map of the form $h(x) = \alpha x + \beta$.
6. (Devaney p. 48, 3) A point p is *recurrent* for f if, for any open interval J about p, there exists $n > 0$ such that $f^n(p) \in J$. Clearly all periodic points are recurrent.
   a. Give an example of a non-periodic recurrent point for F_μ when $\mu > 2 + \sqrt{5}$.
   b. Give an example of a non-wandering point or F_μ, which is not recurrent.

7. (Devaney p. 52, 4) Prove that $T(x) = \tan(x)$ is chaotic on the entire line, despite the fact that there are a dense set of points at which an iterate of $T(x)$ fails to be defined.
8. (Devaney p. 52, 5) Prove that the baker-map

$$B(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

is chaotic on $[0, 1]$.
9. (Devaney p. 59, 2) Let $T_{-1}(x) = x^3 + x$. Prove that T_{-1} is not structurally stable.
10. (Devaney p. 59, 11) We may define a notion of linear structural stability for linear maps by replacing the notion o topological conjugacy by that of linear conjugacy. Two linear maps $T_1, T_2 : \mathbb{R} \mapsto \mathbb{R}$ are linearly conjugate if there is a linear map L such that $T_1 \circ L = L \circ T_2$. $T_1(x) = ax$ is linearly stable if there is a neighborhood N about a such that such that if $b \in N$, then $T_2(x) = bx$ is linearly conjugate to T_1. Find all linearly stable maps and identify all element of a given conjugacy class.
11. (Devaney p. 59, 12) (Hartman's theorem) Let p be a hyperbolic fixed point for f with $f'(p) = \lambda$ and $0 < |\lambda| < 1$. Prove that f is locallly topologically conjugate to its derivative map $x \mapsto \lambda x$ as described in Devaney, Theorem 9.8.
12. (Devaney p. 68, 5) Construct a piecewise linear map with period $2n + 1$.
13. (Devaney p. 68, 8) Construct a map that has periodic points of period 2^j for $j < l$ but no points of period 2^l.