Model predictive control with integrated experiment design for output error systems

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Abstract— **Model predictive control has become an increasingly popular control strategy thanks to the ability to handle constrained systems. Obtaining the required models through system identification is often a time consuming and costly process. Applications oriented experiment design is a means of reducing this effort but is often formulated in terms of the input's spectral properties. Therefore, time domain constraints are difficult to enforce. In this contribution we combine MPC with experiment design to formulate a control problem where excitation constraints are included. The benefits are that time domain constraints are respected while the experiment design criteria are fulfilled. The method is evaluated on a numerical example.**

I. Introduction

MODEL predictive control (MPC) has since its introduction grown more and more popular in many fields, in particular in process industry. MPC was adopted early on by petrochemical industries and is now probably used in all modern refineries [1]. The key properties that have led to the success of MPC are the abilities to easily handle multivariate systems and to incorporate constraints on inputs, outputs and states. As the name indicates, a model is used to predict the process response and a suitable input is calculated accordingly. As a result, the performance of the MPC strongly depends on how well the model reflects the dominant plant dynamics.

It has been reported that the most expensive and time consuming part of MPC commissioning is the process modeling. Estimates of the part of the cost and time of commissioning related to modeling range up to 90 % [2], [3]. As the popularity of MPC increases, time and cost efficient modeling becomes important. Most MPC solutions used today, employ system identification with pseudo random binary excitation signals to obtain process models [4], even for multivariate processes.

Optimal input design is a tool that has been shown to be able to significantly reduce the required experimental effort in control applications [5]. In [6] the idea that the excitation should enhance important process properties while attenuating the less important properties is discussed and formalized. A general framework for applications oriented experiment design is presented in [7], building on the least-costly experiment design paradigm put forward in [8]. Building on this framework, [9] presented an initial idea for optimal experiment design when the intended model use is MPC. Some MPC related issues, relating to the fact that constrained MPC is a nonlinear controller, were addressed. This idea was further developed in [10], [11], where an experiment design algorithm for open loop identification of models for MPC is presented. In this formulation, the experiment design is formulated in the frequency domain, therefore time domain hard constraints on the input signals are difficult to handle.

Two ways around this problem have been suggested. One possibility is to disregard constraints in the design part and only include them during signal generation. There are many algorithms available for design of binary signals with a prescribed spectrum, e.g., [12], [13]. Recently a receding horizon algorithm for signal generation under input and ouput constraints has been proposed [14]. Another possibility is to do experiment design in the time domain. Two recent works, which have inspired this work, are [15], [16].

In this contribution we present a method where the experiment design is included in the MPC formulation as an extra constraint. The purpose of this constraint is to ensure that the applied input excites the system so that a high quality model can be obtained from operational data. This allows for online identification with the MPC in the loop. Our proposed method has similarities with the persistently exciting MPC proposed in [17], [18]; these works, however focus on ensuring persistence of excitation rather than on obtaining an input signal suitable for identification. A more recent contribution along the same lines is [19] where several relaxations of a persistently exciting MPC method are presented. This last contribution, however, does not consider the final purpose of the estimated model explicitly.

An advantage of having the MPC running during the identification experiment is that time domain constraints can be handled. This should be a desirable property for constrained systems. The proposed MPC leads to a non-convex optimization problem. However, through standard techniques, a relaxed convex problem can be formulated. Before closing we also point to the work of Zhu, e.g. [3], related to identification for MPC, which relies on asymptotic variance expressions (in both model order and sample size) for input design; this work, however, does not take into account the specific features of MPC (such as input and state constraints).

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A. Notation

The symbol $\mathbb{E}\{\cdot\}$ denotes the expectation operator. S_n^+ is the cone of symmetric, positive semi-definite matrices of dimensions $n \times n$. For two symmetric matrices X and Y, $X \succeq Y$ means $X - Y \in S_n^+$. For vectors, inequalities are interpreted element wisely, e.g., $\xi \geq 0$ means that all elements of ξ are non-negative. The trace of X is denoted $\operatorname{tr} X$ and $\operatorname{diag} X$ is the vector of the diagonal elements of X. For vectors, x_i denotes element i and for matrices, $X_{i,j}$ denotes element j of row i.

B. Structure of paper

The paper has the following structure. In Section II we go through the relevant system identification and MPC background. In Section III we present the applications oriented experiment design formulation. Section IV introduces our proposed model predictive controller with experiment design constraints together with a convex relaxation thereof. The controller is tested on a numerical example in Section V. Finally, in Section VI conclusions are drawn and future research directions outlined.

II. PRELIMINARIES

A. System identification

Consider a discrete time, linear time-invariant dynamic system described by an output error model of the form

$$x(t+1) = A(\theta)x(t) + B(\theta)u(t), \tag{1a}$$
$$y(t) = C(\theta)x(t) + e(t), \tag{1b}$$

where $\theta \in \mathbb{R}^{n_\theta}$ is an unknown parameter vector, $x(t) \in \mathbb{R}^d$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ the output, and $e(t)$ the innovations; a zero-mean, random process with covariance matrix $\mathbb{E}\left\{e(t)e^T(s)\right\} = \Lambda_e \delta_{t,s}$. We assume that there exists a vector θ_o which corresponds to the true system parameters. We use the prediction error method with quadratic cost for system identification and $\hat{\theta}_N$ denotes the estimate resulting from N samples of input–output data, $Z_N = \{u(t), y(t)\}_{t=1}^N$, see [20].

For an unbiased estimator the Cramér-Rao inequality provides a lower bound on the covariance matrix for the parameter estimation error. Given data from time m to time n, this bound is given by the inverse of the Fisher information matrix $\mathcal{I}_m^n(\theta_o)$, where the latter is defined by

$$\mathcal{I}_m^n(\theta) = \sum_{t=m}^{n} \mathbb{E}\left\{\psi(t,\theta)\Lambda_e^{-1}\psi^T(t,\theta)\right\}, \tag{2}$$

$$\psi_i(t,\theta) = \frac{\mathrm{d}\hat{y}(t)}{\mathrm{d}\theta_i}, \tag{3}$$

$$\psi(t,\theta) = \begin{bmatrix}\psi_1(t,\theta) & \cdots & \psi_{n_\theta}(t,\theta)\end{bmatrix}^T. \tag{4}$$

The reason that we start the sum in (2) at m is that we will later consider the parts of the information matrix related to past and future data. For a given data record, Z^N, the Fisher information matrix can be estimated by

$$I_m^n(\theta) = \sum_{t=m}^{n} \psi(t,\theta)\Lambda_e^{-1}\psi^T(t,\theta). \tag{5}$$

In fact, it holds under fairly mild conditions that

$$\lim_{n\to\infty} \frac{1}{n} I_m^n(\theta) = \lim_{n\to\infty} \frac{1}{n} \mathcal{I}_m^n(\theta), \text{ almost surely.} \tag{6}$$

Now, under very general conditions [21], it holds that

$$\mathcal{I}_1^N(\theta_o)^{1/2}(\hat{\theta}_N - \theta_o) \in As\mathcal{N}(0, I). \tag{7}$$

This implies that

$$\hat{\theta}_N \in U(\alpha) = \left\{\theta \,:\, [\theta - \theta_o]^T \mathcal{I}_1^N(\theta_o)\,[\theta - \theta_o] \leq \chi_\alpha^2(n_\theta)\right\}, \tag{8}$$

with probability α. Here $\chi_\alpha^2(n)$ is the α-percentile of the χ^2-distribution with n degrees of freedom. We call $U(\alpha)$ the *identification ellipsoid*.

The dependence of A, B, C and ψ on θ will henceforth be omitted to simplify notation.

B. Model predictive control

In MPC, the input is determined by solving an optimization problem where the impact of the control signal on the plant is predicted using a system model. A common cost function used in the MPC is given by

$$J = \sum_{k=1}^{N_y} \|y(k) - r(k)\|_Q^2 + \sum_{k=1}^{N_u} \|\Delta u(k)\|_R^2, \tag{9}$$

where $r(k)$ is a reference trajectory, $\Delta u(k) = u(k) - u(k-1)$ is the control update, N_y and N_u are the prediction and control horizons, respectively, and Q and R are two tunable matrix weights. The model in the MPC is augmented with constant disturbances on each output to compensate for wrong model gains. At time instant t, a sequence of inputs is found by solving the optimization problem

$$\begin{aligned}
\underset{\{u(k)\}_{k=1}^{N_u}}{\text{minimize}} \quad & J \\
\text{subject to} \quad & x(k+1) = Ax(k) + Bu(k), \\
& y(k) = Cx(k),\ k = 1, \ldots, N_y, \\
& x(1) = \hat{x}(t), \\
& \Delta u(1) = u(1) - u^\star(t-1), \\
& u_{min} \leq\ u(k) \leq u_{max},\ k = 1, \ldots, N_u, \\
& y_{min} \leq\ y(k+1) \leq y_{max}\ k = 1, \ldots, N_y.
\end{aligned} \tag{10}$$

Here $\hat{x}(t)$ is the estimated system state at time t, obtained either from direct measurements or an observer, and $u^\star(t-1)$ is the input signal applied to the system at time $t-1$. Even though the solution to (10) is a sequence of inputs, only the first input is applied to the system and the optimization is performed again in the next time step, according to the receding horizon principle. The MPC formulation is presented in further detail in [22].

III. APPLICATIONS ORIENTED INPUT DESIGN

The quality of a model will influence the performance of a control application where the model is used. The applications oriented experiment design formulation relies on an application cost as a measure of performance degradation due to mismatch between model and system. We denote the application cost by V_{app}. We can assume without loss of generality that $V_{app}(\theta) \geq 0$ and that $V_{app}(\theta_o) = 0$ is a minimum. In the case of parameterized models such as (1), V_{app} is a function of the model parameters θ. A model is considered acceptable if the degradation is sufficiently small. This gives a set of acceptable models or parameters

$$\Theta_{app}(\gamma) = \left\{ \theta \ : \ V_{app}(\theta) \leq \frac{1}{\gamma} \right\}, \tag{11}$$

where γ is an application specific constant which determines the accuracy of the model. We can make a convex approximation of Θ_{app} using a second order Taylor expansion of V_{app}, using $V_{app}(\theta_o) = V'_{app}(\theta_o) = 0$. Hence, the set of acceptable parameters (11) can be approximated by the ellipsoidal set

$$\mathcal{E}_{app} = \left\{ \theta \ : \ [\theta - \theta_{\text{o}}]^T V''_{app}(\theta_{\text{o}})[\theta - \theta_{\text{o}}] \leq \frac{2}{\gamma} \right\}. \tag{12}$$

We call this the *application ellipsoid*.

The aim of the applications oriented input design is to find an input that with high probability, α, results in acceptable parameter estimates while at the same time minimizing the cost of the identification experiment, i.e.,

$$\begin{array}{ll} \underset{\text{input}}{\text{minimize}} & \text{Experimental cost} \\ \text{subject to} & P\left\{ \hat{\theta}_N \in \Theta_{app}(\gamma) \right\} \geq \alpha. \end{array} \tag{13}$$

The problem (13) is, however, in general non convex and not computationally tractable. In [7], approximating the chance constraint in (13) by

$$U(\alpha) \subseteq \mathcal{E}_{app}, \tag{14}$$

is suggested. The approximation (14) is equivalent to

$$\mathcal{I}_1^N \succeq \frac{\gamma \chi^2_\alpha(n_\theta)}{2} V''_{app}(\theta_0), \tag{15}$$

which is a linear matrix inequality (LMI) in the elements of $\mathcal{I}_1^N$. There are other choices for the approximation of the chance constraint in (13), a discussion on chance constraints in input design is found in [23].

The choice of the application cost and the corresponding γ is highly application specific. Some general ideas for suitable choices in the case of MPC are discussed in [11] and not further elaborated on here. We will refer to (15) as the *experiment design constraint*.

IV. MPC WITH EXPERIMENT DESIGN CONSTRAINTS

In this section we present an MPC formulation which includes input constraints that arise in the application oriented input design formulation. The idea is to let the MPC compute an input that minimizes the control cost while at the same time excites the system enough for a system identification experiment of length N to produce an acceptable model. To achieve this, we include constraint (15) into the MPC formulation (10).

We propose the following MPC formulation to be solved at time t:

$$\begin{array}{ll} \underset{\{u(k)\}_{k=1}^{N_u}}{\text{minimize}} & J \\ \text{subject to} & x(k+1) = Ax(k) + Bu(k), \\ & y(k) = Cx(k), \ k = 1, \ldots, N_y, \\ & x(1) = \hat{x}(t), \\ & \Delta u(1) = u(1) - u^\star(t-1), \\ & u_{min} \leq \ u(k) \leq u_{max}, \ k = 1, \ldots, N_u, \\ & y_{min} \leq \ y(k+1) \leq y_{max} \ k = 1, \ldots, N_y. \\ & \mathcal{I}_1^{t+N_y}(\theta_o) \succeq \kappa(t) \dfrac{\gamma \chi^2_\alpha(n)}{2} V''_{app}(\theta_0). \end{array} \tag{16}$$

The last constraint of (16) is added to ensure that the resulting information matrix, over the MPC prediction horizon, fulfills the application specifications. $\kappa(t)$ is a scaling factor which we set to

$$\kappa(t) = \frac{\min(N, t + N_y)}{N}. \tag{17}$$

This choice of scaling ensures that at time N, the constraint (15) is fulfilled. The implication is that after N samples, the information matrix is such that the application constraint is satisfied with probability α.

A. Information matrix

To incorporate the constraint (15) in the MPC formulation, we need to relate $\mathcal{I}_1^N(\theta_o)$ to the input $u(t)$. Assume that we are at time instant t and want to run the MPC optimization. $\mathcal{I}_1^{t+N_y}(\theta_o)$ can then be split into

$$\mathcal{I}_1^{t+N_y}(\theta_o) = \mathcal{I}_1^t(\theta_o) + \mathcal{I}_{t+1}^{t+N_y}(\theta_o) \tag{18}$$

where $\mathcal{I}_1^t(\theta_o)$ depends on available data while $\mathcal{I}_{t+1}^{t+N_y}(\theta_o)$ will be the predicted addition to the information matrix by the control input. Since we have data up to time $t-1$, Z_{t-1}, and an estimate $\hat{\theta}$ available, we make the approximation

$$\mathcal{I}_1^{t+N_y}(\theta_o) \approx I_1^t(\hat{\theta}) + I_{t+1}^{t+N_y}(\hat{\theta}) \tag{19}$$

Remark: Here we use the estimated parameter $\hat{\theta}$ in lieu of the true parameter values. Such estimates can, for instance, be available from the commisioning of the MPC. We assume that the estimates are sufficiently good for the experiment design. Our approach can easily be extended to a genuine adaptive experiment design, but this is considered future work.

To calculate (19), we proceed along the lines of [24]. First, we find $\psi(t)$ by differentiating (1).

$$\frac{\mathrm{d}x(t+1)}{\mathrm{d}\theta_i} = \frac{\mathrm{d}A}{\mathrm{d}\theta_i} x(t) + A \frac{\mathrm{d}x(t)}{\mathrm{d}\theta_i} + \frac{\mathrm{d}B}{\mathrm{d}\theta_i} u(t), \tag{20}$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}\theta_i} = \frac{\mathrm{d}C}{\mathrm{d}\theta_i} x(t) + C \frac{\mathrm{d}x(t)}{\mathrm{d}\theta_i}. \tag{21}$$

Second, by introducing

$$\mathcal{A} \triangleq \begin{bmatrix} A & 0 & 0 & 0 \\ \frac{\mathrm{d}A}{\mathrm{d}\theta_1} & A & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \frac{\mathrm{d}A}{\mathrm{d}\theta_{n_\theta}} & 0 & 0 & A \end{bmatrix}, \mathcal{B} \triangleq \begin{bmatrix} B \\ \frac{\mathrm{d}B}{\mathrm{d}\theta_1} \\ \vdots \\ \frac{\mathrm{d}B}{\mathrm{d}\theta_{n_\theta}} \end{bmatrix},$$

$$\mathcal{C} \triangleq \begin{bmatrix} \frac{\mathrm{d}C}{\mathrm{d}\theta_1} & C & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \frac{\mathrm{d}C}{\mathrm{d}\theta_{n_\theta}} & 0 & 0 & C \end{bmatrix},$$

$$\xi(t) \triangleq \begin{bmatrix} x(t) & \frac{\mathrm{d}x(t)}{\mathrm{d}\theta_1} & \cdots & \frac{\mathrm{d}x(t)}{\mathrm{d}\theta_{n_\theta}} \end{bmatrix}^T,$$

$$\bar{\psi}(t) \triangleq \begin{bmatrix} \psi_1^T(t) & \cdots & \psi_{n_\theta}^T(t) \end{bmatrix}^T,$$

we can form the augmented state space

$$\xi(t+1) = \mathcal{A}\xi(t) + \mathcal{B}u(t), \tag{22}$$

$$\bar{\psi}(t) = \mathcal{C}\xi(t). \tag{23}$$

This in turn gives us

$$\bar{\psi}(t)\bar{\psi}^T(t) = \mathcal{C}\xi(t)\xi^T(t)\mathcal{C}^T. \tag{24}$$

The elements of $I_t^t = \psi(t)\Lambda_e^{-1}\psi^T(t)$ can now be found from

$$\begin{aligned}\left(\psi(t)\Lambda_e^{-1}\psi^T(t)\right)_{i,j} &= \psi_i^T(t)\Lambda_e^{-1}\psi_j(t) \\ &= \operatorname{tr}\psi_j(t)\psi_i^T(t)\Lambda_e^{-1}.\end{aligned} \tag{25}$$

Remark: These calculations were simplified because of our choice to work with output error models. For more general model structures, one would have to consider the expectation of the expression (24) which is involved due to the correlation between u and e in closed loop and the non linear nature of the MPC.

B. Quadratic formulation

Now we seek to write the MPC with excitation constraints formulation as a quadratic program. Firstly, the MPC cost function (9) can be written as

$$J(t) = \|\bar{y} - \bar{r}\|_{\mathcal{Q}} + \|\Gamma\bar{u} - \bar{u}^\star(t)\|_{\mathcal{R}}, \tag{26}$$

where $\bar{u} \triangleq \begin{bmatrix} u^T(N_u), \dots, u^T(1) \end{bmatrix}^T$, $\bar{y}$ and $\bar{r}$ are defined analogously, $\bar{u}^\star(t) \triangleq [0, \dots, 0, u^\star(t-1)]^T$, $\mathcal{Q}$ and $\mathcal{R}$ are block diagonal matrices with Q and R on the diagonals, respectively, and dimensions commensurate with $\bar{y}$ and $\bar{u}$ and

$$\Gamma \triangleq \begin{bmatrix} 1 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -1 \\ 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{27}$$

The constraints of (10) give that $\bar{y} = \Psi\hat{x}(t) + \Upsilon\bar{u}$, with

$$\Psi \triangleq \begin{bmatrix} CA^{N_y} \\ CA^{N_y-1} \\ \vdots \\ CA \end{bmatrix}, \Upsilon \triangleq \begin{bmatrix} CB & CAB & \cdots & CA^{N_y-1}B \\ 0 & CB & \cdots & CA^{N_y-2}B \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & CB \end{bmatrix}. \tag{28}$$

By defining

$$\mathcal{H} \triangleq \Upsilon^T\mathcal{Q}\Upsilon + \Gamma^T\mathcal{R}\Gamma, \tag{29}$$

$$\mathcal{G}(t) \triangleq 2\left[\Psi\hat{x}(t) - \bar{r}\right]^T\mathcal{Q}\Upsilon - 2\bar{u}^{\star T}(t)\mathcal{R}\Gamma, \tag{30}$$

the cost function (26) can be written as

$$J(t) = \bar{u}^T\mathcal{H}\bar{u} + \mathcal{G}(t)\bar{u} + \text{constant}. \tag{31}$$

The constant term does not influence the optimization. Even though the cost function is a scalar quadratic function of the decision variables, the MPC problem (16) is not a quadratic program. This is due to the added experiment design constraint which is quadratic in $\bar{u}$, and typically makes the problem non-convex and computationally difficult. We therefore consider a convex relaxation of the formulation.

C. Convex relaxation

We introduce the lifting variable $U \in S_{N_y}^+$ and add the constraint $U = \bar{u}\bar{u}^T$, which can be written as

$$\begin{bmatrix} U & \bar{u} \\ \bar{u}^T & 1 \end{bmatrix} \succeq 0, \qquad \operatorname{rank}\begin{bmatrix} U & \bar{u} \\ \bar{u}^T & 1 \end{bmatrix} = 1. \tag{32}$$

We reformulate the MPC optimization problem in the variables U and $\bar{u}$. First, we rewrite the cost function as

$$J(t) = \operatorname{tr}\mathcal{H}U + \mathcal{G}(t)\bar{u}, \tag{33}$$

due to the cyclic property of the trace.

Second, we reformulate the constraints in the variables U and $\bar{u}$. We need to consider the $\mathcal{I}_t^{t+N_y}$ term in the experiment design constraint. Consider first

$$\begin{aligned}\xi(t+1)\xi^T(t+1) &= \mathcal{A}\xi(t)\xi^T(t)\mathcal{A}^T + \mathcal{B}u(t)u^T(t)\mathcal{B}^T \\ &\quad + \mathcal{A}\xi(t)u^T(t)\mathcal{B}^T + \mathcal{B}u(t)\xi^T(t)\mathcal{A}^T,\end{aligned} \tag{34}$$

where, at time t, the only decision variable is $u(t) = \bar{u}_{N_u}$ since $\xi(t)$ depends on past data. Hence (34) can be written as

$$\begin{aligned}\xi(t+1)\xi^T(t+1) &= \mathcal{A}\xi(t)\xi^T(t)\mathcal{A}^T + \mathcal{B}U_{N_u,N_u}\mathcal{B}^T \\ &\quad + \mathcal{A}\xi(t)\bar{u}_{N_u}^T\mathcal{B}^T + \mathcal{B}\bar{u}_{N_u}\xi^T(t)\mathcal{A}^T,\end{aligned} \tag{35}$$

which is linear in U and $\bar{u}$. Now consider

$$\begin{aligned}\xi(t+k)\xi^T(t+k) &= \mathcal{A}\xi(t+k-1)\xi^T(t+k-1)\mathcal{A}^T \\ &\quad + \mathcal{B}u(t+k-1)u^T(t+k-1)\mathcal{B}^T \\ &\quad + \mathcal{A}\xi(t+k-1)u^T(t+k-1)\mathcal{B}^T \\ &\quad + \mathcal{B}u(t+k-1)\xi^T(t+k-1)\mathcal{A}^T.\end{aligned} \tag{36}$$

Since $\xi(t+k-1)\xi^T(t+k-1)$ is linear in U, so is

$$\mathcal{A}\xi(t+k-1)\xi^T(t+k-1)\mathcal{A}^T.$$

The second term,

$$\mathcal{B}u(t+k-1)u^T(t+k-1)\mathcal{B}^T = \mathcal{B}U_{N_y-k+1,N_y-k+1}\mathcal{B}^T$$

is also linear in U. Only the two last terms of (36) remain to be analyzed. Iterating (22) gives

$$\xi(t+k) = \mathcal{A}^k\xi(t) + \sum_{i=0}^{k-1}\mathcal{A}^i\mathcal{B}u(t+k-i-1). \tag{37}$$

Hence

$$\mathcal{A}\xi(t+k-1)u^T(t+k-1)\mathcal{B}^T = \mathcal{A}^k\xi(t)u^T(t+k-1)\mathcal{B}^T + \sum_{i=0}^{k-1}\mathcal{A}^i\mathcal{B}u(t+k-i-1)u^T(t+k-1)\mathcal{B}^T = \mathcal{A}^k\xi(t)\bar{u}^T_{N_y-k+1}\mathcal{B}^T + \sum_{i=0}^{k-1}\mathcal{A}^i\mathcal{B}U_{N_y-k+i+1,N_y-k+1}\mathcal{B}^T, \tag{38}$$

which is linear in U and $\bar{u}$. Therefore, $\xi(t+k)\xi^T(t+k)$ is linear in U and $\bar{u}$ and hence, $\bar{\psi}(t)\bar{\psi}^T(t) = \mathcal{C}\xi(t)\xi^T(t)\mathcal{C}^T$ is also linear in U and $\bar{u}$. As a result, the experiment design constraint is an LMI in the decision variables U and $\bar{u}$ and therefore a convex constraint.

The input and output constraints are also reformulated and the relaxed MPC formulation becomes

$$\begin{aligned}
&\underset{U,\bar{u}}{\text{minimize}} \quad \operatorname{tr}\mathcal{H}U + G(t)\bar{u}\\
&\text{subject to}\\
&\qquad \operatorname{diag} U - \bar{u}u_{max} - \bar{u}u_{min} \leq -u_{max}u_{min},\\
&\qquad \operatorname{diag} \Upsilon U \Upsilon^T - \Upsilon\bar{u}\tilde{y}_{max} - \Upsilon\bar{u}\tilde{y}_{min} \leq \tilde{y}_{max}\tilde{y}_{min},\\
&\qquad \tilde{y}_{min,max} = y_{min,max} - \Psi\hat{x}(t),\\
&\qquad I_1^{t-1} + I_t^{t+N_y} \succeq \kappa(t)\frac{\gamma\chi^2_\alpha(n)}{2}V''_{app}(\theta_0)\\
&\qquad \begin{bmatrix} U & \bar{u} \\ \bar{u}^T & 1 \end{bmatrix} \succeq 0,
\end{aligned}$$

where the rank constraint which forces $U = \bar{u}\bar{u}^T$ has been dropped. The solution to this relaxed problem is the matrix U and the vector $\bar{u}$. An input can be found by drawing a sample of the random variable

$$\tilde{u} = \bar{u} + D^T\zeta, \qquad DD^T = U, \qquad \zeta \in \mathcal{N}(0, I), \tag{39}$$

as suggested in [15], [25]. The input applied to the process can then be extracted from the last elements of $\tilde{u}$.

V. Numerical Example

In this example we illustrate the algorithm on a simulation example. We consider a system consisting of two interconnected tanks. An upper tank is connected to a pump with input $u(t)$. The tank has a hole in the bottom with free flow into a lower tank, which also has a hole with free flow out of the tank. The level in the lower tank is the output, $y(t)$. The system is modelled using the output error model

$$\begin{aligned}
x(t+1) &= \begin{bmatrix} \theta_3 & \theta_4 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)\\
y(t) &= \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix} x(t) + e(t).
\end{aligned} \tag{40}$$

The true system is given by the parameter values $\begin{bmatrix} 0.12 & 0.059 & -0.74 & 0.14 \end{bmatrix}^T$ and the noise variance $\mathbb{E}\left\{e^2(t)\right\} = 0.01$. The goal is to control the level in the lower tank using MPC with the following settings: $N_y = N_u = 5$, $Q_y = I$, $Q_u = 0.001I$. The considered scenario is such that the identification is started at steady state operating conditions of the plant.

We use the true parameter values to get the initial model. As the application cost we choose

$$V_{app}(\theta) = \sum_{t=1}^{T} \|y(t,\theta_o) - y(t,\theta)\|_2^2, \tag{41}$$

over a step response of the system with the MPC running. Hence, we want the identified model to give the step response close to what we would get had the true system parameters been available.

A. Identification Experiment

We set the length of each individual experiment to $N = 100$ samples, the accuracy $\gamma = 200$ and a probability of $\alpha = 99\%$ for the confidence ellipsoid $U(\alpha)$.

We consider the situation that the system is in a state of steady operations during the identification experiment. In other words we only try to control the system output around the steady state level while satisfying constraints. The identification is performed around this equilibrium level with maximum input and output deviations of 2 and 1 units, respectively, i.e., $-u_{min} = u_{max} = 2$ and $-y_{min} = y_{max} = 1$.

We perform a Monte Carlo trial with 500 simulations. The gathered data is used to identify a second order output error model using the System Identification Toolbox in Matlab [26]. For comparison we also run the MPC without the experiment design constraint and try to identify the system from operational data. The application cost is then evaluated for the resulting model.

B. Results

The Monte Carlo trial resulted in models which fulfill the performance specifications in 98.8 % of the cases while identifying a model from normal operating data gave acceptable models in 69.6 % of the cases.

Input data generated by MPC with experiment design constraint together with the resulting output of the system, from one Monte Carlo simulation, are shown in Figure 1. Both signals lie within the white area of the plots which shows that the input and output constraints have been satisfied. The noisy input is required to satisfy the experiment design constraint. To reduce the variance of $u(t)$, a lower value of γ or longer experiment time would be needed. We also show the input–output data from a simulation without the experiment design constraint active to illustrate the price of adding the extra excitation in Figure 2.

It is interesting to compare the closed loop MPC with experiment design constraint to experiment design techniques where the input spectrum is designed and a signal generated by filtering noise through a spectral factor, see e.g. [27], [7], [11]. The optimal input spectrum for an open loop experiment gives the minimal input variance, 0.28, which guarantees the performance specifications. This should be compared to the average input variance of 0.36 for the proposed method. The resulting output variance was on average 0.043. This comparison is straight forward since for output error models, the open loop experiment is optimal and we cannot reduce the required variance of the input signal by closing the loop. Under normal operating conditions,

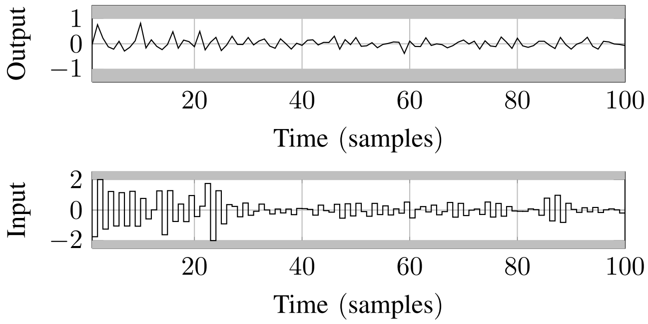

Fig. 1. The output and input of the system (40) using MPC with the experiment design constraint as described in the example. Both signals satisfy the constraint. The price to be paid to be able to identify the model is the variance of both input and output signals.

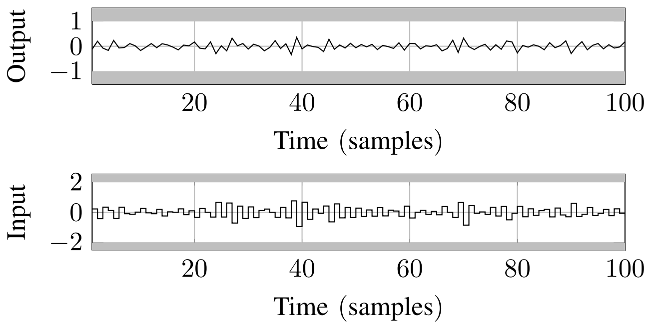

Fig. 2. The output and input of the system (40) using regular MPC without any constraint for excitation. The variance of the signals are around 2–3 times smaller than what can be seen in Figure 2.

when no extra excitation is added, the input and output variances were 0.12 and 0.020, respectively. It is clear that this input variance is not sufficient to meet the performance specifications.

In Figure 3 we show the Bode diagram of one resulting model together with 3 standard deviations of the estimate. The experiment design constraint has resulted in an input that gives a model with low uncertainty in the frequency range 1.5 to 2.5 rad/s. The frequency response at lower frequencies is estimated with lower accuracy. This can be explained by the fact that the MPC can compensate for a missmatch of the model and system gains.

VI. Conclusions

We have presented an MPC formulation with integrated experiment design for output error models. Ideas from applications oriented experiment design have been used to add a constraint in the MPC. The constraints relates to the information matrix of the estimated model parameters. This added constraint ensures that the input signal excites the system sufficiently to identify a model that gives desired application performance. The price to be paid is that the signals become “noisy” compared to when no extra excitation is needed.

Future research directions include extending the results to more general model classes where the correlation between input and noise has to be considered. We also believe that the method, with suitable changes, will work well for iterative identification where also the experiment design constraint can be updated. This will be further investigated.

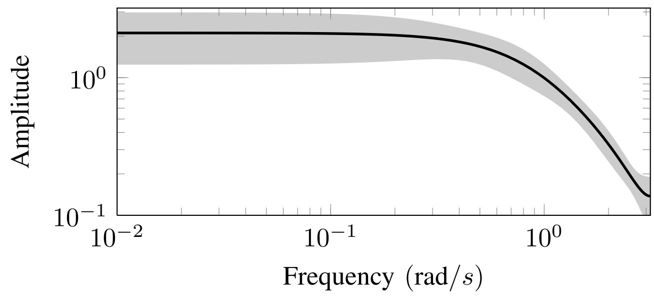

Fig. 3. Bode diagram of the model of (40) estimated from $N = 100$ samples of data generated by the MPC with experiment design constraints. The light gray area shows 3 standard deviations of the estimate.

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