Stochastic Process

- Stationary process
- Markov process
- Markov-1 source
- Entropy rate

Stochastic Process

- Stochastic process $\{X_i\}$ is an indexed sequence of random variables $X_1, X_2, \ldots, X_n$ with joint PMF

$$f_{X_1,X_2,\ldots,X_n} = \Pr\{(X_1, X_2, \ldots, X_n) = (x_1, x_2, \ldots, x_n)\}$$

$$\forall \quad (x_1, x_2, \ldots, x_n) \in \mathcal{X}^n$$

- A stochastic process is said to be stationary if the joint PMF is invariant with respect to shifts

$$f_{X_1,X_2,\ldots,X_n} = f_{X_{1+l},X_{2+l},\ldots,X_{n+l}} \quad \forall \quad l \in \mathcal{Z}$$

Markov Process

- A stochastic process is said to be a Markov process or a Markov chain if

$$f_{X_{n+1}|X_n,X_{n-1},\ldots,X_1} = f_{X_{n+1}|X_n}$$

- The PMF of a Markov process can be written as

$$f_{X_1,X_2,\ldots,X_n} = f_{X_1} f_{X_2|X_1} f_{X_3|X_2} \cdots f_{X_n|X_{n-1}}$$

- The Markov process is said to be time invariant if

$$f_{X_{n+1}|X_n} = f_{X_1|X_0} \quad \forall \quad n$$

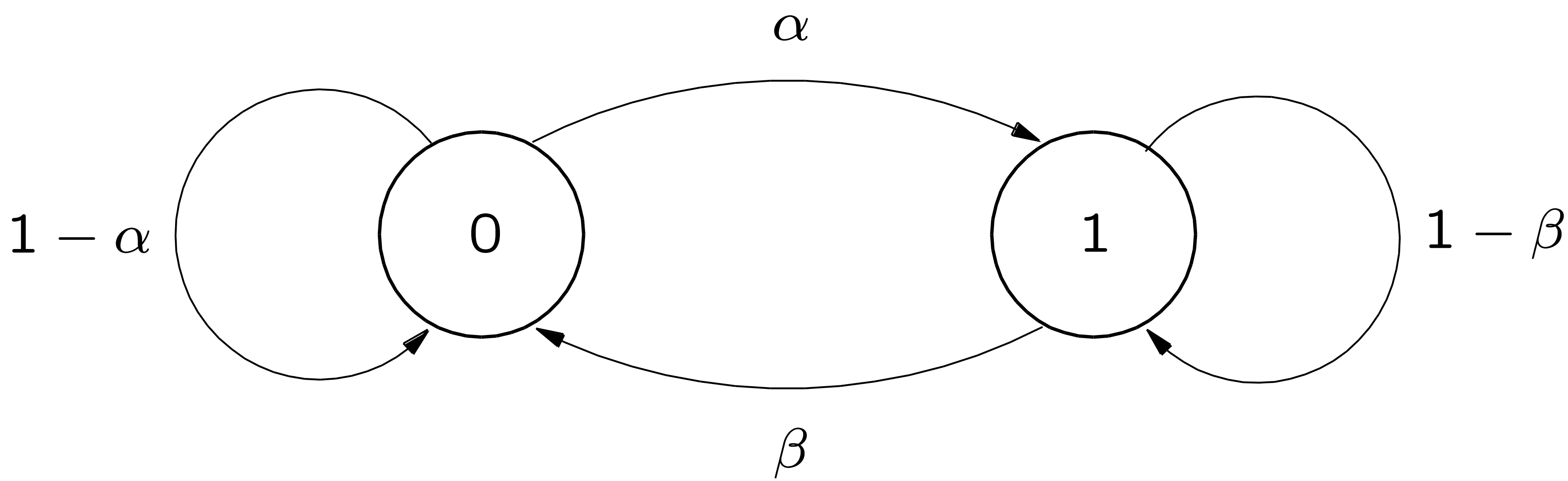
Markov-p Process

- A stochastic process is said to be Markov-p, with p ≥ 1, if

$$f_{X_{n+1}|X_n,X_{n-1},\ldots,X_1} = f_{X_{n+1}|X_n,X_{n-1},\ldots,X_{n+1-p}}$$

- Order-p conditional PMF describes Markov-p process fully

Markov-1 Source: Two-State Markov Chain



- Probability transition matrix

$$\mu_{i+1} = \mu_i P_{i+1|i} \qquad\qquad P_{i+1|i} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

- Stationary state probabilities:

$$\mu_\infty = \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}$$

Entropy Rate

- How does the entropy of the sequence X^n grow with n?
- Entropy rate of a stochastic process $\{X_i\}$ is defined by

$$H(\{X_i\}) = \lim_{n\to\infty} \frac{1}{n} H(X_1, X_2, \ldots, X_n)$$

- Using the chain rule

$$H(\{X_i\}) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} H(X_i | X_{i-1}, \ldots, X_1)$$

Examples of Entropy Rates

- Process is a sequence of i.i.d. random variables X

$$H(\{X_i\}) = \lim_{n\to\infty} \frac{nH(X)}{n} = H(X)$$

- Time-invariant Markov chain

$$H(\{X_i\}) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} H(X_i|X_{i-1}) = H(X_1|X_0)$$