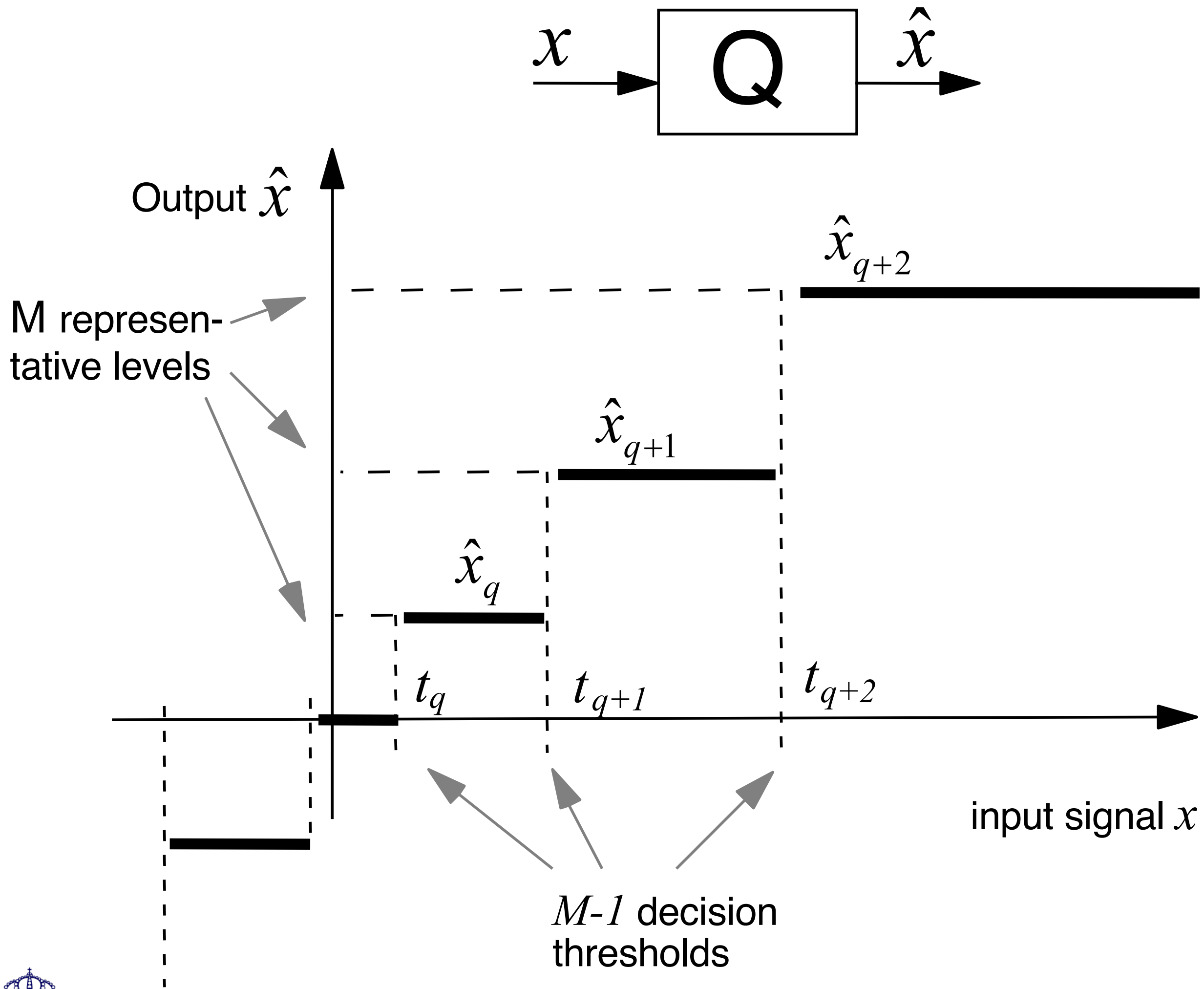
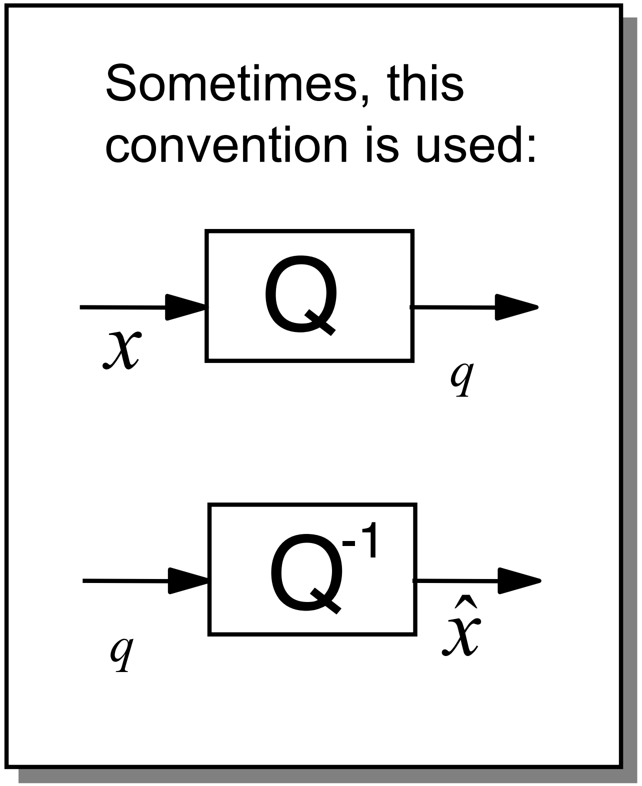
Quantization

- Scalar quantizer
- Lloyd-Max scalar quantizer
  - Quantizer design algorithm
  - High rate approximation
- Entropy-constrained scalar quantizer
  - Quantizer design algorithm
  - High rate approximation
- Vector quantization
- Lattice vector quantization

Quantization

Input-output characteristic of a scalar quantizer





Lloyd-Max Scalar Quantizer

- Problem: For a signal x with given PDF $f_X(x)$ find a quantizer with M representative levels such that

$$d = MSE = E\left[\left(X - \hat{X}\right)^2\right] \rightarrow \min .$$

- Solution: Lloyd-Max quantizer

  [Lloyd, 1957][Max, 1960]

  - *M-1* decision thresholds exactly half-way between representative levels.
  - *M* representative levels in the centroid of the PDF between two successive decision thresholds.
  - Necessary condition

$$t_q = \frac{1}{2}\left(\hat{x}_{q-1} + \hat{x}_q\right) \quad q = 1, 2, \ldots, M - 1$$

$$\hat{x}_q = \frac{\int_{t_q}^{t_{q+1}} x f_X(x)\,dx}{\int_{t_q}^{t_{q+1}} f_X(x)\,dx} \quad q = 0, 1, \ldots, M - 1$$

Iterative Lloyd-Max Quantizer Design

1. Guess initial set of representative levels $\hat{x}_q \quad q = 0, 1, 2, \ldots, M\text{-}1$
2. Calculate decision thresholds

$$t_q = \frac{1}{2}\left(\hat{x}_{q-1} + \hat{x}_q\right) \quad q = 1, 2, \ldots, M\text{-}1$$

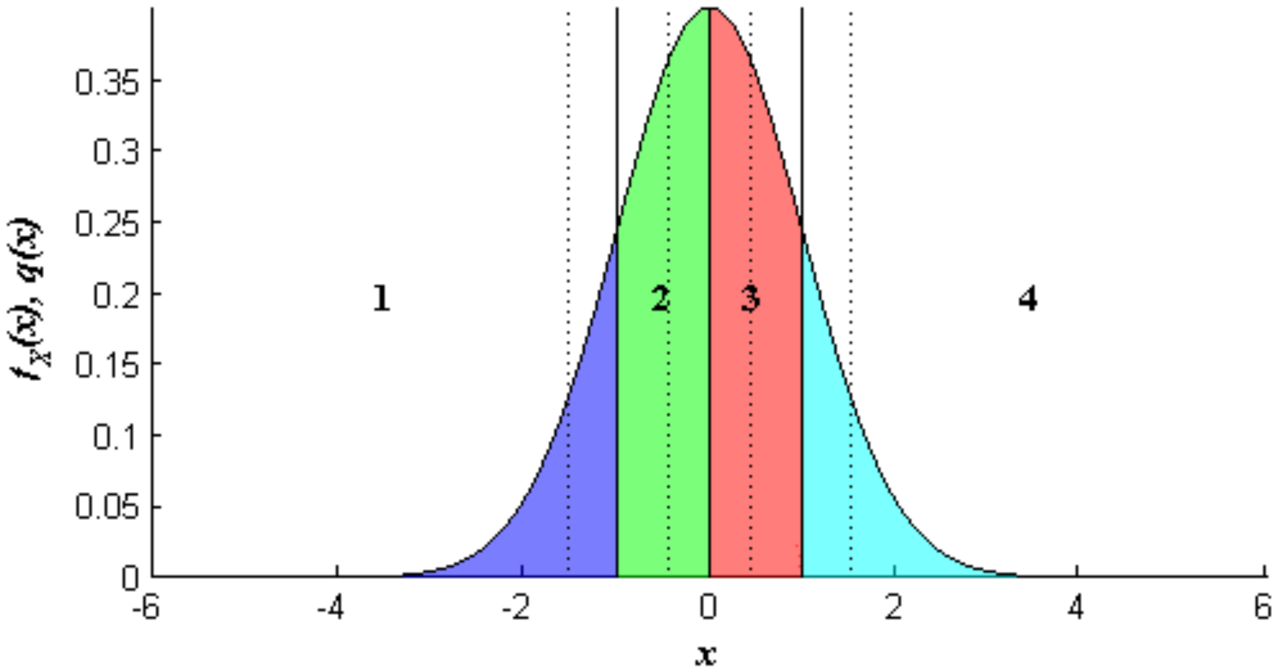
3. Calculate new representative levels

$$\hat{x}_q = \frac{\int_{t_q}^{t_{q+1}} x \cdot f_X(x)dx}{\int_{t_q}^{t_{q+1}} f_X(x)dx} \quad q = 0, 1, \ldots, M\text{-}1$$

4. Repeat **2.** and **3.** until no further distortion reduction

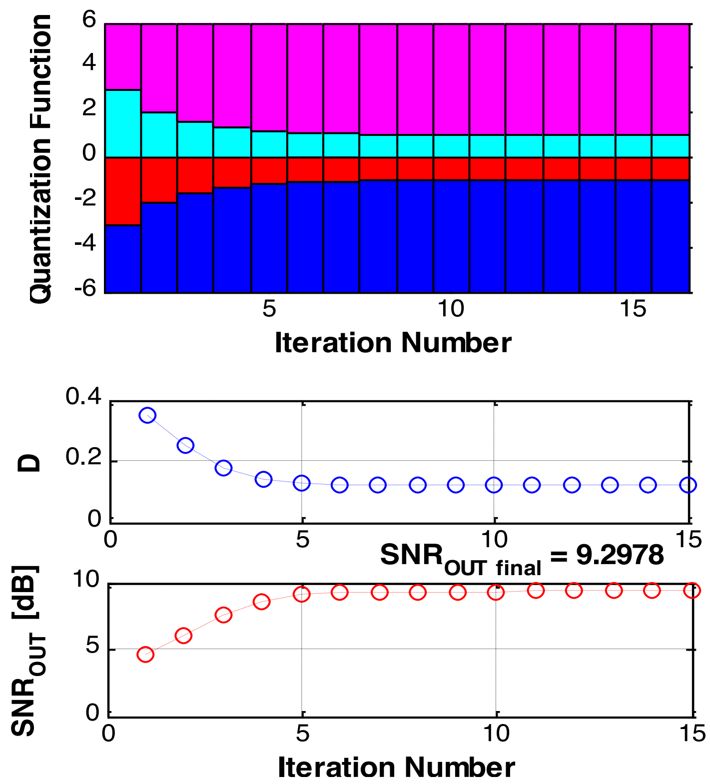
Example: Lloyd-Max Design I

- X zero-mean, unit-variance Gaussian r.v.
- Design scalar quantizer with 4 quantization indices with minimum expected distortion D^*
- Optimum quantizer, obtained with the Lloyd algorithm
  - Decision thresholds -0.98, 0, 0.98
  - Representative levels –1.51, -0.45, 0.45, 1.51
  - $D^* = 0.12$
  - $D^* = 9.30$ dB

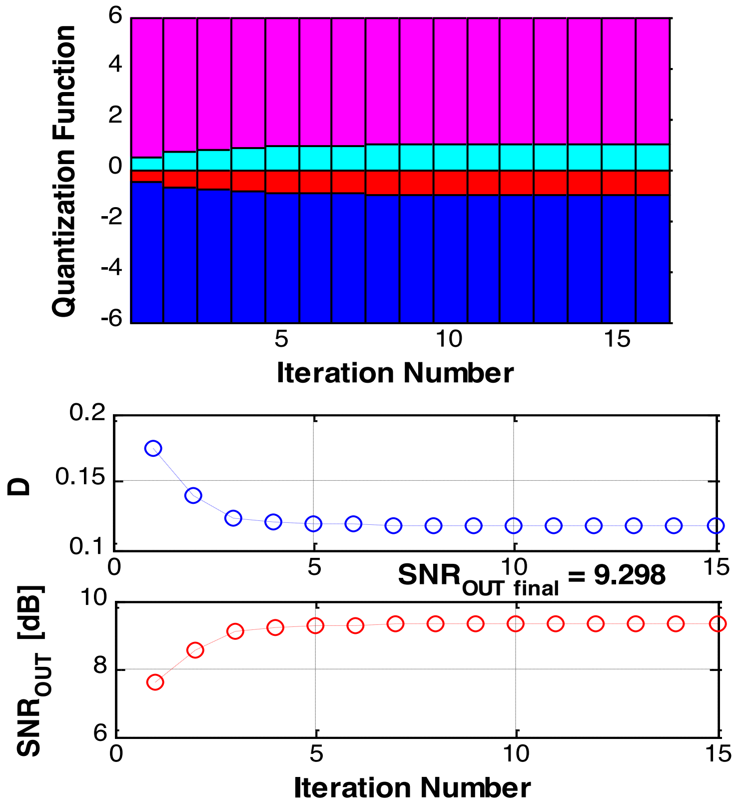


Example: Lloyd-Max Design II

- Convergence
  - Initial quantizer A:
    decision thresholds -3, 0, 3



  - Initial quantizer B:
    decision thresholds -0.5, 0, 0.5



  - After 6 iterations, in both cases (D-D*)/D* < 1%

Lloyd Algorithm with Training Data

1. Guess initial set of representative levels $\hat{x}_q;\ q = 0,1,2,\ldots,M-1$
2. Assign each sample x_i in training set ***T*** to closest representative $\hat{x}_q$

$$B_q = \{x \in \boldsymbol{T} : Q(x) = q\} \qquad q = 0,1,2,\ldots,M-1$$

3. Calculate new representative levels

$$\hat{x}_q = \frac{1}{\|B_q\|} \sum_{x \in B_q} x \quad q = 0,1,\ldots,M-1$$

4. Repeat **2.** and **3.** until no further distortion reduction

Lloyd-Max Quantizer Properties

- Zero-mean quantization error

$$E\left[\left(X-\hat{X}\right)\right]=0$$

- Quantization error and reconstruction decorrelated

$$E\left[\left(X-\hat{X}\right)\hat{X}\right]=0$$

- Variance subtraction property

$$\sigma_{\hat{X}}^2=\sigma_X^2-E\left[\left(X-\hat{X}\right)^2\right]$$

High Rate Approximation I

- Let Δ_i be the step size of the scalar quantizer in cell i
- Local density of centroids c_i: $g(c_i) = \frac{1}{\Delta_i}$
- Centroid density function

$$\int_{\mathcal{R}} g(x)dx = M$$

M representative levels

- Expected distortion

$$D = \sum_i p_i \frac{\Delta_i^2}{12} \approx \frac{1}{12} \int_{\mathcal{R}} g^{-2}(x) f_X(x) dx$$

- Optimal constrained resolution scalar quantizer

$$\min_g D(g) \quad \text{s.t.} \quad \int_{\mathcal{R}} g(x)dx = M$$

High Rate Approximation II

- Approximate solution of the "Max quantization problem," assuming high rate and smooth PDF *[Panter, Dite, 1951]*

$$\frac{1}{g(x)} = \frac{1}{M} \frac{\int_{\mathcal{R}} \sqrt[3]{f_X(x)}dx}{\sqrt[3]{f_X(x)}}$$

Distance between two successive quantizer representative levels

Probability density function of *x*

- Approximation for the quantization error variance:

$$d = E\left[\left(X - \hat{X}\right)^2\right] \approx \frac{1}{12M^2}\left[\int_x \sqrt[3]{f_X(x)}dx\right]^3$$

High Rate Approximation III

- High-rate distortion-rate function for scalar Lloyd-Max quantizer:

$$d(R) \cong \varepsilon^2 \sigma_X^2 \, 2^{-2R} \qquad \longleftarrow \quad M = 2^R$$

$$\text{with} \quad \varepsilon^2 \sigma_X^2 = \frac{1}{12} \left[\int_x \sqrt[3]{f_X(x)} dx \right]^3$$

- Some example values for ε^2

uniform	1
Laplacian	$\frac{9}{2} = 4.5$
Gaussian	$\frac{\sqrt{3}\pi}{2} \cong 2.721$

High Rate Approximation IV

- Partial distortion theorem: Each interval makes an (approximately) equal contribution to overall mean-squared error.

$$\Pr\left\{t_i \le X < t_{i+1}\right\} E\left[\left(X-\hat{X}\right)^2 \middle| t_i \le X < t_{i+1}\right]$$
$$\cong \Pr\left\{t_j \le X < t_{j+1}\right\} E\left[\left(X-\hat{X}\right)^2 \middle| t_j \le X < t_{j+1}\right] \quad \text{for all } i, j$$

Entropy-Constrained Scalar Quantizer

- Lloyd-Max quantizer optimum for fixed-rate encoding. How can we do better for variable-length encoding of the quantizer index?
- Problem: For a signal x with given pdf $f_X(x)$ find a quantizer

$$d = MSE = E\left[\left(X - \hat{X}\right)^2\right] \rightarrow \min. \quad \text{s.t.} \quad R = H\left(\hat{X}\right) = -\sum_{q=0}^{M-1} p_q \log_2 p_q$$

- Solution: Lagrangian cost function

$$J = d + \lambda R = E\left[\left(X - \hat{X}\right)^2\right] + \lambda H\left(\hat{X}\right) \rightarrow \min.$$

Iterative Entropy-Constrained SQ Design

1. Guess initial set of representative levels $\hat{x}_q;\ q = 0,1,2,\ldots,M\text{-}1$ and corresponding probabilities p_q
2. Calculate *M-1* decision thresholds

$$t_q = \frac{\hat{x}_{q\text{-}1} + \hat{x}_q}{2} - \lambda \frac{\log_2 p_{q-1} - \log_2 p_q}{2\left(\hat{x}_{q\text{-}1} - \hat{x}_q\right)} \qquad q = 1,2,\ldots,M\text{-}1$$

3. Calculate *M* new representative levels and probabilities p_q

$$\hat{x}_q = \frac{\int_{t_q}^{t_{q+1}} x f_X(x)dx}{\int_{t_q}^{t_{q+1}} f_X(x)dx} \qquad q = 0,1,\ldots,M\text{-}1$$

4. Repeat **2.** & **3.** until no further reduction in Lagrangian cost

ECSQ Design with Training Data

1. Guess initial set of representative levels $\hat{x}_q;\ q = 0,1,2,\ldots,M\text{-}1$ and corresponding probabilities p_q
2. Assign each sample x_i in training set ***T*** to representative minimizing Lagrangian cost $J_{x_i}(q) = (x_i - \hat{x}_q)^2 - \lambda \log_2 p_q$

$$B_q = \{x \in \boldsymbol{T} : Q_\lambda(x) = q\} \qquad q = 0,1,2,\ldots,M\text{-}1$$

3. Calculate new representative levels and probabilities p_q

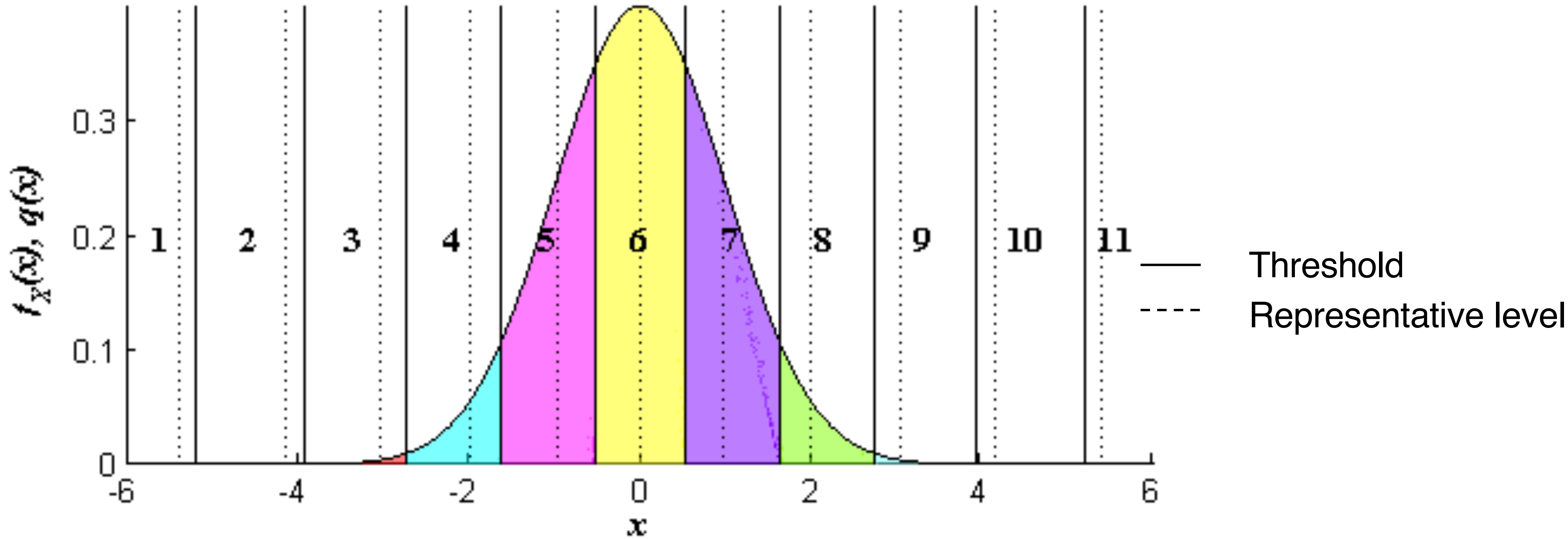
$$\hat{x}_q = \frac{1}{\|B_q\|} \sum_{x \in B_q} x \quad q = 0,1,\ldots,M\text{-}1$$

4. Repeat **2.** and **3.** until no further reduction in overall cost

$$J = \sum_{x_i} J_{x_i} = \sum_{x_i} (x_i - Q(x_i))^2 - \lambda \log_2 p_{q(x_i)}$$

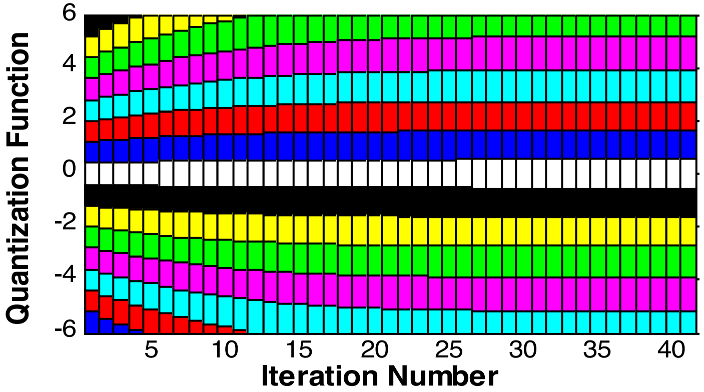
Example: ECSQ Design I

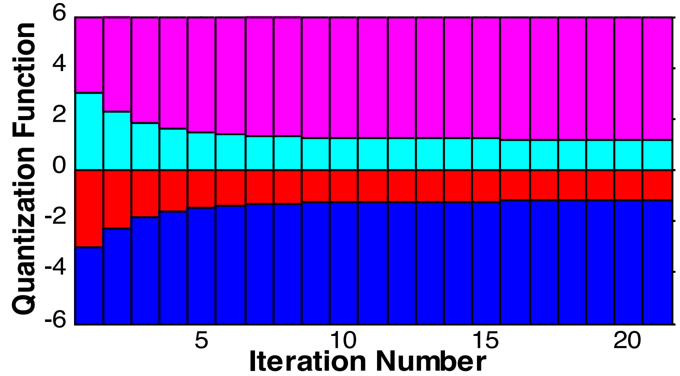
- X zero-mean, unit-variance Gaussian r.v.
- Design entropy-constrained scalar quantizer with rate $R \simeq 2$ bits, and minimum distortion $D*$
- Optimum quantizer, obtained with the entropy-constrained Lloyd algorithm
  - 11 intervals (in [-6,6]), almost uniform
  - $D*$=0.09 (10.53 dB), R=2.0035 bits (compare to fixed-length example)

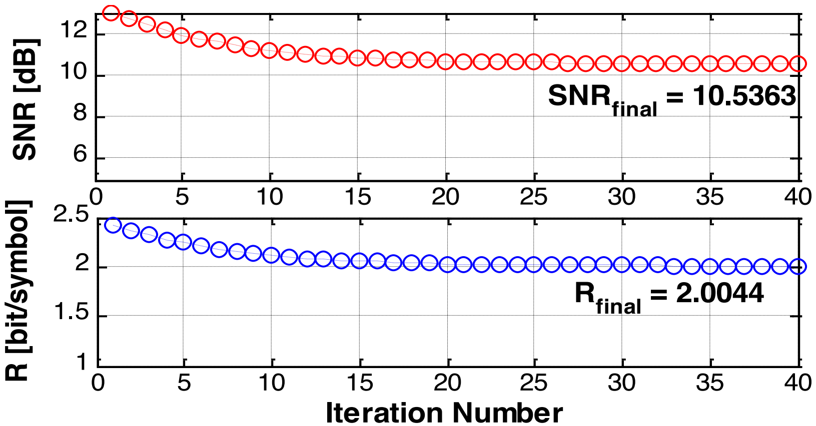


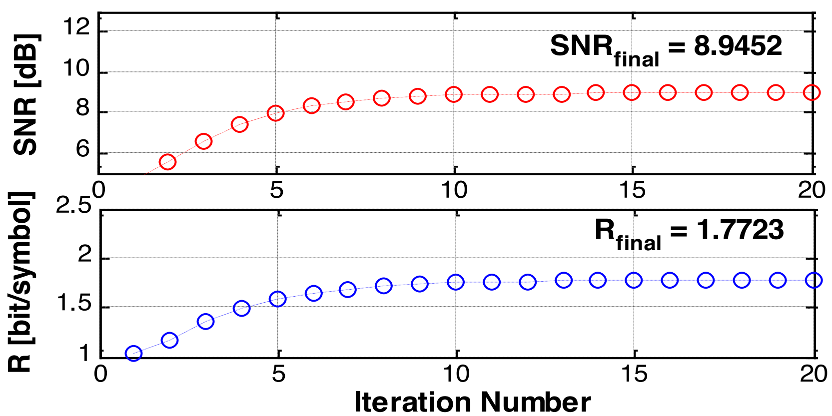
Example: ECSQ Design II

- Same Lagrangian multiplier is used in all experiments
  - Initial quantizer A, 15 intervals in [-6,6], with the same length
  - Initial quantizer B, 4 intervals in [-6,6], with the same length









High Rate Results for ECSQ

- For MSE distortion and high rates, uniform quantizers (followed by entropy coding) are optimum *[Gish, Pierce, 1968]*
- Entropy and distortion for smooth PDF and fine quantizer interval Δ

$$H\left(\hat{X}\right) \cong h\left(X\right) - \log_2 \Delta$$

$$d \cong \int_{-\Delta/2}^{\Delta/2} \varepsilon^2 d\varepsilon = \frac{\Delta^2}{12}$$

- Distortion rate function

$$d\left(R\right) \cong \frac{1}{12} 2^{2h\left(X\right)} 2^{-2R}$$

is factor $\frac{\pi e}{6}$ or 1.53 dB from Shannon Lower Bound

$$D\left(R\right) \geq \frac{1}{2\pi e} 2^{2h\left(X\right)} 2^{-2R}$$

Comparison: High Rate Performance of SQ

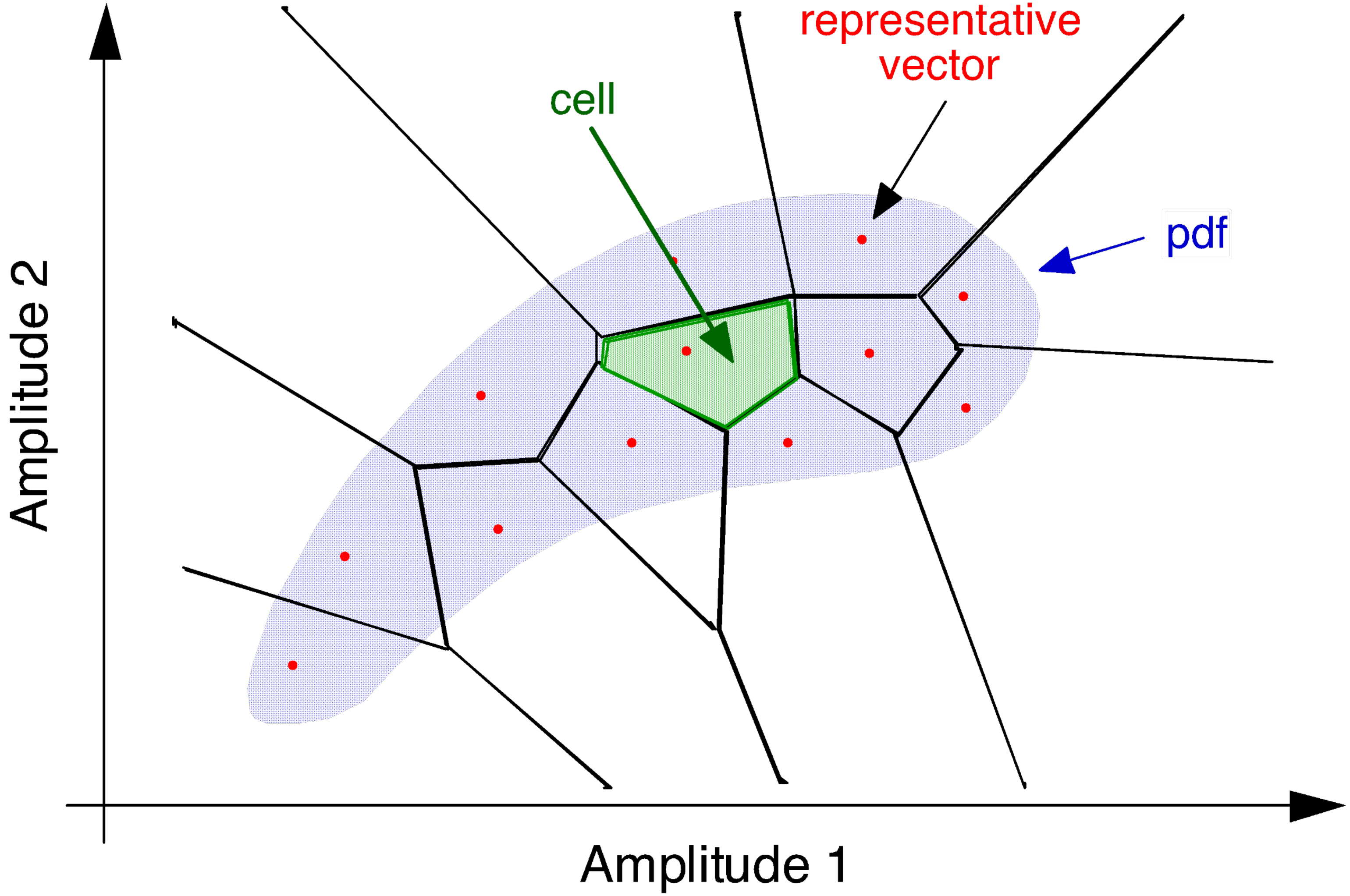
- High-rate distortion-rate function

$$d(R) \cong \varepsilon^2 \sigma_X^2 2^{-2R}$$

- Scaling factor ε^2

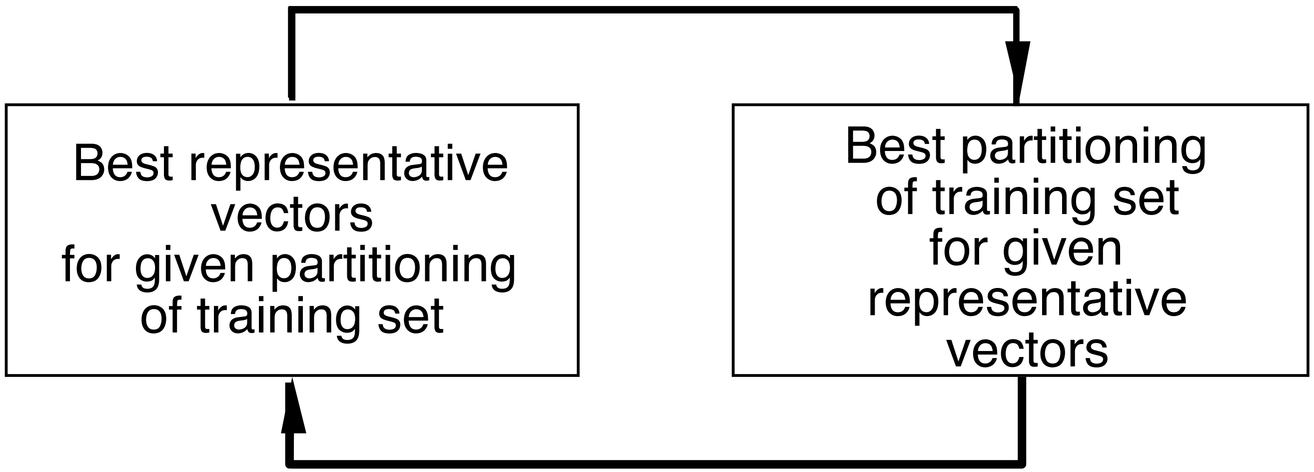
	Shannon LowBd	Lloyd-Max	Entropy-coded
Uniform	$\frac{6}{\pi e} \cong 0.703$	1	1
Laplacian	$\frac{e}{\pi} \cong 0.865$	$\frac{9}{2} = 4.5$	$\frac{e^2}{6} \cong 1.232$
Gaussian	1	$\frac{\sqrt{3}\pi}{2} \cong 2.721$	$\frac{\pi e}{6} \cong 1.423$

Vector Quantization



LBG Algorithm

- Lloyd algorithm generalized for VQ *[Linde, Buzo, Gray, 1980]*



- Assumption: fixed code word length
- Code book unstructured: full search

Design of Entropy-Constrained VQ

- Extended LBG algorithm for entropy-coded VQ
  [Chou, Lookabaugh, Gray, 1989]
- Lagrangian cost function: solve unconstrained problem rather than constrained problem

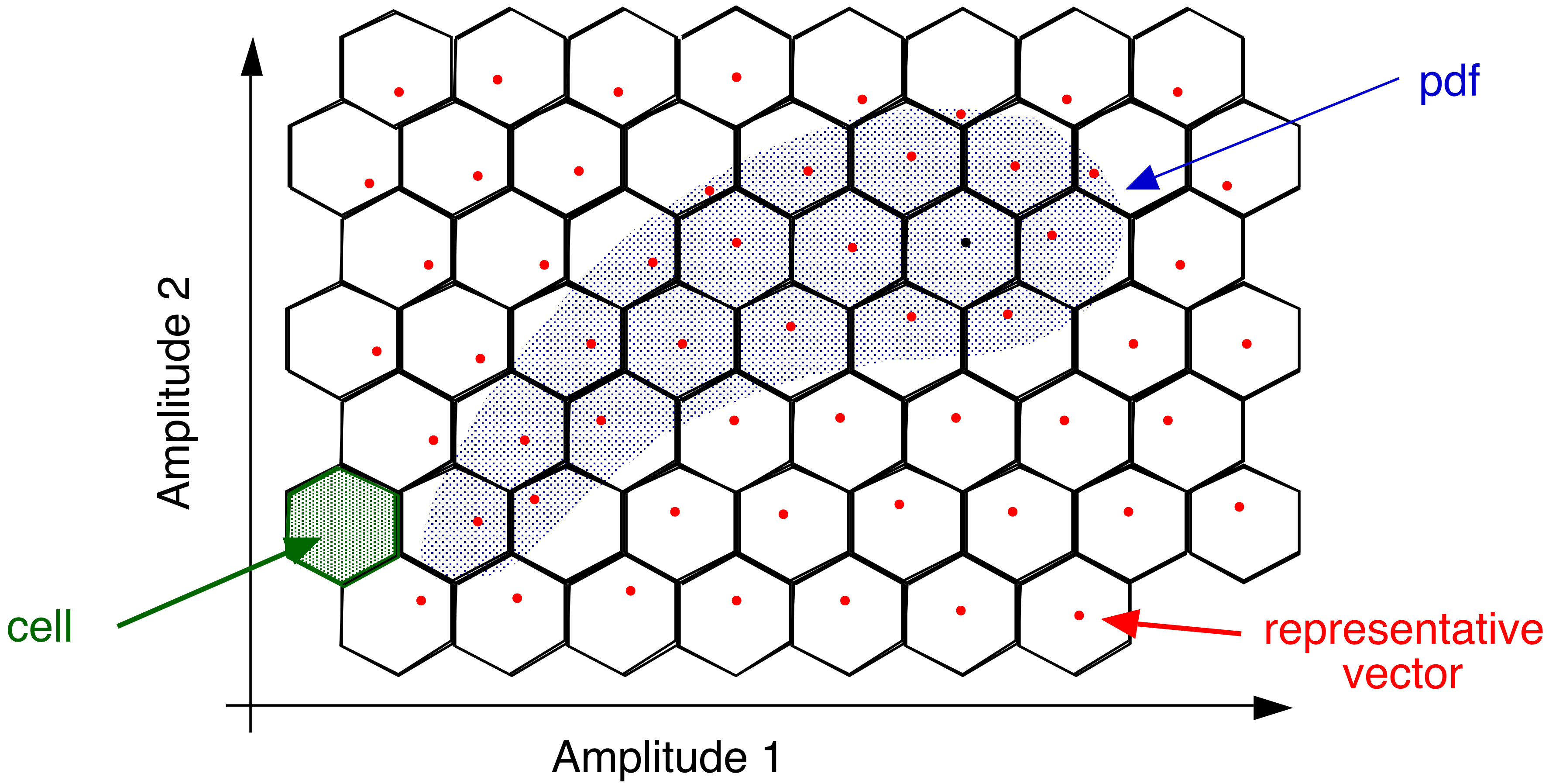
$$J = d + \lambda R = E\left[\left\|X - \hat{X}\right\|^2\right] + \lambda H\left(\hat{X}\right) \to \min.$$

- Unstructured code book: full search for

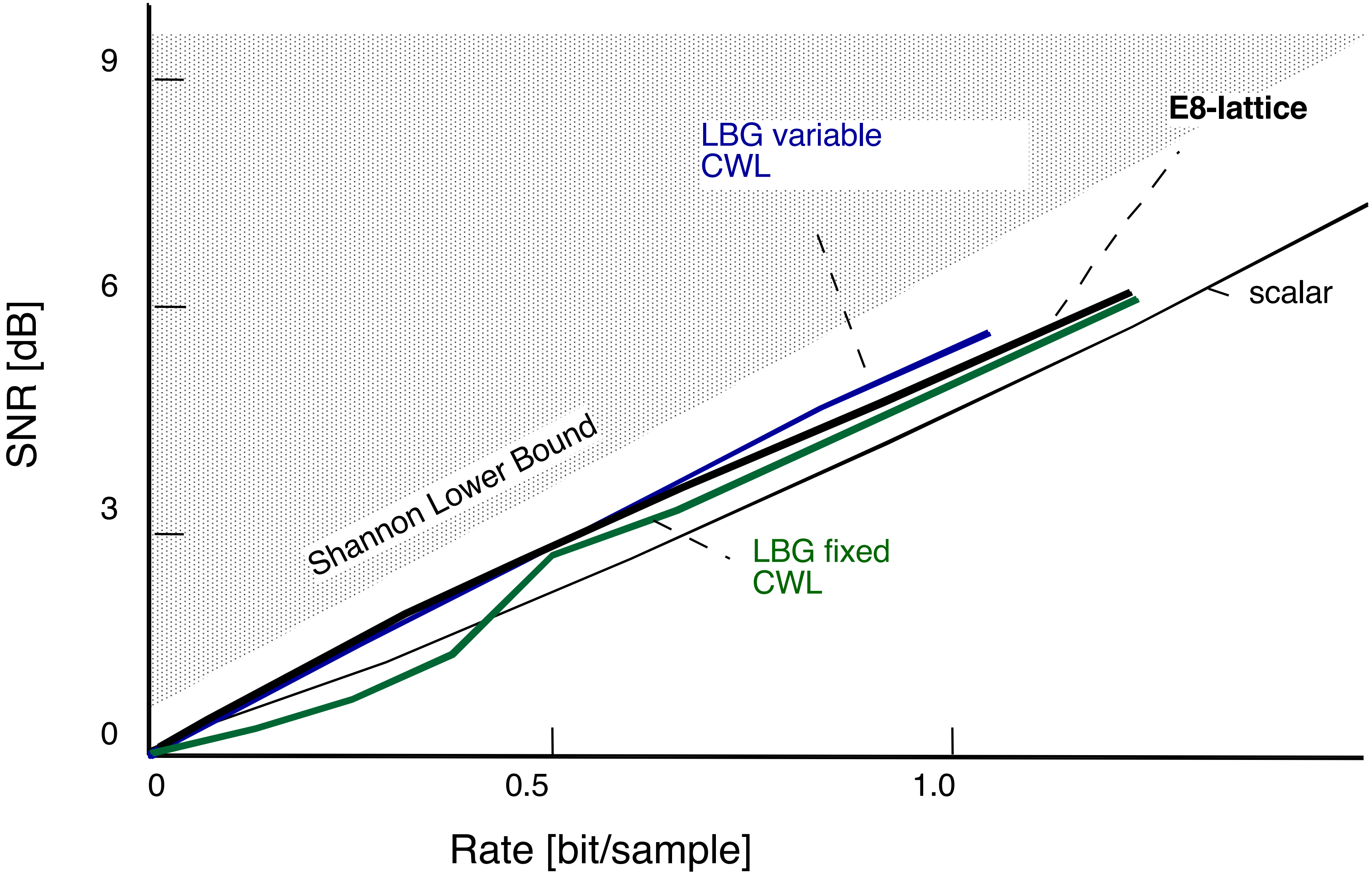
$$J_{x_i}\left(q\right) = \left\|x_i - \hat{x}_q\right\|^2 - \lambda \log_2 p_q$$

The most general coder structure:
Any source coder can be interpreted as VQ with VLC!

Lattice Vector Quantization



8D VQ of Memoryless Laplacian Source



8D VQ of a Gauss-Markov Source