Experimental study:

Applications oriented input design - MPC for Reference Tracking of Quadruple Water Tanks

Mariette Annergren

Introduction

The experimental study was performed on a quadruple water tanks process at the Department of Automatic Control, Lund University. More specifically, process S:005 was used. The experimental set-up was introduced in [1]. A layout of the water tanks process is shown in Figure 1. The main components are two lower tanks, two upper tanks and two pumps. Pump 1 delivers water into Tank 1 and Tank 4, while pump 2 delivers water to Tank 2 and Tank 3. Pressure sensors are located at the bottom of each tank. The signals from the sensors in the two lower tanks are the output signals of the process. They provide information about the water levels. The input signals are the voltages applied to the two pumps.

The purpose of this study is to exemplify the applications oriented input design framework. The purpose is *not* to investigate the water tanks process nor MPC control per se. The considered water tanks process is well-studied, see [1]. A physical model can easily be derived from first principles and its parameters are not difficult to identify. However, in this study we want to identify the model parameters in an automated fashion *and* with the intended

application in mind. We will see that the input design, without any meddling from the user, suggests an input signal that is well-motivated from our knowledge of the plant. The designed input signal highlights the process dynamics that are of importance for the application and hides the unimportant one.

The theoretical background in this study is sparse. For more details on system identification and applications oriented input design see for example [2], [3], [4], and the references therein. For an introduction to MPC, see [5].

The softwares used are `Matlab`, `Simulink` and `cvx`. `cvx` is a package for specifying and solving convex programs [6], [7]. The solver used in `cvx` is set to `sdpt3` [8].

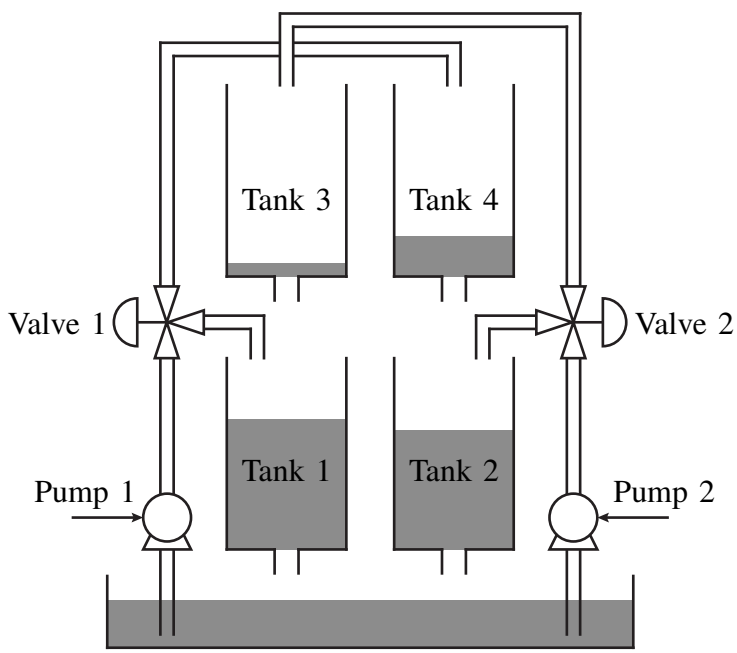

Figure 1. Water tanks process. Water is pumped from the basin into the four tanks. The flow from pump 1 fills Tanks 1 and 4 while the flow from pump 2 fills Tanks 2 and 3. The flow is divided between the tanks according to the settings of the two valves.

Water tanks model

Standard MPC uses a linear and discrete time model. The water tanks model used in this study is

$$\mathcal{M}(\theta): \quad x(t+1) = A(\theta)x(t) + B(\theta)u(t), \tag{1a}$$

$$y(t) = C(\theta)x(t) + e(t), \tag{1b}$$

with

$$A = e^{A_c}, \; B = \int_0^1 e^{A_c(1-t)} B_c dt,$$

and

$$A_c = \begin{bmatrix} -\tau_1 & 0 & \tau_3 & 0 \\ 0 & -\tau_2 & 0 & \tau_4 \\ 0 & 0 & -\tau_3 & 0 \\ 0 & 0 & 0 & -\tau_4 \end{bmatrix}, \; B_c = \begin{bmatrix} \frac{k_1\gamma_1}{A} & 0 \\ 0 & \frac{k_2\gamma_2}{A} \\ 0 & \frac{k_2(1-\gamma_2)}{A} \\ \frac{k_1(1-\gamma_1)}{A} & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} l_1 & 0 & 0 & 0 \\ 0 & l_2 & 0 & 0 \end{bmatrix}, \; \tau_i = \frac{a_i}{A}\sqrt{\frac{g}{2{x_i}^o}},$$

as given in [1]. The model is derived using zero order hold sampling and a sampling rate of 1 Hz. The state vector is $x(t) = [x_1(t) \; x_2(t) \; x_3(t) \; x_4(t)]^{\mathrm{T}}$. The component x_i is the deviation of the water level of Tank i from the operating point x_i^o expressed in centimeters. The input signal is $u(t) = [u_1(t) \; u_2(t)]^{\mathrm{T}}$. The component u_j is the deviation of the voltage of pump j from the operating point expressed in volts. The output signal is $y(t) = [y_1(t) \; y_2(t)]^{\mathrm{T}}$, where y_i is the

deviation of the pressure in volts of Tank i from the operating point y_i^o affected by noise. The measurement noise $e(t) = [e_1(t)\ e_2(t)]^{\mathrm{T}}$ is assumed to be zero mean white Gaussian noise with covariance matrix Λ. The states at time t can be estimated from the measurement $y(t)$ using a Kalman filter.

The physical meaning and values of the parameters in model (1) are listed in Table I. The parameters a_1, a_2, a_3, a_4, γ_1, γ_2, k_1, k_2, and covariance matrix Λ are estimated from a long identification experiment using white noise with low power as excitation signal. The covariance matrix of the excitation signal is $0.1 I_{2\times 2}$. The number of data samples used are 3426, collected at 1 Hz. That is, the identification experiment is almost an hour long. All other parameter values in the model are taken from the process specification.

We denote the parameters θ, the true parameter values θ^0, the initial estimate θ^{init} and the parameter values estimated using N data points $\hat{\theta}_N$. They all lie in $\mathbb{R}^n$. We refer to estimates obtained using input design as *optimal estimates* and using white noise as *white estimates*.

TABLE I

PHYSICAL PARAMETERS OF THE QUADRUPLE WATER TANKS PROCESS.

Parameter	Description	Value	Standard deviation
a_1	cross sectional area of outlet of Tank 1 (cm^2)	0.0649	0.0010
a_2	cross sectional area of outlet of Tank 2 (cm^2)	0.0594	0.0021
a_3	cross sectional area of outlet of Tank 3 (cm^2)	0.3211	0.0737
a_4	cross sectional area of outlet of Tank 4 (cm^2)	0.1353	0.0245
A_i	cross sectional area of Tank $i = 1, \ldots, 4$ (cm^2)	4.9	-
γ_1	fraction of flow from pump 1 to lower tank	0.7283	0.0094
γ_2	fraction of flow from pump 2 to lower tank	0.7271	0.0079
k_1	voltage to volumetric flow rate constant of pump 1 (cm^3/s/V)	2.362	0.0370
k_2	voltage to volumetric flow rate constant of pump 2 (cm^3/s/V)	1.797	0.0377
l_i	water level to voltage constant of sensor $i = 1, 2$ (V/cm)	0.5	-
Λ	covariance matrix of measurement noise	$\begin{bmatrix} 0.0051 & 0.0010 \\ 0.0010 & 0.0142 \end{bmatrix}$	-
x^0	steady-state values of water levels (cm)	$[13.5\ 13.2\ 0.6\ 1.2]^{\mathrm{T}}$	-
u^0	steady-state values of input signals (V)	$[4.4\ 4.6]^{\mathrm{T}}$	-

Examples

In this study we describe and present results from five different examples of designed identification experiments. The examples are briefly described in Table II.

TABLE II

FIVE DIFFERENT TYPES OF DESIGNED IDENTIFICATION EXPERIMENTS.

Examples	Description
Example 1	The parameters to be estimated are $\theta = [\gamma_1 \ k_1]^{\mathrm{T}}$. The intial estimate is $\theta^{\text{init}} = [0.7 \ 1.6]^{\mathrm{T}}$. The MPC is without integral action.
Example 2	The parameters to be estimated are $\theta = [a_1 \ a_2 \ a_3 \ a_4 \ \gamma_1 \ \gamma_2 \ k_1 \ k_2]^{\mathrm{T}}$. The intial estimate is $\theta^{\text{init}} = [0.03 \ 0.03 \ 0.5 \ 0.5 \ 0.7 \ 0.7 \ 1.6 \ 1.6]^{\mathrm{T}}$. The MPC is without integral action.
Example 3	The parameters to be estimated are $\theta = [a_1 \ a_2 \ a_3 \ a_4 \ \gamma_1 \ \gamma_2 \ k_1 \ k_2]^{\mathrm{T}}$. The intial estimate is $\theta^{\text{init}} = [0.03 \ 0.03 \ 0.05 \ 0.05 \ 0.7 \ 0.7 \ 1.6 \ 1.6]^{\mathrm{T}}$. The MPC is without integral action.
Example 4	The parameters to be estimated are $\theta = [\gamma_1 \ k_1]^{\mathrm{T}}$. The intial estimate is $\theta^{\text{init}} = [0.7 \ 1.6]^{\mathrm{T}}$. The MPC is with integral action.
Example 5	The parameters to be estimated are $\theta = [a_1 \ a_2 \ a_3 \ a_4 \ \gamma_1 \ \gamma_2 \ k_1 \ k_2]^{\mathrm{T}}$. The intial estimate is $\theta^{\text{init}} = [0.03 \ 0.03 \ 0.05 \ 0.05 \ 0.7 \ 0.7 \ 1.6 \ 1.6]^{\mathrm{T}}$. The MPC is with integral action.

Application cost

For more details of the concepts introduced in this section, see for example [3] and [4].

We use MPC to control the water tanks process. The objective of the controller is to perform reference tracking of the water levels in the two lower tanks. The reference signals are shown in Figure 2. The signals are deviations from the steady state values measured in centimeters.

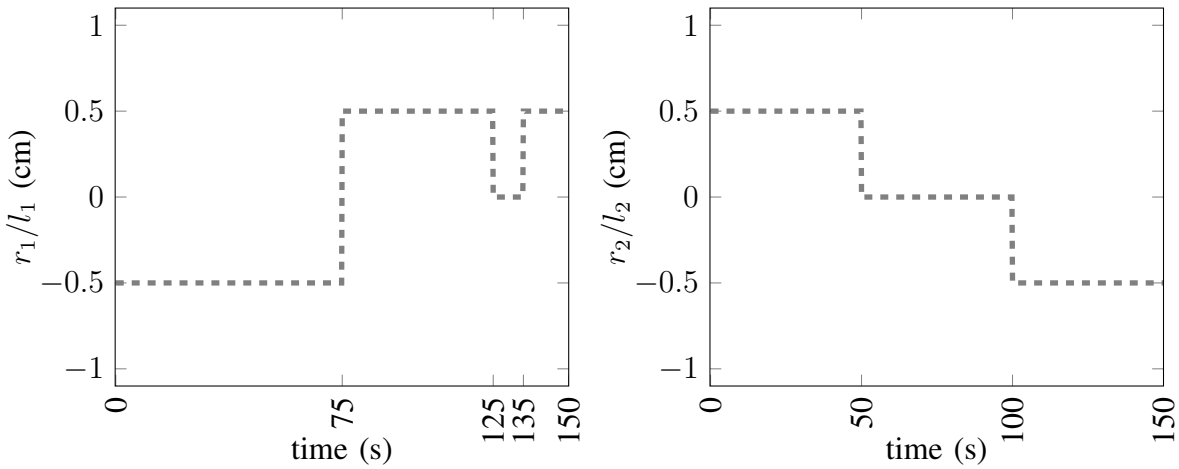

Figure 2. References for output signals. The reference signals r_1/l_1 and r_2/l_2 are shown.

The performance of the controller improves when the model is able to predict the true output of the system more accurately. Thus, an application cost that punishes the output error is chosen. That is,

$$V_{\text{app}}(\theta) = \frac{1}{N_{\text{app}}} \sum_{t=1}^{N_{\text{app}}} \frac{1}{2} \|y(t,\theta) - y(t,\theta^0)\|^2, \tag{2}$$

with $N_{\text{app}} = 150$. Here, $y(t,\theta)$ is the closed-loop output signal at time t using an MPC model with parameter values equal to the values of θ. Thus, $y(t,\theta^0)$ is the closed-loop output signal at

time t using an MPC model based on the true parameter values.

The application set is

$$\Theta_{\text{app}}(\gamma) = \left\{ \theta \ | V_{\text{app}}(\theta) \leq \frac{1}{\gamma} \right\}, \tag{3}$$

and the ellipsoidal approximation of the application set is

$$\mathcal{E}_{\text{app}}(\gamma) = \left\{ \theta \ | \frac{1}{2} \ (\theta - \theta^0)^{\mathrm{T}} V''_{\text{app}}(\theta^0)(\theta - \theta^0) \leq \frac{1}{\gamma} \right\}. \tag{4}$$

We refer to the inequality in (3) as the *application requirement* and to the inequality in (4) as the *second order approximation of the application requirement*.

The upper bound on performance degradation, γ, is chosen as

$$\gamma = \frac{100}{V(\theta^0, r(t))},$$

with

$$V(\theta^0, r(t)) = \frac{1}{N_{\text{app}}} \sum_{t=1}^{N_{\text{app}}} \frac{1}{2} \|r(t) - y(t, \theta^0)\|^2,$$

where $r(t) = [r_1(t) \ r_2(t)]^{\mathrm{T}}$ is the reference signal. Meaning, we allow a performance degradation of 1% from the performance level obtained using the true parameters in the model. The value of γ in the different experiments are shown in Table III. The value of γ varies since different initial estimates of θ^0 and different MPC controllers are used.

TABLE III

VALUE OF γ IN THE DIFFERENT EXAMPLES.

Examples	Example 1	Example 2	Example 3	Example 4	Example 5
Value of γ	1.8×10^4 V^{-2}	1.6×10^4 V^{-2}	1.8×10^4 V^{-2}	2.2×10^4 V^{-2}	2.2×10^4 V^{-2}

To summarize, the estimated model is to be used in an MPC that performs reference tracking. We want the closed-loop output signal, when using the estimated model in the MPC, to be close to the output signal obtained when using the true system as model in the MPC. How *close* the signals should be to each other and what measure to use is set by the value of γ and the definition of the application cost (2).

Control strategy

The controller implemented is an MPC. The input is calculated using the cost function

$$J(t)=\sum_{i=0}^{N_y}\|\hat{y}(t+i|t)-r(t+i)\|_{Q_y}^2+\sum_{i=0}^{N_u}\|\hat{u}(t+i|t)-r_u(t+i)\|_{Q_u}^2+\sum_{i=0}^{N_u}\|\hat{u}(t+i+1|t)-\hat{u}(t+i|t)\|_{Q_{\Delta_u}}^2,$$

and solving the optimization problem

$$\begin{aligned} &\underset{u(t)}{\text{minimize}}\ J(t), \\ &\text{subject to } \hat{y}\in\mathcal{Y}, \\ &\hat{u}\in\mathcal{U}, \end{aligned}$$

where $\hat{y}$ and $\hat{u}$ are estimated using the model (1), measurements and state estimates. We set $N_y = N_u = 10$, $Q_y = 100I_{2\times2}$, $Q_u = 10I_{2\times2}$,

$$\mathcal{U} = \{-3.9 \le u_1 \le 5.62,\ -4.1 \le u_2 \le 5.4, -2.0 \le \Delta u_1 \le 2.0,\ -2.0 \le \Delta u_2 \le 2.0\}$$

and

$$\mathcal{Y} = \{-13.5 \le y_1 \le 1.5,\ -13.2 \le y_2 \le 1.9,\ -0.6 \le y_3 \le 14.4,\ -1.2 \le y_4 \le 13.8\}.$$

Here, Δu_i denotes the difference between two consecutive values of u_i. In Example 1-3, we set $Q_{\Delta_u} = 20I_{2\times2}$ and we do not have integral action. In Example 4 and 5, we set $Q_{\Delta_u} = 0I_{2\times2}$ and we do have integral action.

The constraints on $\hat{u}$ and $\hat{y}$ correspond to the physical limitations of the process. The maximum allowed pump voltage is 10 V. The minimum allowed voltage is 0.5 V. The minimum voltage is nonzero so that the tubes are always filled with water. The maximum allowed water level is to prevent overflow of the tanks. The allowed level is approximately five centimeters lower than the water tank edges. The minimum allowed water tank level is zero, that is we allow the tanks to be empty.

The input reference signal, r_u, is set to the input signal that brings the system to r at steady state, see [9].

When running the MPC application on the process, we use the second order low pass filter

$$G_{\text{filter}}(s) = \begin{bmatrix} \frac{1}{0.25s^2+s+1} \\ \\ \frac{1}{0.25s^2+s+1} \end{bmatrix}$$

on the measured output signal. The filter is the same as the one used on process SN:005 in the laboratory exercises in the engineering program at the Department of Automatic Control, Lund University. When simulating the water tanks process, we have no noise so no filter is needed.

Identification experiments

The number of samples used in the identification experiments is denoted N. For a sufficiently large N, the estimates $\hat{\theta}_N$ are contained inside

$$\mathcal{E}_{\text{SI}} = \left\{ \theta \mid (\theta - \theta^0)^{\text{T}} \bar{\mathbf{I}}_{\text{F}} (\theta - \theta^0) \leq \frac{\chi^2_\alpha(n)}{N} \right\},$$

with probability α. Here, $\chi^2_\alpha(n)$ is the α-percentile of the χ^2-distribution with n degrees of freedom and $\bar{\mathbf{I}}_{\text{F}}$ is the average information matrix. We call $\mathcal{E}_{\text{SI}}$ the identification ellipsoid. We

choose $\alpha = 0.95$ in the χ^2-distribution and the experiment length to 300 samples ($N = 300$) in all the examples. That is, the identification experiments are five minutes long. For more details on system identification, see [2].

Spectrum of input signal

We define the spectrum of the input signal as an FIR spectrum. The spectrum is denoted $\Phi_{ij}(\omega)$ for input signal j and output signal i. We use finite dimensional parametrization and the KYP lemma to obtain tractable convex optimization problems in the applications oriented input design framework. The objective of the optimization problems is to minimize the cost of the identification experiment. The cost is defined as the power of the excitation signal in all five examples. The decision variables of the optimization problems are the optimal input spectrum parameters, denoted $c_0, \ \ldots, \ c_M$ for some user-specified M. If $M = 1$, the spectrum is forced to be flat. That is, the input signal is white noise. For more details on the spectrum design, see for example [3] and [4].

Validation of method

The applications oriented input design method is validated in several ways:

- We check that the initial estimates used in the design do not already fulfill the application cost when evaluated in simulation using the true parameter values (the values obtained from the long identification experiment and the process specification).
- We compare two types of application ellipsoids. The first type is designed based on the initial estimates and the second type is designed based on the true parameter values.

- We discuss which parameters that are important for the application. The *importance* of a parameter is evaluated based on the possible range of its value. The range is defined as both the absolute and relative deviation from the true parameter value. However, this only gives a hint of the important directions in the parameter space. In general it is difficult to rank the parameters in order of importance for $n > 3$.
- We compare two types of identification ellipsoids. The first type is designed using a system based on the initial estimates. The second type is designed using a system based on the true parameter values.
- We check if the optimal estimates fulfill the application cost in simulation. The application cost is evaluated on the linearized system given by (1) based on the initial estimates and based on the true parameter values.
- We check if the optimal estimates fulfill the second order approximation of the application requirement. Also here, the application cost is evaluated on the linearized system based on the initial estimates and the true parameter values.
- We check if the optimal estimates fulfill the application cost when evaluated on the process. That is, we run the application with a model based on the true parameter values and with a model based on the optimal estimates. We then calculate the application cost. Of course, here we have noise present which we do not have in simulation and the noise realization differs between the two runs. Consequently, we cannot expect the application requirement to be fulfilled as stated. However, we can compare the result obtained from the optimal estimates and from the white estimates.
- We do the same comparisons as above but for estimates using a white input signal. We use the same experiment length as in the optimal experiments, that is $N = 300$. This length

is, according to theory, much smaller than the experiment length required for the white estimates to have the same statistical properties as the optimal estimates.

- We compare the performance of the optimal estimates and the white estimates. That is, we compare the values of the different versions of application costs stated above.
- We compare four types of experiment lengths. The first experiment length is the user-defined N on which the optimal input spectrum is based. The remaining experiments lengths are the minimum lengths that, theoretically, give the same statistical properties of the estimates as the first experiment length. The second experiment length is the minimum length obtained if the input signal is defined as white noise of equal power to the optimal input signal and the parameters are set to the initial estimates in the design. The third is the minimum experiment length obtained when using the optimal spectrum parameters and the true parameter values in the design. The fourth is the minimum experiment length obtained if the input signal is defined as white noise of equal power to the optimal input signal and the parameters are set to the true parameter values in the design.

Example 1 - Estimating $\theta = [\gamma_1 \ k_1]^{\mathrm{T}}$ with no integral action in the MPC

We want to estimate the values of γ_1 and k_1, that is $\theta = [\gamma_1 \ k_1]^{\mathrm{T}}$. All other parameter values are set as specified in Table I. We use an MPC with no integral action.

Check of initial estimates

We use as initial estimates the values from the process specification, that is $\theta^{\text{init}} = [0.7 \ 1.6]^{\mathrm{T}}$. From experience of this particular process, we know that these parameter values have changed over time. Before performing the input design, we check if θ^{init} fulfills the application requirement. That is, we evaluate the application cost in simulation using a system based on θ^0 and an MPC model based on θ^{init}. In this check, we get that θ^{init} does not fulfill the application requirement. Meaning, we can continue with the example.

Application ellipsoid

We calculate two types of application ellipsoids. The first type is evaluated in simulation using a system and MPC model based on θ^{init}. The second type is evaluated in simulation using a system and MPC model based on θ^0. The ellipsoids are depicted in Figure 3. We see that, in terms of allowed absolute deviation, it is more important to estimate γ_1 accurately than k_1 for the considered application. That is, to fulfill the application requirement it is more important to know the division of water between Tank 1 and Tank 4, than to know the voltage to volumetric flow rate constant of pump 1. This is intuitive since the MPC can more easily compensate for an error in k_1 than an error in γ_1. The parameter k_1 is only present in the MPC model as a scaling

factor of u_1, while the presence of γ_1 is a bit more delicate, see (1). If k_1 is larger than the true values we get less water in Tank 1 and Tank 4, and consequently also in Tank 2, than predicted by the model, and the other way around. The MPC can, to a certain degree, compensate for this model error by increasing or decreasing u_1. However, if γ_1 is larger than the true values we get less water in Tank 1, more water in Tank 4 and also more water in Tank 2, than predicted by the model, and vice versa. The MPC cannot compensate for this error by only using u_1. It needs to involve u_2, which invokes the coupling of the system. In addition we note that, although the two ellipsoids are different, they have approximately the same directions of and ratio between the semi-axes. Meaning, they both yield the interpretation that γ_1 is more important to know with high accuracy than k_1.

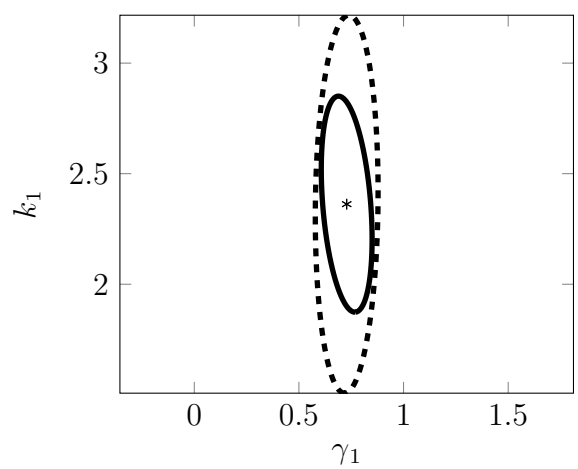

Figure 3. Example 1 – Application ellipsoids. The application ellipsoids based on θ^{init} and θ^0 are displayed as (▬) and (- - -), respectively. The true parameter values are denoted ($*$).

In Table IV, we give the range of each parameter value that may fulfill the second order approximation of the application requirement based on and centered around θ^{init}. In Table V, we give the range of each parameter value that may fulfill the second order approximation of the

application requirement when based on θ^{init} and centered around θ^0, and based on and centered around θ^0. That is, we show the largest possible offset to θ^{init} (Table IV) or θ^0 (Table V) when projecting all of the eigenvectors of the properly scaled Hessian, V''_{app}, onto each axis of the parameter space. The offset is given in absolute values and as percentage of θ^{init} (Table IV) or θ^0 (Table V). We see that it is more important to estimate γ_1 than k_1, both when considering the absolute and relative measure, and both when centered at θ^{init} and θ^0.

TABLE IV

EXAMPLE 1 – POSSIBLE RANGE OF EACH PARAMETER VALUE IN THE INITIAL DESIGN.

θ	θ^{init}	Possible range based on θ^{init}	
γ_1	0.7	0.7 ± 0.1214	$(\pm 17\%)$
k_1	1.6	1.6 ± 0.4880	$(\pm 31\%)$

TABLE V

EXAMPLE 1 – POSSIBLE RANGE OF EACH PARAMETER VALUE.

θ	θ^0	Possible range based on θ^{init}		Possible range based on θ^0	
γ_1	0.7283	0.7283 ± 0.1214	$(\pm 17\%)$	0.7283 ± 0.1500	$(\pm 21\%)$
k_1	2.362	2.362 ± 0.4880	$(\pm 21\%)$	2.362 ± 0.8550	$(\pm 36\%)$

Optimal spectrum

We calculate the optimal spectrum using 40 spectrum parameters, $c_0, \ldots, c_{39}$, where $c_i \in \mathbb{R}^{2\times 2}$. The optimal spectrum is based on θ^{init}. The spectrum turns out to give an optimal u_2 that remains at its steady state level. This makes sense since the only way for the output

signals to be influenced by γ_1 and k_1 is by varying u_1. Also, by not varying u_2, the optimal input signal effectively hides parts of the system that are unimportant for the identified model to fulfill the application requirement. The optimal spectrum is shown in Figure 4, along with the optimal spectrum based on θ^0. We see that the two spectra are similar, although the design based on θ^{init} requires a higher power. The trace of c_0 is 0.0030 when based on θ^{init} and 0.0013 when based on θ^0.

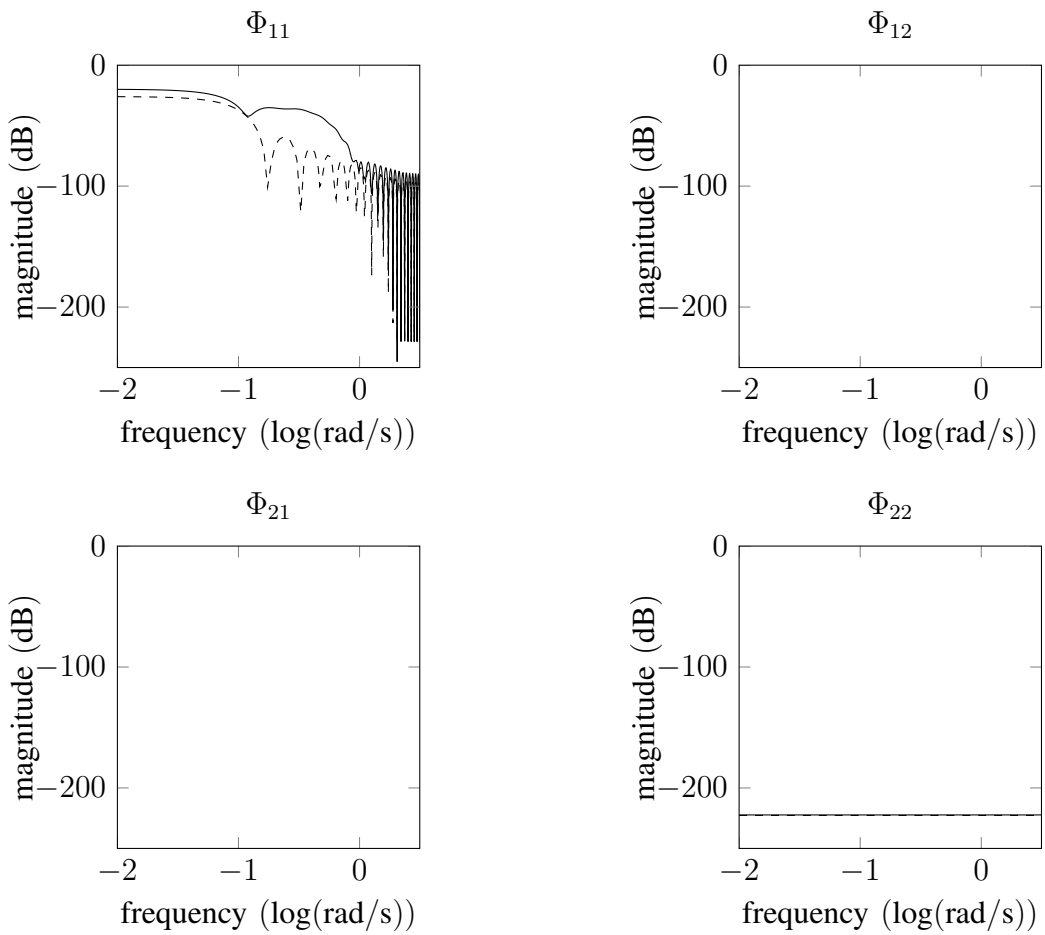

Figure 4. Example 1 – Optimal spectrum. The optimal spectra based on θ^{init} (▬) and θ^0 (▪▪▪) are shown. We see that u_1 and u_2 are uncorrelated and the variance of u_2 is numerically zero for both spectra.

Identification ellipsoid

We calculate two kinds of identification ellipsoids. The first type is designed using a system based on θ^{init}. That is, it is based on the optimal spectrum shown in Figure 4. The second type is designed using a system based on θ^0. The ellipsoids are shown in Figure 5. Note that the two ellipsoids are similar in shape and size, and that we have centered both around θ^0 for a valid comparison.

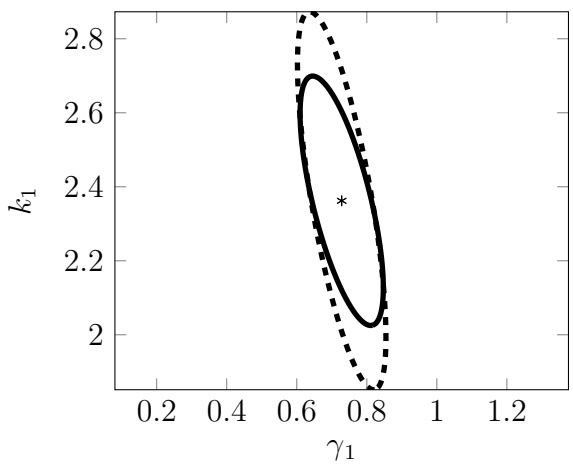

Figure 5. Example 1 – Identification ellipsoids. The identification ellipsoids based on θ^{init} and θ^0 are displayed as (▬) and (- - -), respectively. The true parameter values are denoted ($*$).

Estimates

We estimate θ using the optimal input signal and a white input signal. The white input signal has the same power as the optimal input signal, but divided equally between u_1 and u_2. We perform fifteen identification experiments for each type of signal. A new realization of the input signal is used in each experiment.

The resulting white estimates are shown in Figure 6, along with the identification ellipsoid based on the white spectrum and θ^0. The application ellipsoids based on θ^{init} and θ^0 are also shown for comparison. The white estimates are quite scattered, and only four estimates fulfill the second order approximation of the application requirement based on θ^{init}. The same four estimates also fulfill the second order approximation of the application requirement based on θ^0.

The resulting optimal estimates are shown in Figures 7 and 8. We see that the optimal estimates are more gathered than the white ones. Note also that they are spread out more in the k_1-direction than in the γ_1-direction. Meaning, the optimal estimates follow the shape of the application ellipsoid. Nine of the optimal estimates fulfill the second order approximation of the application requirement based on θ^{init}, see Figure 7. All fifteen of the optimal estimates fulfill the second order approximation of the application requirement based on θ^0, see Figure 8.

For comparison, we also estimate θ using a white input signal with its covariance equal to the optimal c_0. That is, all the power is used to excite u_1. The estimates are shown in Figures 9 and 10. We see that the estimates are still spread out. In fact, no improvement in terms of fulfilling the second order approximation of the application requirement was found.

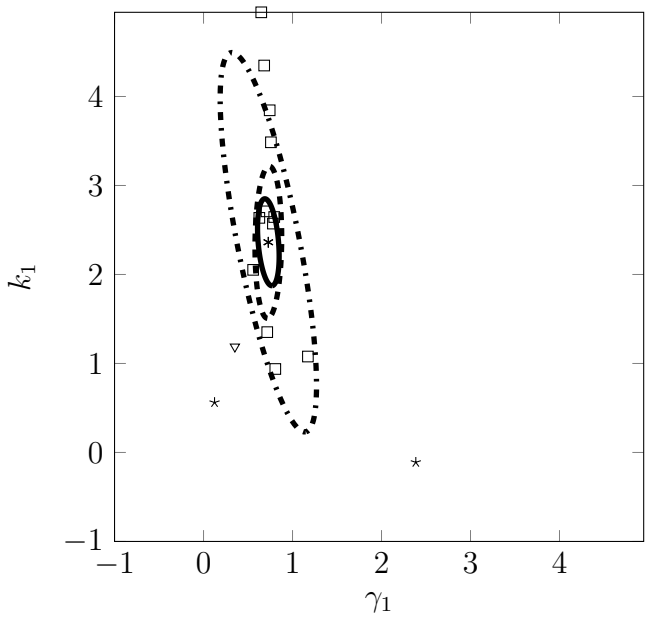

Figure 6. Example 1 – White estimates. The white estimates, denoted (□), (▽) and (⋆) are shown, along with the application ellipsoids based on θ^{init} (▬) and θ^0 (▪ ▪ ▪). Also, the identification ellipsoid based on the white spectrum and θ^0 is shown (·▪·▪). The true parameter values are denoted (∗). The white estimate denoted (▽) corresponds to Experiment 5, and gives an extremely bad application performance. The value of the application cost is more than 100 times the values for all other estimates. The white estimates denoted (⋆) correspond to Experiment 10 and 15 and cannot be evaluated in simulation (they cause simulation break-down due to non-real state values) nor on the process (they cause overflow).

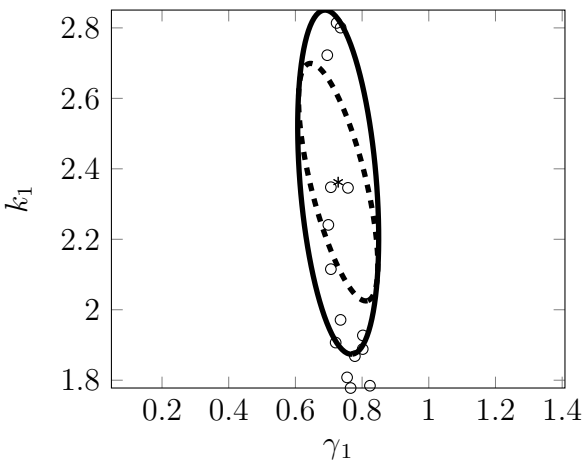

Figure 7. Example 1 – Optimal estimates and ellipsoids based on θ^{init}. The optimal estimates (○) are shown, along with the application ellipsoid (▬) and identification ellipsoid (- - -) based on θ^{init}. The true parameter values are denoted (∗).

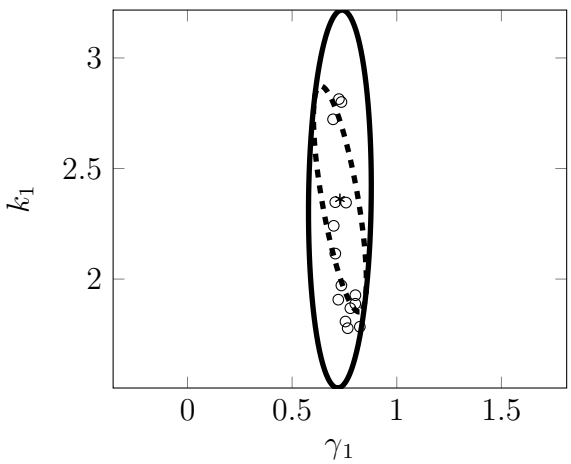

Figure 8. Example 1 – Optimal estimates and ellipsoids based on θ^0. The optimal estimates (○) are shown, along with the application ellipsoid (▬) and identification ellipsoid (- - -) based on θ^0. The true parameter values are denoted (∗).

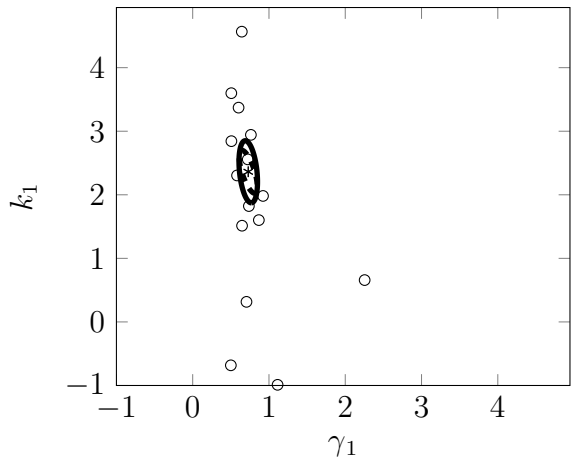

Figure 9. Example 1 – White estimates using optimal power distribution and ellipsoids based θ^{init}. The white estimates (○) are shown, along with the application ellipsoid (▬) and identification ellipsoid (▪▪▪) based on θ^{init}. The true parameter values are denoted (∗).

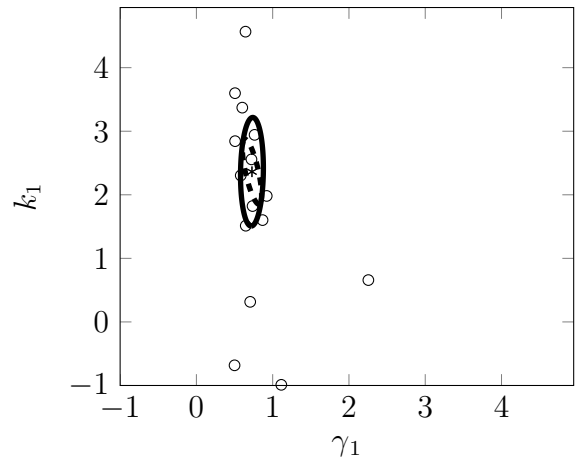

Figure 10. Example 1 – White estimates using optimal power distribution and ellipsoids based on θ^0. The white estimates (○) are shown, along with the application ellipsoid (▬) and identification ellipsoid (▪▪▪) based on θ^0. The true parameter values are denoted (∗).

Application cost

We check the application cost for all white and optimal estimates. We evaluate the application cost on the process, see Figures 11 and 12. Note that the noise increases the level of the application cost. We also evaluate the application cost in simulation based on both θ^{init} and θ^0, see Figures 13–16. (NB! The optimal experiment i is not related to the white experiment i.)

We see in Figure 12 that none of the estimates fulfill the application requirement evaluated on the process. However, the optimal estimates outperform the white estimates. In fact, two of the white estimates cause overflow in Tank 1 and are not included in the evaluation. Five of the optimal estimates and one of the white estimates fulfill the application requirement based on θ^{init} evaluated in simulation, see Figure 14. In Figure 16, we see that thirteen of the optimal estimates and three of the white estimates fulfill the application requirement based on θ^0 evaluated in simulation. The result is better in the latter case since the application cost based on θ^0 gives a larger set of acceptable parameters than the cost based on θ^{init}, see Figure 3. We conclude that

- evaluating the application cost using θ^{init},
- designing the input signal using the second order approximation of the application requirement,
- having noise present,

affect the level of degradation. However, the input design still manage to excite the important dynamics of the system and hide the unimportant dynamics. The optimal estimates in general outperform the white estimates with respect to the value of the application cost, even if the desired value is not obtained.

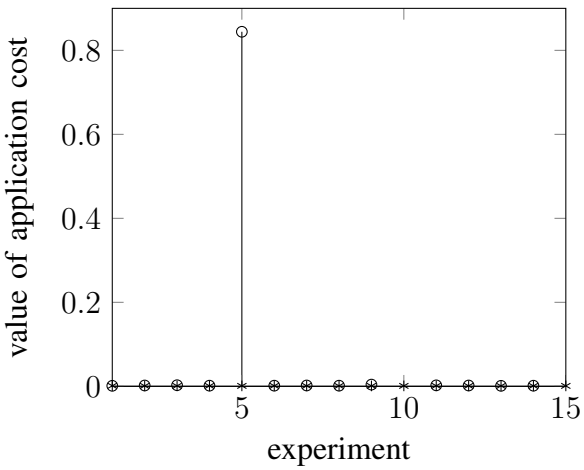

Figure 11. Example 1 – Application cost evaluated on process. The application cost for optimal estimates (∗) and white estimates (◦) are shown. Note the bad performance of the estimate from Experiment 5, also note that the application cost values for the estimates from Experiment 10 and 15 are not included since the cost could not be evaluated.

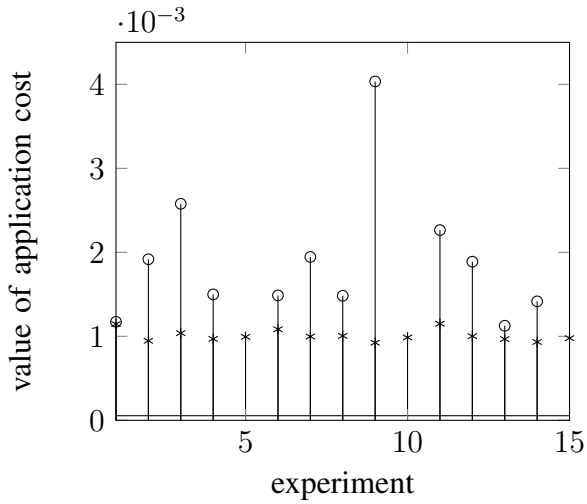

Figure 12. Example 1 – Application cost evaluated on process excluding white Experiment 5, 10 and 15. We see that the optimal estimates in general outperforms the white estimates. However, none of the estimates fulfill the requirement of an application cost lower than $1/\gamma$, denoted (▬).

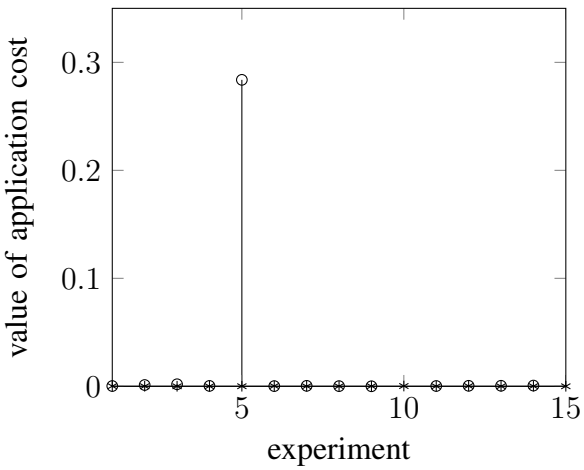

Figure 13. Example 1 – Application cost evaluated on system based on θ^{init}. The application cost for the optimal estimates ($*$) and white estimates ($\circ$) are shown. Note the bad performance of the estimate from Experiment 5, also note that the application cost values for the estimates from Experiment 10 and 15 are not included since the cost could not be evaluated.

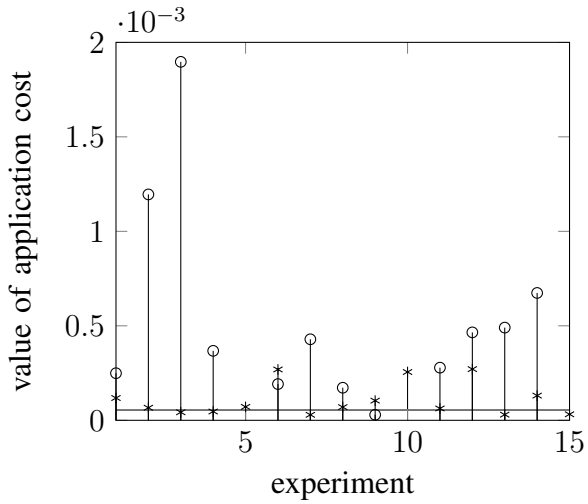

Figure 14. Example 1 – Application cost evaluated on system based on θ^{init} excluding white Experiment 5, 10 and 15. We see that the optimal estimates in general outperforms the white estimates. Five of the optimal estimates and one of the white estimates fulfill the requirement of an application cost lower than $1/\gamma$, denoted (—).

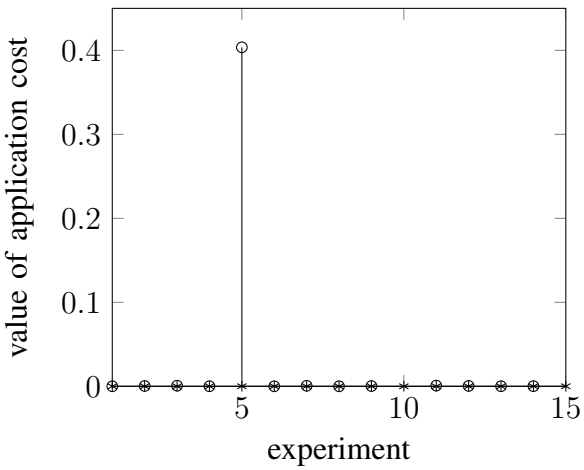

Figure 15. Example 1 – Application cost evaluated on system based on θ^0. The application cost for the optimal estimates ($*$) and white estimates ($\circ$) are shown. Note the bad performance of the estimate from Experiment 5, also note that the application cost values for the estimates from Experiment 10 and 15 are not included since the cost could not be evaluated.

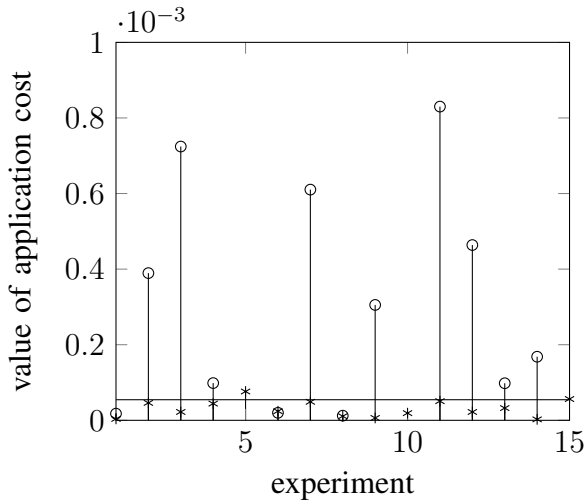

Figure 16. Example 1 – Application cost evaluated on system based on θ^0 excluding white Experiment 5, 10 and 15. We see that the optimal estimates in general outperforms the white estimates. Thirteen of the optimal estimates and three of the white estimates fulfill the requirement of an application cost lower than $1/\gamma$, denoted (—).

Signals from process

The output signals from the process when evaluating the application cost are shown in Figures 17 and 19 for the optimal estimates and Figures 18 and 20 for the white estimates. We have excluded the results from the white Experiments 10 and 15 since they caused overflow in the tanks. In Figures 21-24, we have excluded the white Experiment 5 and zoomed in for comparison. The output signals when θ^0 are used in the MPC model are shown in Figures 25 and 26. All outputs except those corresponding to the white Experiments 5, 10 and 15 fulfill the constraints imposed on the water tank levels. Note that the outputs have been scaled to centimeters instead of volt.

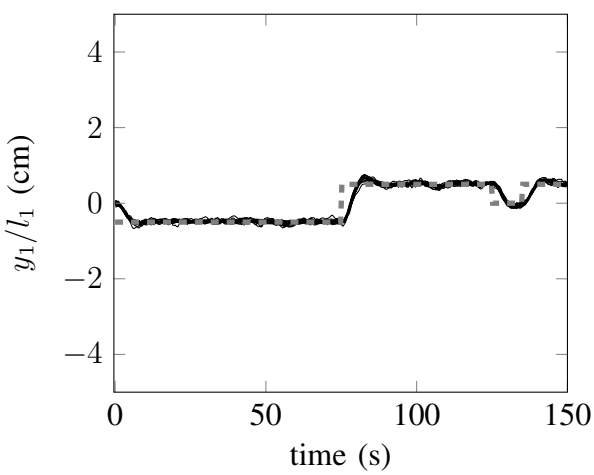

Figure 17. Example 1 – Optimal y_1/l_1 from application evaluation. The optimal output y_1/l_1 (——) from fifteen application evaluations on the process are shown. The reference signal is denoted (- - -).

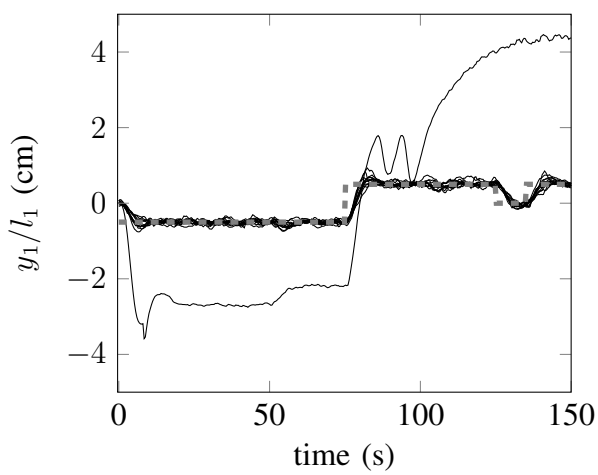

Figure 18. Example 1 – White y_1/l_1 from application evaluation. The white output y_1/l_1 (——) from thirteen application evaluations on the process are shown. The reference signal is denoted (- - -).

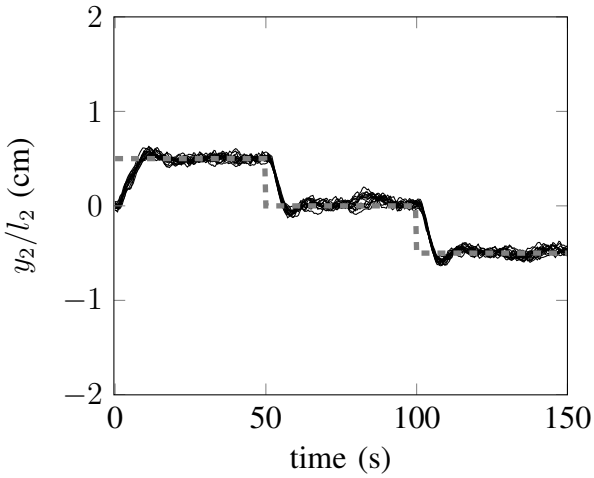

Figure 19. Example 1 – Optimal y_2/l_2 from application evaluation. The optimal output y_2/l_2 (▬) from fifteen application evaluations on the process are shown. The reference signal is denoted (- - -).

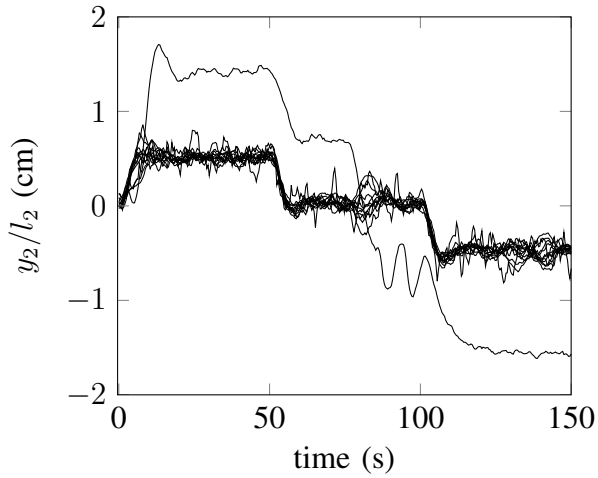

Figure 20. Example 1 – White y_2/l_2 from application evaluation. The white output y_2/l_2 (▬) from thirteen application evaluations on the process are shown. The reference signal is denoted (- - -).

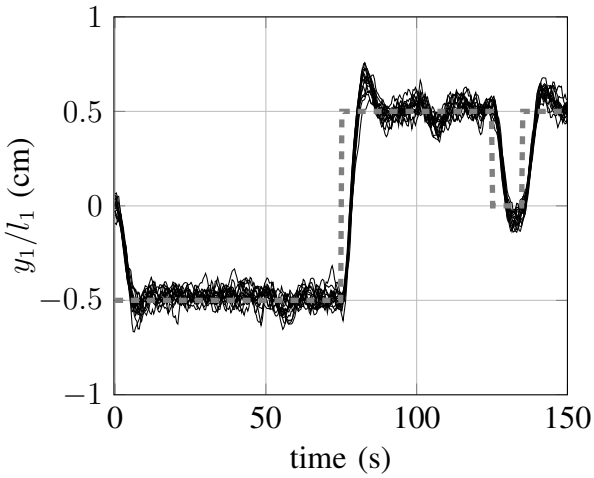

Figure 21. Example 1 – Optimal y_1/l_1 from application evaluation zoomed in. The optimal output y_1/l_1 (▬) from fifteen application evaluations on the process are shown. The reference signal is denoted (- - -).

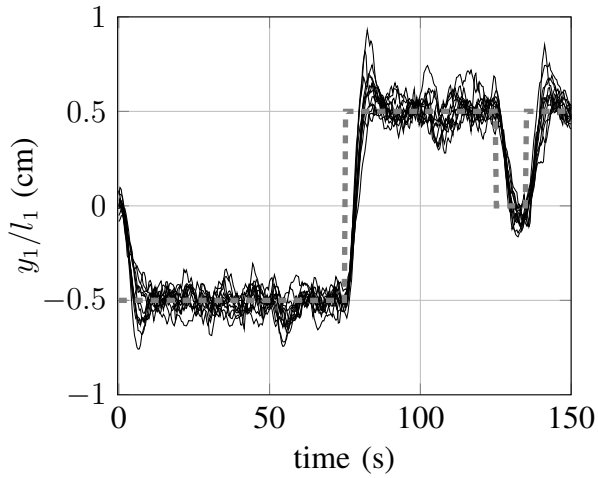

Figure 22. Example 1 – White y_1/l_1 from application evaluation excluding Experiment 5. The white output y_1/l_1 (▬) from twelve application evaluations on the process are shown .The reference signal is denoted (- - -).

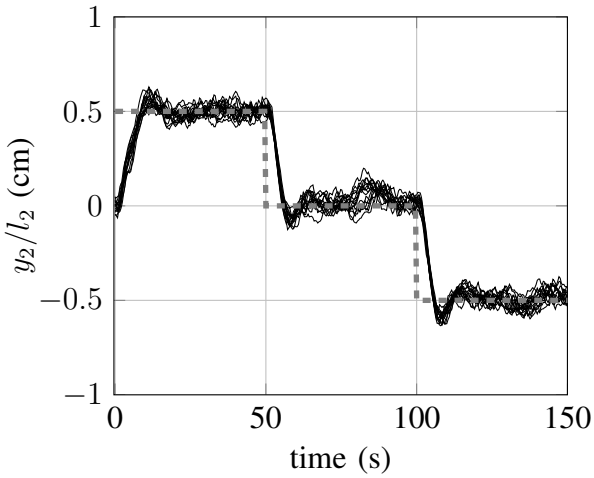

Figure 23. Example 1 – Optimal y_2/l_2 from application evaluation zoomed in. The optimal output y_2/l_2 (▬) from fifteen application evaluations on the process are shown. The reference signal is denoted (- - -).

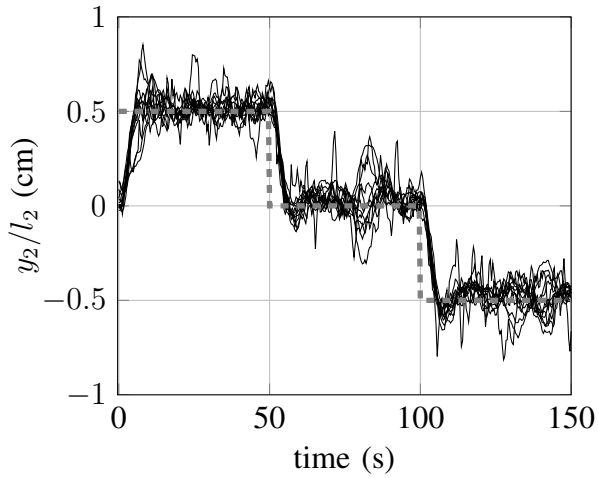

Figure 24. Example 1 – White y_2/l_2 from application evaluation excluding Experiment 5. The white output y_2/l_2 (▬) from twelve application evaluations on the process are shown. The reference signal is denoted (- - -).

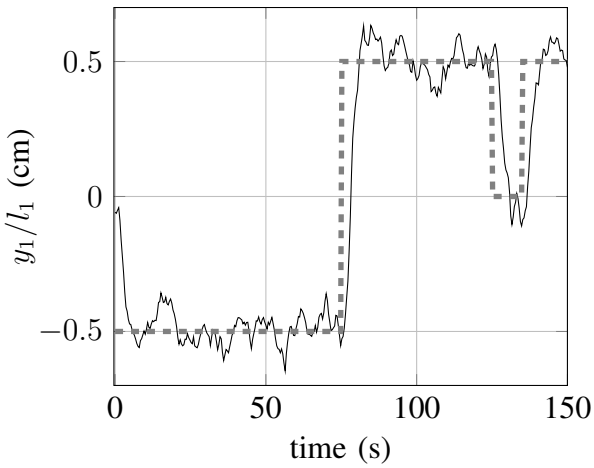

Figure 25. Example 1 – True y_1/l_1 used in application evaluation. The output y_1/l_1 (▬) is obtained from the process using θ^0 in the MPC model. The reference signal is denoted (- - -).

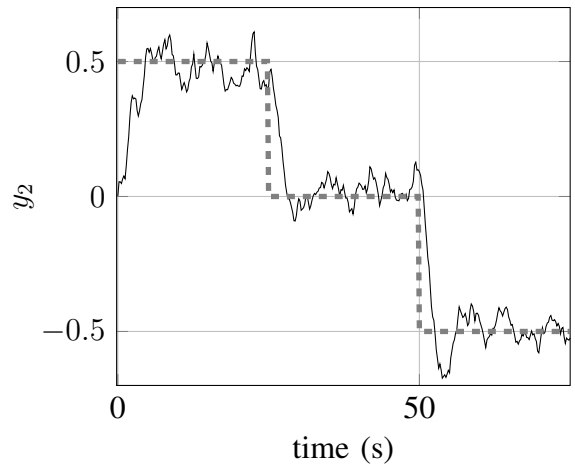

Figure 26. True y_2/l_2 used in application evaluation. The output y_2/l_2 (▬) is obtained from the process using θ^0 in the MPC model. The reference signal is denoted (- - -).

Experiment length

We evaluate four different experiment lengths. The first length, N, is defined by the user. We set it to correspond to a five minutes long identification experiment (300 samples). This is the length used to calculate the optimal spectrum. The second length is the minimum experiment length necessary for the optimal input signal to achieve the desired statistical properties of the resulting estimates when using a system based on θ^0 instead in the optimization problem. We denote the second length N^{true}. The third length is the minimum experiment length necessary for the white input signal to achieve the same statistical properties of the resulting estimates as with the optimal input signal. We calculate the length using a system based on θ^{init} in the optimization problem and denote it $N_{\text{white}}^{\text{initial}}$. The fourth length is the same as $N_{\text{white}}^{\text{initial}}$ but the system is based on θ^0 instead in the optimization problem and it is denoted $N_{\text{white}}^{\text{true}}$.

The values of the experiment lengths are shown in Table VI. The identification ellipsoids corresponding to the different experimental lengths are shown in Figure 27 and 28.

TABLE VI

EXAMPLE 1 – EXPERIMENT LENGTHS.

Experiment length	Value in samples	Value in minutes
N	300	5
N^{true}	131	2
$N_{\text{white}}^{\text{initial}}$	12678	211
$N_{\text{white}}^{\text{true}}$	5513	92

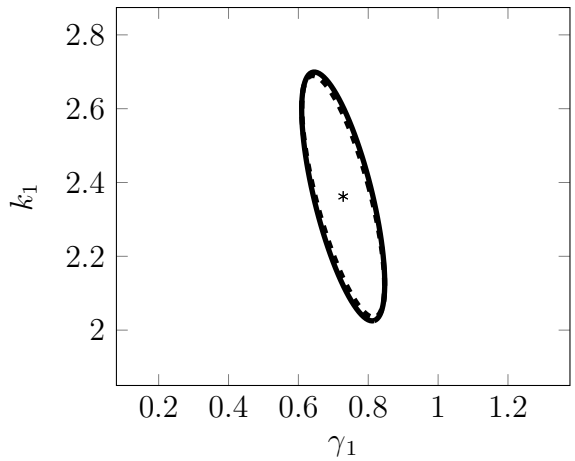

Figure 27. Example 1 – Identification ellipsoids based on N and $N_{\text{white}}^{\text{initial}}$. The identification ellipsoids for the optimal (▬) and white (- - -) spectrum are shown. The two ellipsoids are approximately the same. We achieve the same statistical properties of the white estimates as for the optimal estimates if we elongate the experiment length to $N_{\text{white}}^{\text{initial}}$.

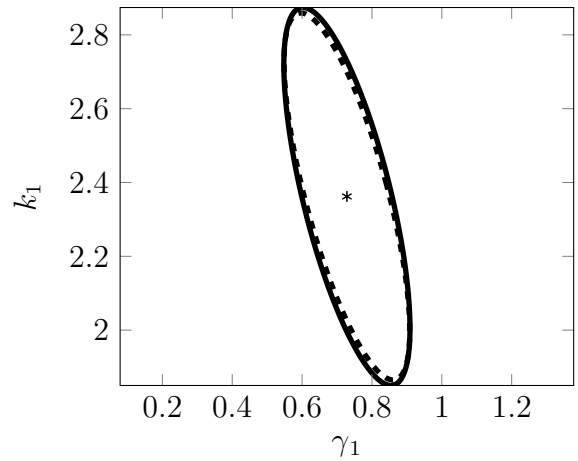

Figure 28. Example 1 – Identification ellipsoids based on N^{true} and $N_{\text{white}}^{\text{true}}$. The identification ellipsoids for the optimal spectrum (▬) and white spectrum (- - -) are shown. Also here the two ellipsoids are approximately the same. The ellipsoids are larger than the ones in Figure 27 due to a larger applications ellipsoid, see Figure 3.

We see that the white input signal requires, according to theory, a longer experiment length to achieve the same performance as the optimal input signal. The white input signal is approximately 42 times longer than the optimal input signal in both cases. It is reasonable that the white input signal requires a longer experiment than the optimal input signal. We divide the input power equally between u_1 and u_2. Meaning, only half of the power is used to excite

the input that matters, u_1. The other half is wasted on u_2. We must elongate the experiment to compensate for using less power in u_1 and for exciting unimportant dynamics of the system. Note also that $N_{\text{white}}^{\text{initial}}$ and N are larger than $N_{\text{white}}^{\text{true}}$ and N^{true}, respectively, which relates to the fact that the application ellipsoid based on θ^0 is larger than the application ellipsoid based on θ^{init}, see Figure 3.

Example 2 - Estimating $\theta = [a_1\ a_2\ a_3\ a_4\ \gamma_1\ \gamma_2\ k_1\ k_2]^{\mathrm{T}}$ with no integral action in the MPC

In Example 2, we estimate $\theta = [a_1\ a_2\ a_3\ a_4\ \gamma_1\ \gamma_2\ k_1\ k_2]^{\mathrm{T}}$. All other parameter values are set as specified in Table I. We use an MPC with no integral action. We estimate 20 spectrum parameters instead of 40 as in Example 1. This is to avoid numerical issues when solving the optimization problem that determines the spectrum parameters. For 40 parameters `cvx` returns an inaccurate solution that only satisfies a relaxed tolerance level. This is not the case for 20 parameters.

Check of initial estimates

We use as initial estimates the values from the process specification slightly modified, that is $\theta^{\text{init}} = [0.03\ 0.03\ 0.5\ 0.5\ 0.7\ 0.7\ 1.6\ 1.6]^{\mathrm{T}}$. The original process specification sets $a_3 = a_4 = 0.03$. We modify θ^{init} to circumvent numerical issues when constructing the information matrix in the optimization problem that determines the spectrum parameters. More specifically, `Matlab` fails to solve a discrete-time Lyapunov equation. Before performing the input design, we check if θ^{init} fulfills the application requirement. In this check, we get that θ^{init} does not fulfill the application requirement.

Application ellipsoid

As in Example 1, we calculate two types of application ellipsoids. The first type is evaluated in simulation using a system and MPC model based on θ^{init}. The second type is

evaluated in simulation using a system and MPC model based on θ^0.

In Tables VII and VIII, we give the range of each parameter value that may fulfill the second order approximation of the application requirement centered at θ^{init} and θ^0, respectively.

We see that the ranges of a_3, a_4 and γ_1 are larger based on θ^{init} than on θ^0, both in terms of absolute and relative deviation. Consequently, the input design might not give estimates of these values that are accurate enough for the application in mind.

In terms of absolute deviation, we conclude that we can rank the parameters in order of increasing importance as a_3, a_4, k_1, k_2, γ_2, γ_1, a_2 and a_1, when the design is based on θ^{init}. The ranking based on θ^0 is a_3, k_1, k_2, a_4,γ_2, γ_1, a_2 and a_1. The possible ranges based on θ^0 confirm the least and most important parameters.

The order of increasing importance is different in terms of relative deviation than absolute deviation. The ranking based on relative deviation and the initial design centered at θ^{init} (Table VII) is a_3, a_4, a_2, k_1, k_2, a_1, γ_2 and γ_1. The ranking based on relative deviation, θ^{init} and centered at θ^0 (Table VIII) is a_4, a_3, k_2, k_1, a_2, a_1, γ_2 and γ_1. The ranking based on relative deviation, θ^0 and centered at θ^0 (Table VIII) is a_3, a_4, a_2, k_1, k_2, a_1, γ_2 and γ_1.

Despite the different rankings, they all state that a_1 and a_2 are more important than a_3 and a_4, and γ_1 and γ_2 are more important than k_1 and k_2 to estimate with higher accuracy. As in Example 1, the MPC can compensate for an error in k_1 using u_1, and by the same reasoning it can compensate for an error in k_2 using u_2. All other parameters require both input signals to compensate for the errors. The outlet areas a_1 and a_2 are crucial for the application requirement. Tank 1 and Tank 2 have the highest water levels of the tanks and consequently give the largest

outflow of water. An error in these areas affects the tracking capability of the MPC.

TABLE VII

EXAMPLE 2 – POSSIBLE RANGE OF EACH PARAMETER VALUE IN THE INITIAL DESIGN.

θ	θ^{init}	Possible range based on θ^{init}	
a_1	0.03	0.03 ± 0.0240	$(\pm 80\%)$
a_2	0.03	0.03 ± 0.0330	$(\pm 110\%)$
a_3	0.5	0.5 ± 7.7117	$(\pm 1542\%)$
a_4	0.5	0.5 ± 4.6647	$(\pm 933\%)$
γ_1	0.7	0.7 ± 0.2200	$(\pm 31\%)$
γ_2	0.7	0.7 ± 0.2271	$(\pm 32\%)$
k_1	1.6	1.6 ± 1.4212	$(\pm 89\%)$
k_2	1.6	1.6 ± 1.3383	$(\pm 84\%\ )$

TABLE VIII

EXAMPLE 2 – POSSIBLE RANGE OF EACH PARAMETER VALUE.

θ	θ^0	Possible range based on θ^{init}		Possible range based on θ^0	
a_1	0.0649	0.0649 ± 0.0240	$(\pm 37\%)$	0.0649 ± 0.0447	$(\pm 69\%)$
a_2	0.0594	0.0594 ± 0.0330	$(\pm 56\%)$	0.0594 ± 0.0565	$(\pm 95\%)$
a_3	0.3211	0.3211 ± 7.7117	$(\pm 2401\%)$	0.3211 ± 3.5924	$(\pm 1119\%)$
a_4	0.1353	0.1353 ± 4.6647	$(\pm 3447\%)$	0.1353 ± 0.3261	$(\pm 241\%)$
γ_1	0.7283	0.7283 ± 0.2200	$(\pm 30\%)$	0.7283 ± 0.1436	$(\pm 20\%)$
γ_2	0.7271	0.7271 ± 0.2271	$(\pm 31\%)$	0.7271 ± 0.2776	$(\pm 38\%)$
k_1	2.362	2.362 ± 1.4212	$(\pm 60\%)$	2.362 ± 2.1284	$(\pm 90\%)$
k_2	1.797	1.797 ± 1.3383	$(\pm 74\%\ )$	1.797 ± 1.4238	$(\pm 79\%)$

Optimal spectrum

We calculate the optimal spectrum using 20 parameters $c_0, \ldots, c_{19}$, where $c_i \in R^{2\times 2}$. The optimal spectrum is shown in Figure 29, along with the optimal spectrum based on θ^0. We see that the spectra differ from each other at lower frequencies. The design based on θ^{init} requires a higher power, trace$(c_0) = 0.0639$, than the design based on θ^0, trace$(c_0) = 0.0368$.

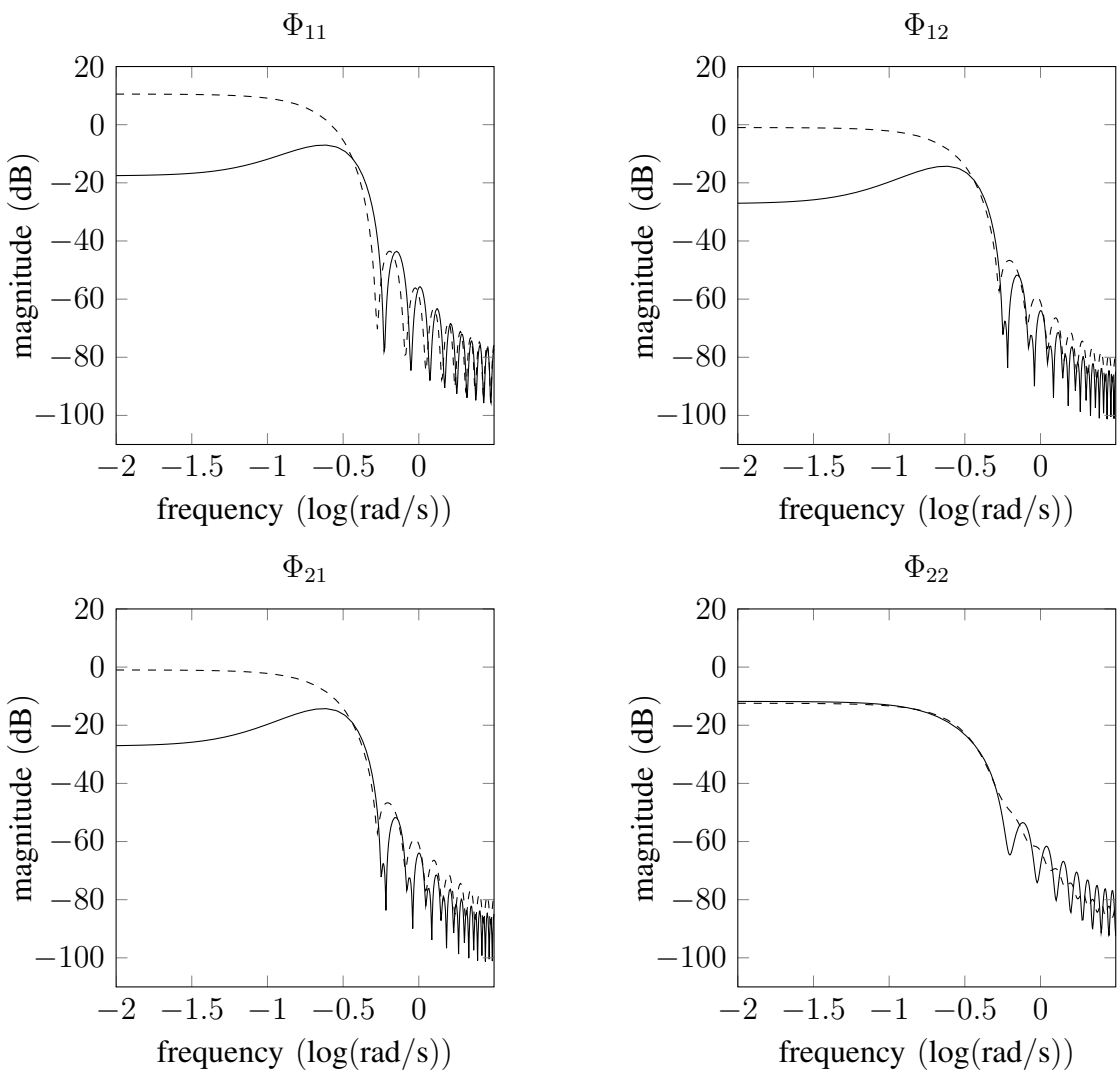

Figure 29. Example 2 – Optimal spectrum. The optimal spectrum based on θ^{init} (▬) and θ^0 (- - -) are shown. We see that u_1 and u_2 are correlated.

Identification ellipsoid

We calculate two kinds of identification ellipsoids. The first is based on the optimal spectrum and θ^0. The second is based on a white spectrum and θ^0. The white spectrum corresponds to a white excitation signal of equal power to the optimal input signal.

In Table IX, we give the possible ranges of each estimated parameter value according to the identification ellipsoids. That is, we show the largest possible offset to θ^0 when projecting all of the eigenvectors of the properly scaled information matrix onto each axis of the parameter space.

We can rank the parameters in order of increasing importance. In terms of absolute deviation both spectra give the same ranking, that is, a_3, k_2, k_1, a_4, γ_2, γ_1, a_2 and a_1.

In terms of relative deviation the optimal spectrum gives a_3, a_4, a_2, k_2, k_1/a_1, γ_2 and γ_1, that is k_1 and a_1 are allowed equal relative deviation. The white spectrum gives a_3, a_4, a_2, k_2, k_1, a_1, γ_2 and γ_1.

The rankings differ from those obtained from the application ellipsoids. However, they all state that a_1 and a_2 are more important than a_3 and a_4, and γ_1 and γ_2 are more important than k_1 and k_2 to estimate with higher accuracy. We also note that the range of a_3 is larger based on the optimal spectrum than on the white spectrum, but the volume of the white identification ellipsoid is 357 times larger than the volume of the optimal identification ellipsoid. Suggesting that the white estimates will be more scattered than the optimal ones.

TABLE IX

EXAMPLE 2 – POSSIBLE RANGE OF EACH ESTIMATED PARAMETER VALUE.

θ	θ^0	Possible range based on θ^0 and optimal spectrum		Possible range based on θ^0 and white spectrum	
a_1	0.0649	0.0649 ± 0.0086	$(\pm 13\%)$	0.0649 ± 0.0179	$(\pm 28\%)$
a_2	0.0594	0.0594 ± 0.0172	$(\pm 29\%)$	0.0594 ± 0.0327	$(\pm 55\%)$
a_3	0.3211	0.3211 ± 1.876	$(\pm 584\%)$	0.3211 ± 1.755	$(\pm 547\%)$
a_4	0.1353	0.1353 ± 0.1930	$(\pm 143\%)$	0.1353 ± 0.5357	$(\pm 396\%)$
γ_1	0.7283	0.7283 ± 0.0602	$(\pm 8\%)$	0.7283 ± 0.1361	$(\pm 19\%)$
γ_2	0.7271	0.7271 ± 0.0789	$(\pm 11\%)$	0.7271 ± 0.1649	$(\pm 23\%)$
k_1	2.362	2.362 ± 0.3064	$(\pm 13\%)$	2.362 ± 0.7065	$(\pm 30\%)$
k_2	1.797	1.797 ± 0.4600	$(\pm 26\%)$	1.797 ± 0.7152	$(\pm 40\%)$

Estimates

As in Example 1, we estimate θ using the optimal input signal and a white input signal. We perform ten identification experiments for each type of signal.

It turns out that it is difficult to estimate reasonable values of a_3 using the optimal signal. In six of the ten estimates, a_3 is larger than 10^5. We believe that this is due to the small water tank level in steady state of Tank 3. The water flow out of Tank 3 is not limited by the outlet area as it would have been had the water level been higher. Three of the optimal estimates fulfill the second order approximation of the application requirement based on θ^0 and none based on θ^{init}. Two of the white estimates have a_3 larger than 10^5. Three of the white estimates fulfill the second order approximation of the application requirement based on θ^0 and one based on θ^{init}.

The white estimate from Experiment 3 gives extremely bad performance.

All ten optimal estimates are displayed in Table X.

TABLE X

EXAMPLE 2 – ESTIMATED PARAMETERS USING OPTIMAL INPUT SIGNAL.

θ	Exp. 1	Exp. 2	Exp. 3	Exp. 4	Exp. 5	Exp. 6	Exp. 7	Exp. 8	Exp. 9	Exp. 10
a_1	0.0586	0.0597	0.0460	0.0502	0.0464	0.0514	0.0633	0.0621	0.0594	0.0782
a_2	0.0567	0.0371	0.0473	0.0898	0.0680	0.0806	0.0539	0.1025	0.0631	0.0866
a_3	7×10^5	8×10^3	0.0234	4×10^5	0.0159	2×10^7	0.2372	0.2343	1×10^6	3×10^6
a_4	0.0420	0.0623	0.0400	0.0542	0.0525	0.0257	0.1869	0.0933	0.0992	0.0957
γ_1	0.6877	0.7615	0.6963	0.6311	0.7256	0.6981	0.8117	0.6409	0.7022	0.7436
γ_2	0.6133	0.6379	0.5997	0.9405	0.6764	0.7791	0.5739	0.9140	0.6398	0.7506
k_1	2.381	2.085	2.6784	2.5100	2.1749	2.5741	2.5631	2.5870	2.4390	2.3307
k_2	1.901	1.537	1.8474	1.8177	2.0980	2.2888	2.0938	1.8615	1.8574	2.0337

Application cost

We check the application cost for all white and optimal estimates. We evaluate the application cost on the process, see Figure 30 and 31. We also evaluate the application cost in simulation based on both θ^{init} and θ^0, see Figures 32 and 35.

In addition, only one of the white models and none of the optimal models fulfill the second order approximation of the application requirement when based on θ^{init}, but three of the white and optimal models fulfill it when based on θ^0.

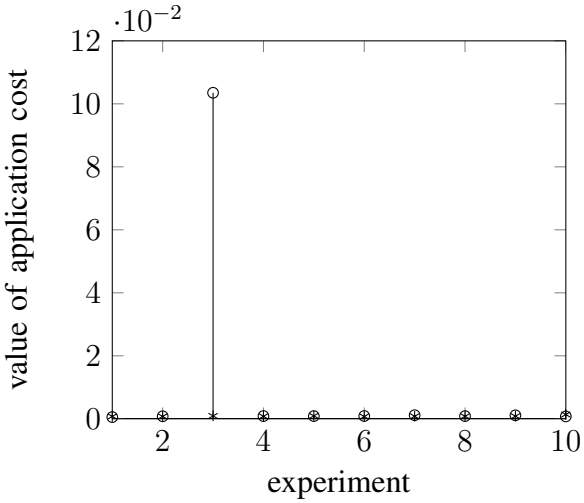

Figure 30. Example 2 – Application cost evaluated on process. The application cost for optimal estimates (∗) and white estimates (◦) are shown. Note the bad performance of the estimate from Experiment 3.

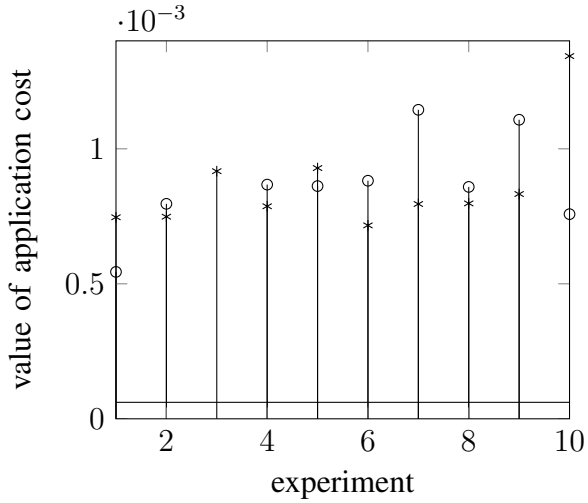

Figure 31. Example 2 – Application cost evaluated on process excluding Experiment 3. The optimal estimates outperforms the white estimates on average. However, there is no significant difference and none of the estimates fulfill the requirement of an application cost lower than $1/\gamma$, denoted (▬).

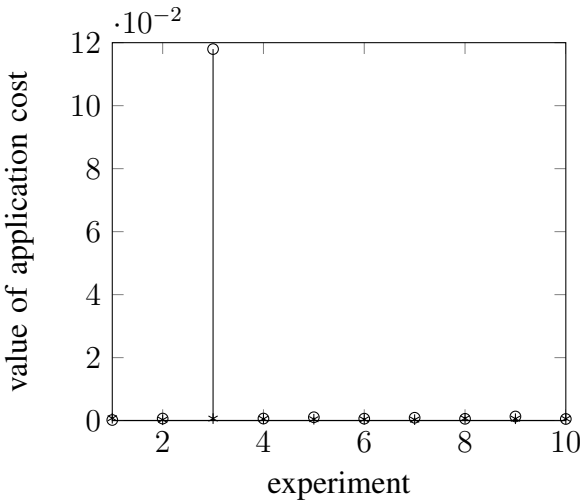

Figure 32. Example 2 – Application cost evaluated in simulation on system based on θ^{init}. The application cost for the optimal estimates ($*$) and white estimates ($\circ$) are shown. Note the bad performance of the estimate from Experiment 3.

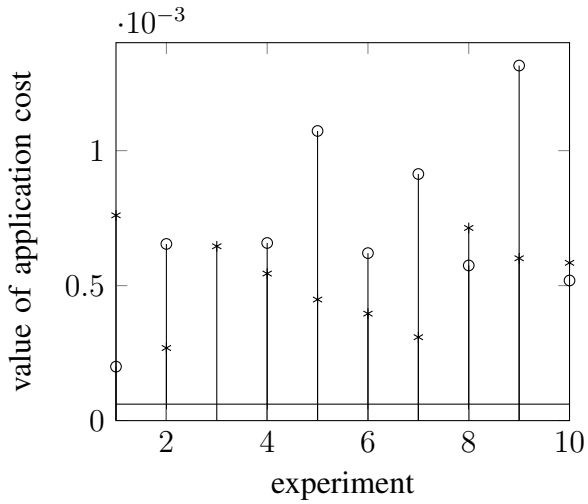

Figure 33. Example 2 – Application cost evaluated in simulation on system based on θ^{init} excluding Experiment 3. The optimal estimates outperforms the white estimates in general, but none of the estimates fulfill the requirement of an application cost lower than $1/\gamma$, denoted (—).

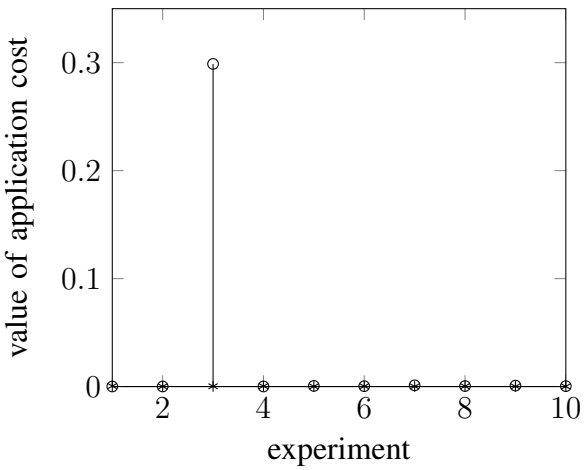

Figure 34. Example 2 – Application cost evaluated in simulation on system based on θ^0. The application cost for the optimal estimates ($*$) and white estimates ($\circ$) are shown. Note the bad performance of the estimate from Experiment 3.

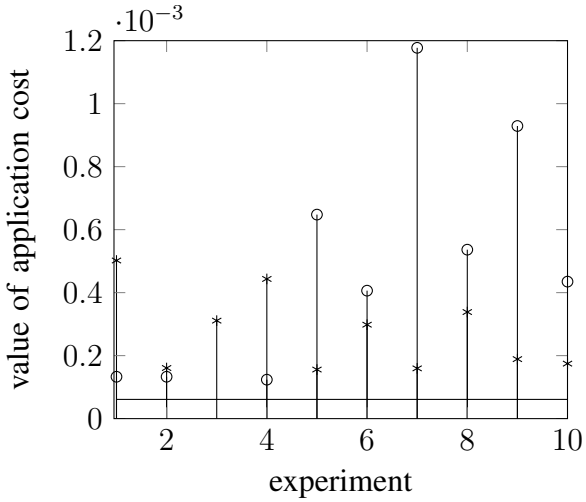

Figure 35. Example 2 – Application cost evaluated on system based on θ^0 excluding Experiment 3. We see that the optimal estimates outperforms the white estimates in general, but none of the estimates fulfill the requirement of an application cost lower than $1/\gamma$, denoted (▬).

Signals from process

The output signals from the process when evaluating the application cost are shown in Figures 36-38 for the optimal estimates and Figures 37-39 for the white estimates. Based on these plots, it is not evident that the MPC models based on optimal estimates perform better than MPC models based on white estimates. When excluding the estimate from Experiment 3, it is difficult to differentiate between the optimal output signals and the white output signals. All outputs fulfill the constraints imposed on the water tank levels. Note that the outputs have been scaled to centimeters instead of volt.

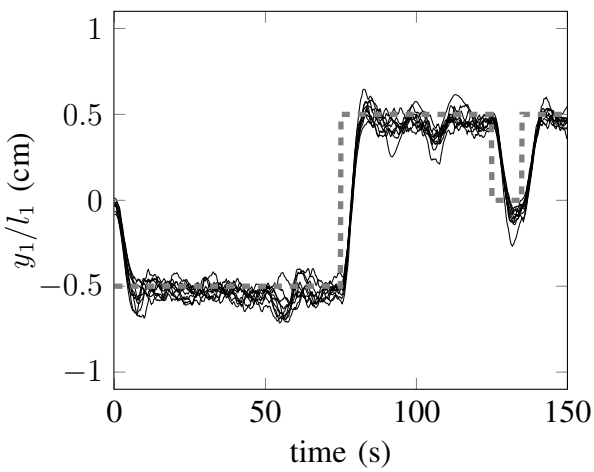

Figure 36. Example 2 – Optimal y_1 from application evaluation. The optimal output y_1/l_1 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

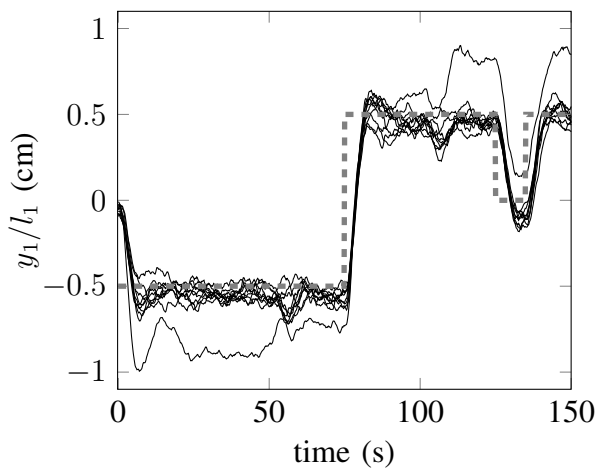

Figure 37. Example 2 – White y_1 from application evaluation. The white output y_1/l_1 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

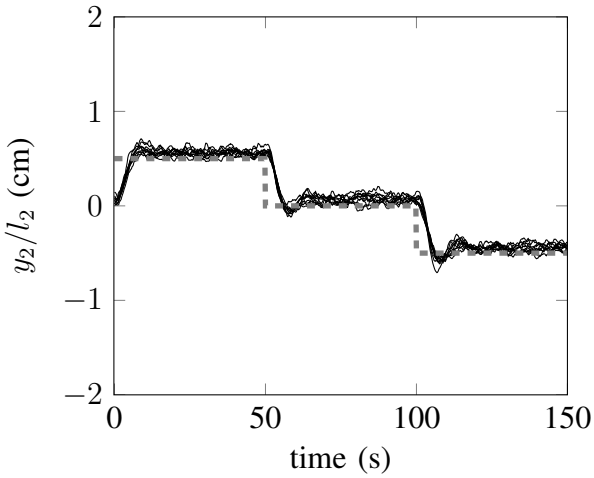

Figure 38. Example 2 – Optimal y_2 from application evaluation. The optimal output y_2/l_2 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

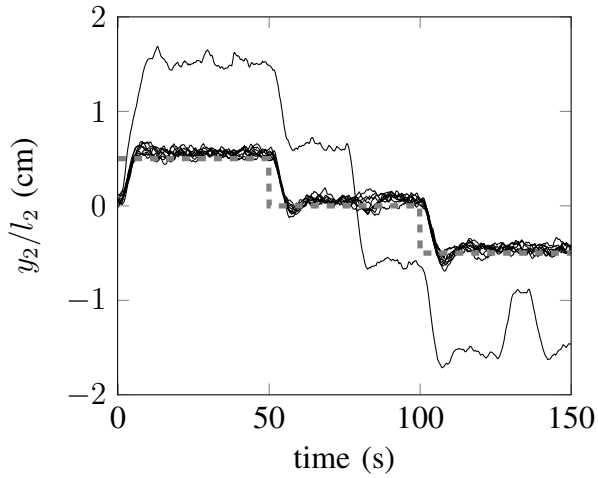

Figure 39. Example 2 – White y_2 from application evaluation. The white output y_2/l_2 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

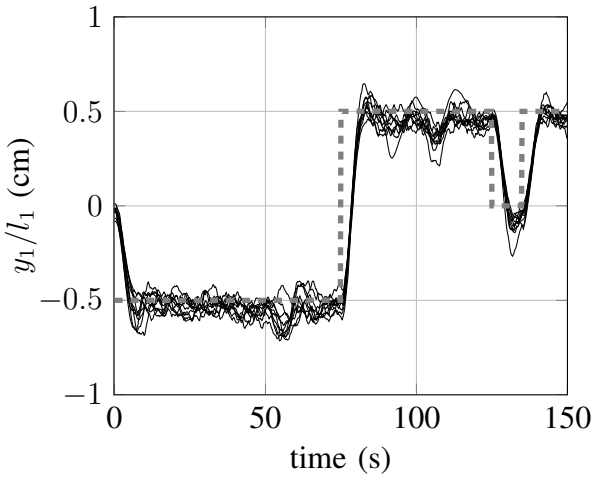

Figure 40. Example 2 – Optimal y_1 from application evaluation zoomed in. The optimal output y_1/l_1 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

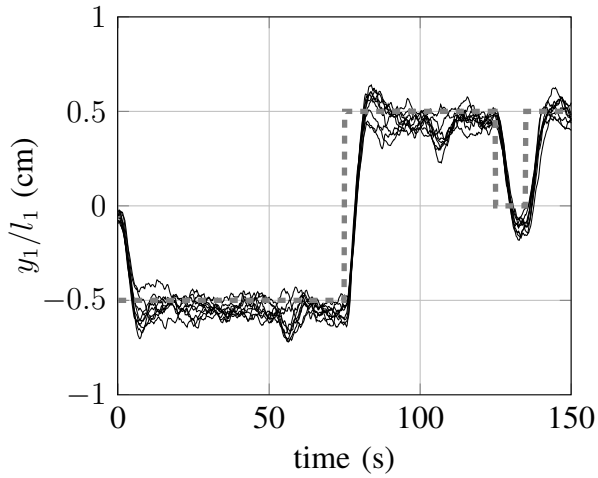

Figure 41. Example 2 – White y_1 from application evaluation excluding Experiment 3. The white output y_1/l_1 (▬) from nine application evaluations on the process are shown. The reference signal is denoted (- - -).

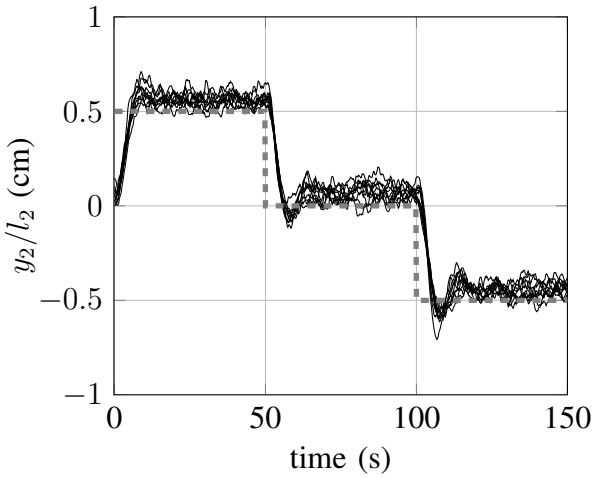

Figure 42. Example 2 – Optimal y_2 from application evaluation zoomed in. The optimal output y_2/l_2 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

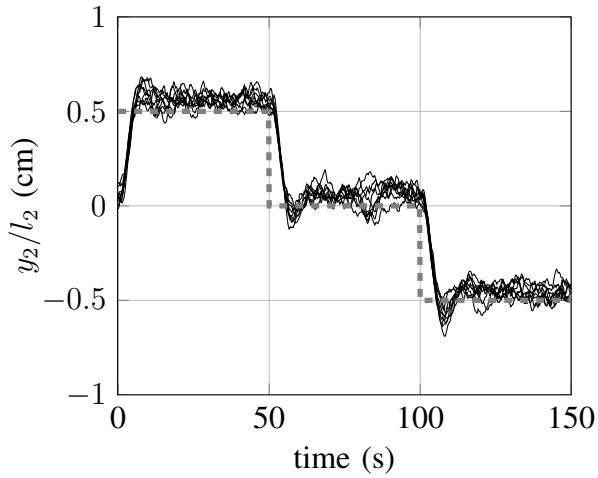

Figure 43. Example 2 – White y_2 from application evaluation excluding Experiment 3. The white output y_2/l_2 (▬) from nine application evaluations on the process are shown. The reference signal is denoted (- - -).

We redo the white identification experiments but with twice as long experiment time, 600 samples. The resulting output signals from the process in the application evaluations are shown in Figures 44 and 45. When comparing the outputs with those in Figures 36-39, we see that it is difficult to differentiate between them. It is also difficult to differentiate between outputs based on estimated models in the MPC and outputs based on the true model in the MPC, see Figures 25 and 26. We conclude that for the θ^{init} used here, we do not gain much, in terms of reference tracking on the real process, from designing the spectrum of the input signal. Even though we double the experiment length we do not see much improvement in the outputs.

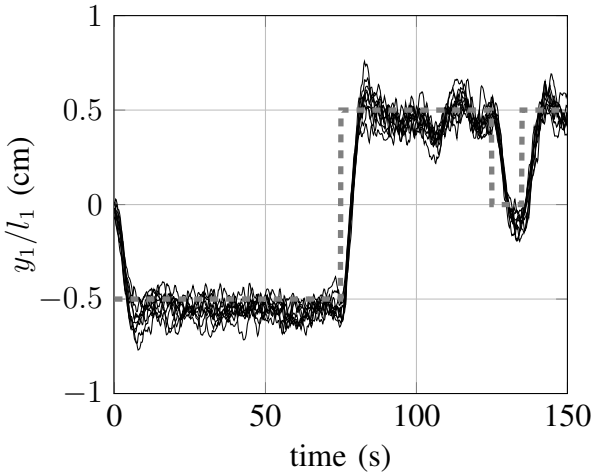

Figure 44. Example 2 – White y_1 from application evaluation using twice as long experiment time. The white output y_1/l_1 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

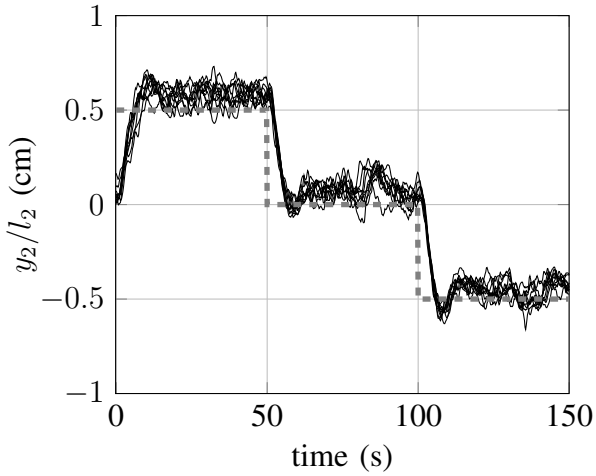

Figure 45. Example 2 - White y_2 from application evaluation using twice as long experiment time. The white output y_2/l_2 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

Experiment length

As in Example 1, we evaluate four different experiment lengths. The values of the experiment lengths are shown in Table XVI. We see that the white input signal requires, according to theory, a longer experiment length to achieve the same performance as the optimal input signal. The white input signal is approximately 3.4 times longer than the optimal input signal when based on the initial estimates and 4.5 times longer when based on the true parameter values.

TABLE XI

EXAMPLE 2 – EXPERIMENTAL LENGTHS.

Experiment length	Value in samples	Value in minutes
N	300	5
$N_{\text{white}}^{\text{initial}}$	1015	17
$N_{\text{white}}^{\text{true}}$	1182	20
N^{true}	216	4

Example 3 - Estimating $\theta = [a_1\ a_2\ a_3\ a_4\ \gamma_1\ \gamma_2\ k_1\ k_2]^{\mathrm{T}}$ with no integral action in the MPC

In Example 3, we once again estimate $\theta = [a_1\ a_2\ a_3\ a_4\ \gamma_1\ \gamma_2\ k_1\ k_2]^{\mathrm{T}}$, but we change θ^{init}. The reason for this is to see how the values of θ^{init} affect the performance of the method.

Check of initial estimates

We use as initial estimates the values from the process specification slightly modified, that is $\theta^{\text{init}} = [0.03\ 0.03\ 0.05\ 0.05\ 0.7\ 0.7\ 1.6\ 1.6]^{\mathrm{T}}$. in Example 2, $a_3 = a_4 = 0.5$.

However, with these initial estimates `cvx` returns an inaccurate solution when calculating the optimal spectrum. As suggested in the `cvx`-manual, we check that the estimates are accurate enough for our purpose. We do so by checking that the linear matrix inequality constraint is approximately active, and it is. The smallest eigenvalue of the matrix should be zero and it is of the order of 10^{-7}.

Before performing the input design, we check if θ^{init} fulfill the application requirement. In this check, we get that θ^{init} do not fulfill the application requirement.

Application ellipsoid

As in Example 1 and 2, we calculate two types of application ellipsoids. The first type is evaluated in simulation using a system and MPC model based on θ^{init}. The second type is evaluated in simulation using a system and MPC model based on θ^0.

In Tables XII and XIII, we give the range of each parameter value that may fulfill the second order approximation of the application requirement centered at θ^{init} and θ^0, respectively.

We see that the range of γ_1 is larger based on θ^{init} than on θ^0. Consequently, the input design might not give estimates of these values that are accurate enough for the application in mind.

In terms of absolute deviation, we conclude that we can rank the parameters in order of increasing importance as k_2, k_1, γ_2, γ_1, a_3, a_4, a_2 and a_1, when the design is based on θ^{init}. The ranking based onθ^0 is, as in Example 2, a_3, k_1, k_2, a_4,γ_2, γ_1, a_2 and a_1.

The order of increasing importance is different in terms of relative deviation than absolute deviation. The ranking based on relative deviation and the initial design centered at θ^{init} (Table XII) is a_3, a_4, a_2, a_1, k_2, k_1, γ_2 and γ_1. The ranking based on relative deviation, θ^{init} and centered at θ^0 (Table XIII) is k_2, a_2, a_4, k_1, a_1, γ_2, a_3 and γ_1. The ranking based on relative deviation, θ^0 and centered at θ^0 (Table XIII) is, as in Example 2, a_3, a_4, a_2, k_1, k_2, a_1, γ_2 and γ_1.

Despite the different rankings, they all state that a_1 and a_2 are more important than a_3 and a_4, and γ_1 and γ_2 are more important than k_1 and k_2 to estimate with higher accuracy. Except the ranking based on relative deviation, θ^{init} and centered at θ^0 (Table XIII). That is, the ranking that is based on the design that is actually used in the identification experiments. Here, a_3 is deemed more important than a_1, and a_4 is deemed more important than a_2. However, the possible range of these parameters are smaller than the ones obtained based on θ^0. The input design can still give estimates of these values that are accurate enough for the application.

TABLE XII

EXAMPLE 3 – POSSIBLE RANGE OF EACH PARAMETER VALUE IN THE INITIAL DESIGN.

θ	θ^{init}	Possible range based on θ^{init}	
a_1	0.03	0.03 ± 0.0233	($\pm 78\%$)
a_2	0.03	0.03 ± 0.0310	($\pm 103\%$)
a_3	0.05	0.05 ± 0.1031	($\pm 206\%$)
a_4	0.05	0.05 ± 0.0693	($\pm 139\%$)
γ_1	0.7	0.7 ± 0.1802	($\pm 26\%$)
γ_2	0.7	0.7 ± 0.2385	($\pm 34\%$)
k_1	1.6	1.6 ± 1.0168	($\pm 64\%$)
k_2	1.6	1.6 ± 1.0865	($\pm 68\%$)

TABLE XIII

EXAMPLE 3 – POSSIBLE RANGE OF EACH PARAMETER VALUE.

θ	θ^0	Possible range based on θ^{init}		Possible range based on θ^0	
a_1	0.0649	0.0649 ± 0.0233	($\pm 36\%$)	0.0649 ± 0.0422	($\pm 65\%$)
a_2	0.0594	0.0594 ± 0.0310	($\pm 52\%$)	0.0594 ± 0.0534	($\pm 90\%$)
a_3	0.3211	0.3211 ± 0.1031	($\pm 32\%$)	0.3211 ± 3.3951	($\pm 1057\%$)
a_4	0.1353	0.1353 ± 0.0693	($\pm 51\%$)	0.1353 ± 0.3082	($\pm 228\%$)
γ_1	0.7283	0.7283 ± 0.1802	($\pm 25\%$)	0.7283 ± 0.1357	($\pm 19\%$)
γ_2	0.7271	0.7271 ± 0.2385	($\pm 33\%$)	0.7271 ± 0.2624	($\pm 36\%$)
k_1	2.362	2.362 ± 1.0168	($\pm 43\%$)	2.362 ± 2.0115	($\pm 85\%$)
k_2	1.797	1.797 ± 1.0865	($\pm 60\%$)	1.797 ± 1.3456	($\pm 75\%$)

Optimal spectrum

We calculate the optimal spectrum using 20 parameters $c_0, \ldots, c_{19}$, where $c_i \in R^{2\times 2}$. The optimal spectrum is shown in Figure 46, along with the optimal spectrum based on θ^0. We see

that the spectrum differs from the one obtained in Example 2. The design based on θ^{init} requires a smaller power, trace$(c_0) = 0.0234$, than the design based on θ^0, trace$(c_0) = 0.0412$.

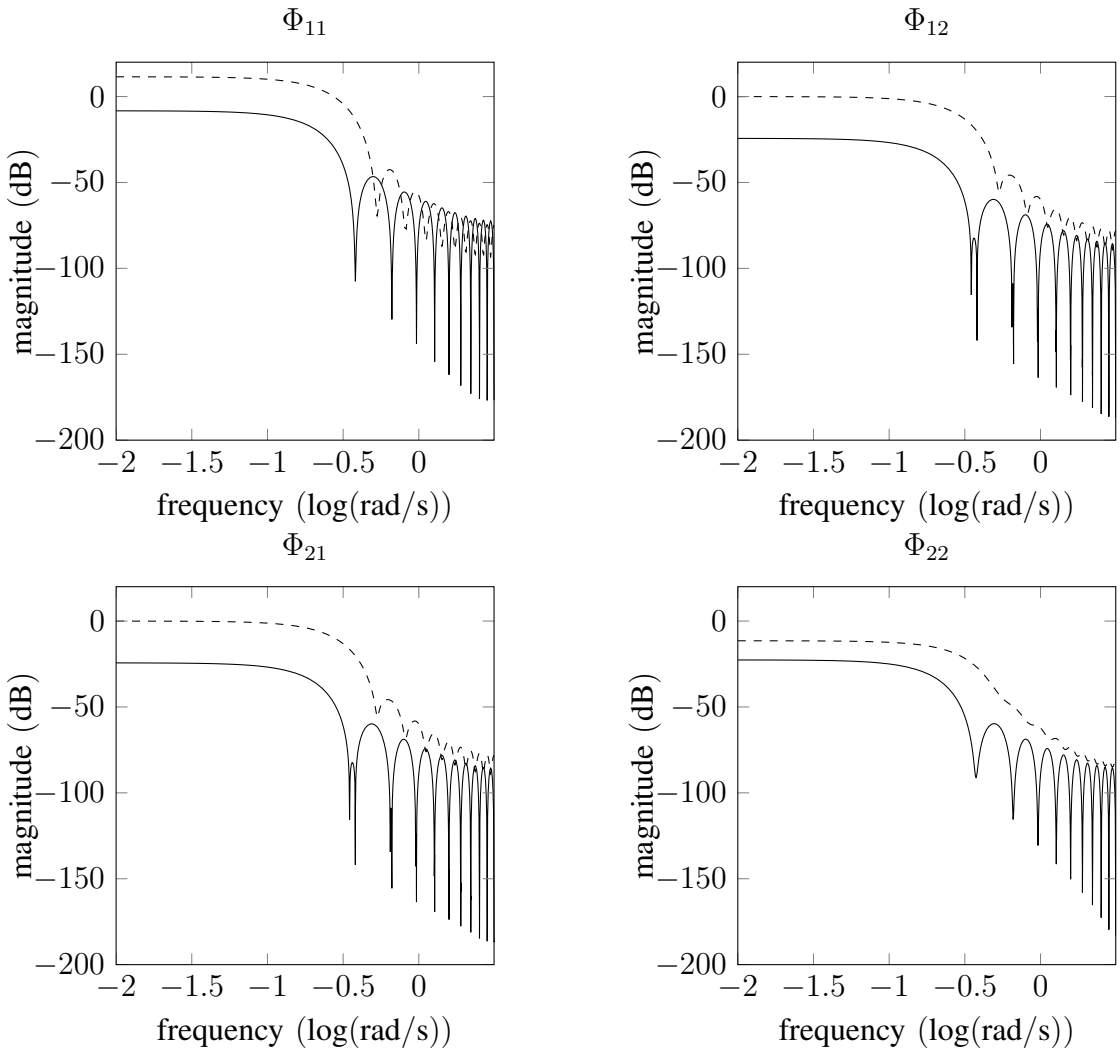

Figure 46. Example 3 – Optimal spectrum. The optimal spectrum based on θ^{init} (▬) and θ^0 (- - -) are shown. We see that u_1 and u_2 are correlated.

Identification ellipsoid

As in Example 2, we calculate two kinds of identification ellipsoids. The first is based on the optimal spectrum and θ^0. The second is based on a white spectrum and θ^0. The white

spectrum corresponds to a white excitation signal of equal power to the optimal input signal.

In Table XIV, we give the possible range of each estimated parameter value according to the identification ellipsoid.

We can rank the parameters in order of increasing importance. In terms of absolute deviation both spectra give the same ranking, that is, a_3, k_2, k_1, a_4, γ_2, γ_1, a_2 and a_1. In terms of relative deviation the optimal spectrum gives a_3, a_4, a_2, k_2, k_1/γ_2, a_1 and γ_1, that is k_1 and γ_2 are allowed equal relative deviation. The white spectrum gives a_3, a_4, a_2, k_2, k_1, a_1, γ_2 and γ_1.

The rankings differ from those obtained from the application ellipsoids. However, they all state that a_1 and a_2 are more important than a_3 and a_4, and γ_1 and γ_2 are more important than (or equally important to) k_1 and k_2 to estimate with higher accuracy. We also see that the identification ellipsoid based on the optimal spectrum has a smaller possible range of each parameter than the ellipsoid based on the white spectrum, except for a_3. The volume of the white identification ellipsoid is 1123 times larger than the volume of the optimal identification ellipsoid. The optimal ellipsoid is in turn 18 times larger than the optimal ellipsoid obtained in Example 2.

TABLE XIV

EXAMPLE 3 – POSSIBLE RANGE OF EACH ESTIMATED PARAMETER VALUE.

θ	θ^0	Possible range based on θ^0 and optimal spectrum		Possible range based on θ^0 and white spectrum	
a_1	0.0649	0.0649 ± 0.0072	$(\pm 15\%)$	0.0649 ± 0.0295	$(\pm 45\%)$
a_2	0.0594	0.0594 ± 0.0308	$(\pm 46\%)$	0.0594 ± 0.0540	$(\pm 91\%)$
a_3	0.3211	0.3211 ± 3.523	$(\pm 1097\%)$	0.3211 ± 2.898	$(\pm 903\%)$
a_4	0.1353	0.1353 ± 0.3734	$(\pm 276\%)$	0.1353 ± 0.8845	$(\pm 654\%)$
γ_1	0.7283	0.7283 ± 0.0947	$(\pm 13\%)$	0.7283 ± 0.2247	$(\pm 31\%)$
γ_2	0.7271	0.7271 ± 0.1368	$(\pm 19\%)$	0.7271 ± 0.2723	$(\pm 37\%)$
k_1	2.362	2.362 ± 0.4429	$(\pm 19\%)$	2.362 ± 1.1665	$(\pm 49\%)$
k_2	1.797	1.797 ± 0.8084	$(\pm 45\%)$	1.797 ± 1.1810	$(\pm 66\%)$

Estimates

We estimate θ using the optimal input signal and the white input signal. We perform ten identification experiments for each type of signal. A new realization of the input signal is used in each experiment.

It is difficult to estimate reasonable values of a_4 using the optimal signal. In four of the ten estimates it is larger than 10^3. We believe that this is due to the small water tank level in steady state of Tank 4 (as in Tank 3 in Example 2). Four of the optimal estimates fulfill the second order approximation of the application requirement based on θ^0 and three on θ^{init}. None of the white estimates fulfill the second order approximation of the application requirement based on θ^0 nor based on θ^{init}.

All ten optimal estimates are displayed in Table XV.

TABLE XV

EXAMPLE 3 – ESTIMATED PARAMETERS USING OPTIMAL INPUT SIGNAL.

θ	Exp. 1	Exp. 2	Exp. 3	Exp. 4	Exp. 5	Exp. 6	Exp. 7	Exp. 8	Exp. 9	Exp. 10
a_1	0.0580	0.0678	0.0713	0.0592	0.0696	0.0685	0.0551	0.0434	0.1171	0.0423
a_2	0.0667	0.0840	0.0508	0.0614	0.0863	0.0678	0.0579	0.0604	0.0600	0.0532
a_3	0.1779	1.7553	0.0337	0.0047	0.0271	0.0003	0.0980	0.1217	0.0286	0.0235
a_4	0.0990	1.4637	0.0522	2×10^5	2×10^5	7×10^3	0.0547	0.0359	7×10^3	0.0284
γ_1	0.7128	0.6955	0.6947	0.7247	0.7006	0.6994	0.6704	0.6754	0.7771	0.6233
γ_2	0.7760	0.6165	0.6822	0.5234	0.7245	1.9012	0.7133	0.6621	0.4249	0.7256
k_1	1.9743	2.5614	1.7458	1.9282	2.0216	2.1272	2.2686	2.4342	1.8671	1.5674
k_2	1.7399	2.2983	1.6126	2.2392	2.0662	0.5208	1.7740	2.0622	3.2981	1.7347

Application cost

We check the application cost for all white and optimal estimates. We evaluate the application cost on the process, see Figures 47 and 48. We also evaluate the application cost in simulation based on both θ^{init} and θ^0, see Figures 49–52.

In addition, none of the white models fulfill the second order approximation of the application requirement when based on the initial estimate or the true parameter values, but three of the optimal models fulfills it when based on the initial estimate and four of them when based on the true parameter values.

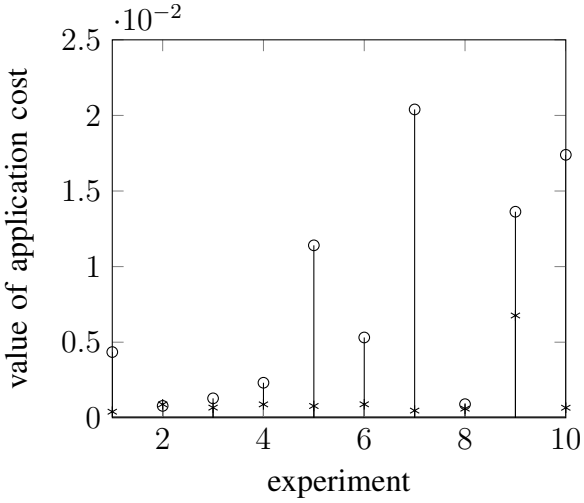

Figure 47. Example 3 – Application cost evaluated on process. The application cost for optimal estimates (∗) and white estimates (○) are shown.

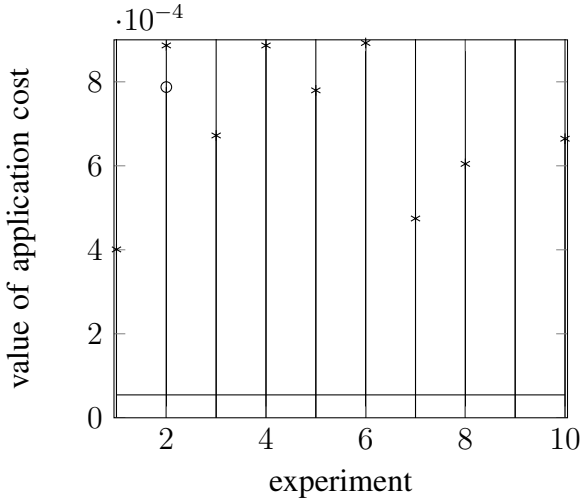

Figure 48. Example 3 – Application cost evaluated on process zoomed in. We see that the optimal estimates on average outperforms the white estimates. However, none of the estimates fulfill the requirement of an application cost lower than $1/\gamma$, denoted (▬).

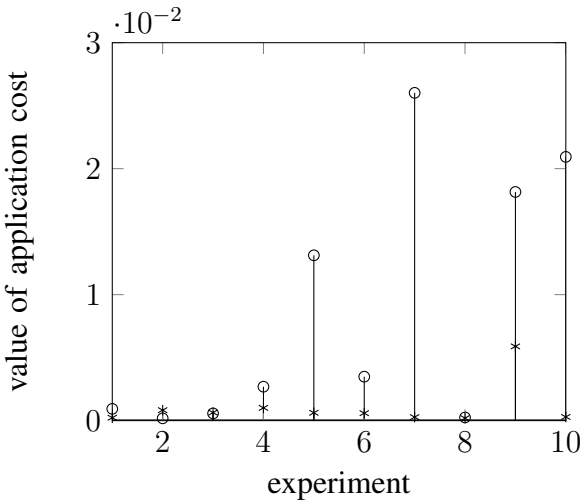

Figure 49. Example 3 – Application cost evaluated on system based on initial estimates. The application cost for the optimal estimates ($*$) and white estimates ($\circ$) are shown.

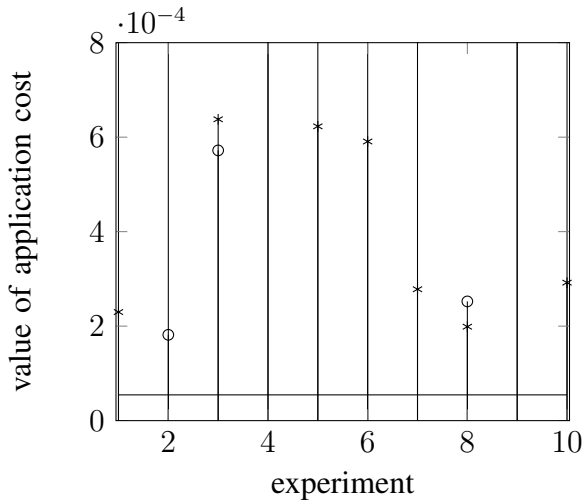

Figure 50. Example 3 – Application cost evaluated on system based on initial estimates zoomed in. We see that the optimal estimates in general outperforms the white estimates, but none of the estimates fulfill the requirement of an application cost lower than $1/\gamma$, denoted (—).

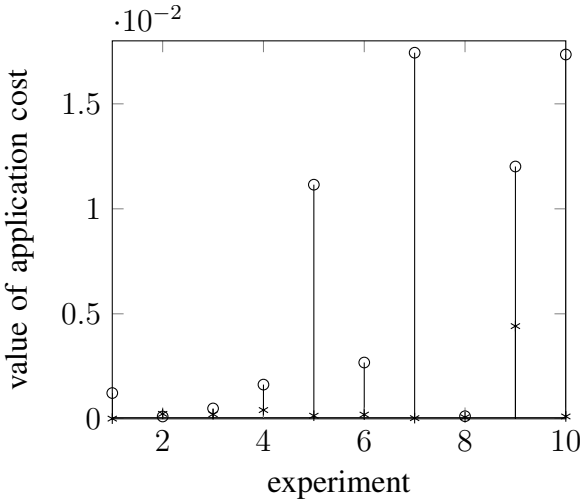

Figure 51. Example 3 – Application cost evaluated on system based on true parameter values. The application cost for the optimal estimates (∗) and white estimates (○) are shown.

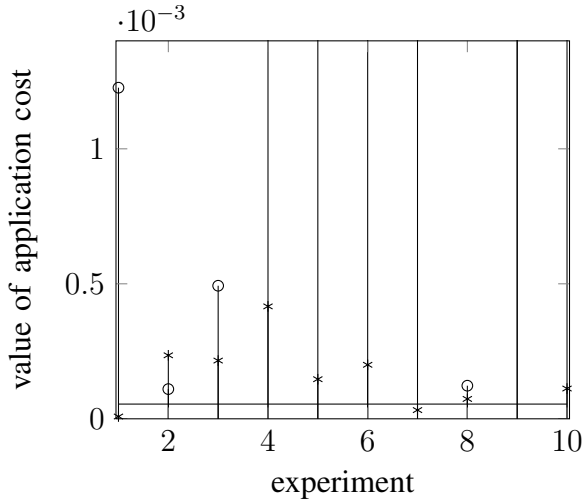

Figure 52. Example 3 – Application cost evaluated on system based on true parameter values zoomed in. We see that the optimal estimates in general outperforms the white estimates, but only two of the optimal estimates fulfills the requirement of an application cost lower than $1/\gamma$, denoted (▬).

Signals from process

The output signals from the process when evaluating the application cost are shown in Figures 53-55 for the optimal estimates and Figures 54-56 for the white estimates. We see that the MPC models based on optimal estimates performs better than MPC models based on white estimates. All outputs fulfill the constraints imposed on the water tank levels.

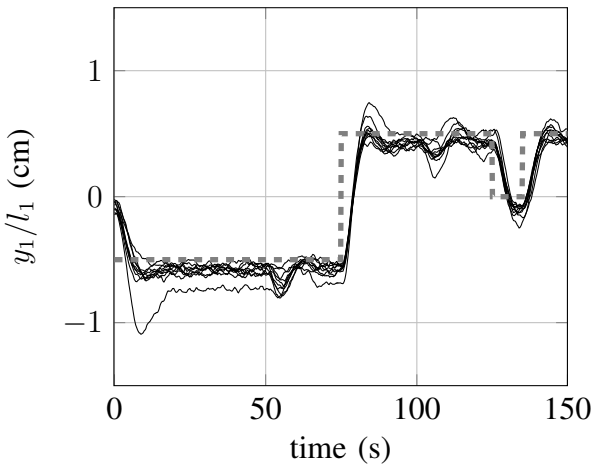

Figure 53. Example 3 – Optimal y_1 from application evaluation. The optimal output y_1/l_1 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

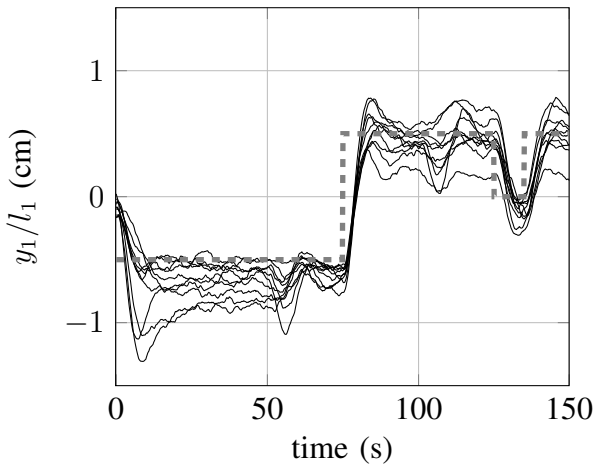

Figure 54. Example 3 – White y_1 from application evaluation. The white output y_1/l_1 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

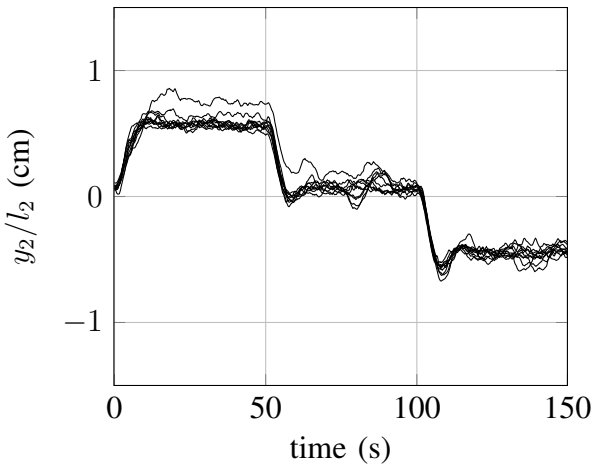

Figure 55. Example 3 – Optimal y_2 from application evaluation. The optimal output y_2/l_2 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

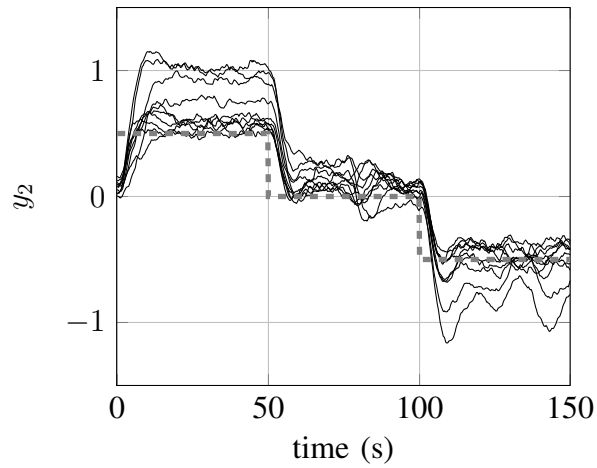

Figure 56. Example 3 – White y_2 from application evaluation. The white output signal y_2/l_2 (▬) ten application evaluations on the process are shown. The reference signal is denoted (- - -).

Experiment length

As in Example 1 and 2, we evaluate four different experiment lengths. The values of the experiment lengths are shown in Table XVI. We see that the white input signal requires, according to theory, a longer experiment length to achieve the same performance as the optimal input signal. The white input signal is approximately 16.3 times longer than the optimal input signal when based on the initial estimates and 4.3 times longer when based on the true parameter values.

TABLE XVI

EXAMPLE 3 – EXPERIMENTAL LENGTHS.

Experiment length	Value in samples	Value in minutes
N	300	5
$N_{\text{white}}^{\text{initial}}$	4878	81
$N_{\text{white}}^{\text{true}}$	3606	60
N^{true}	846	14

We perform a long identification experiment using the white input signal. The experiment length is set to $N_{\text{white}}^{\text{initial}}$, that is 4878 samples. We identify four different estimates of the parameter θ. The estimates are based on 25 %, 50 %, 75 % and 100 % of the collected data. That is, 1219, 2439, 3658 and 4878 samples. The four estimates are shown in Table XVII.

TABLE XVII

EXAMPLE 3 – ESTIMATED PARAMETERS USING OPTIMAL INPUT SIGNAL.

θ	Exp. 1: $N = 1219$	Exp. 2: $N = 2439$	Exp. 3: $N = 3658$	Exp. 4: $N = 4878$
a_1	0.1318	0.1083	0.0887	0.0940
a_2	0.0544	0.0845	0.1040	0.0881
a_3	0.0131	0.0179	0.0277	0.0171
a_4	0.0043	0.1664	0.0432	0.0749
γ_1	3.3486	0.7749	0.7444	0.8019
γ_2	0.5261	0.7981	0.9393	0.8266
k_1	0.4185	1.8340	1.8864	1.7465
k_2	2.0627	1.1150	0.9227	1.0977

In Figures 57 and 58, we see the values of the second order approximation of the application requirement based on the θ^0 for each estimate. We see that the fourth estimate, corresponding to $N_{\text{white}}^{\text{initial}}$ samples, almost fulfills the application requirement. The value of the cost for the fourth estimate is 5.52×10^{-5} and the value of $1/\gamma$ is 5.45×10^{-5}. In Figures 59 and 60, we see the output signals y_1 and y_2 from the process when using an MPC model based on the first and fourth estimates, respectively. We see that the output signals corresponding to the fourth estimate are on average closer to the reference than the output signals corresponding to the first estimate.

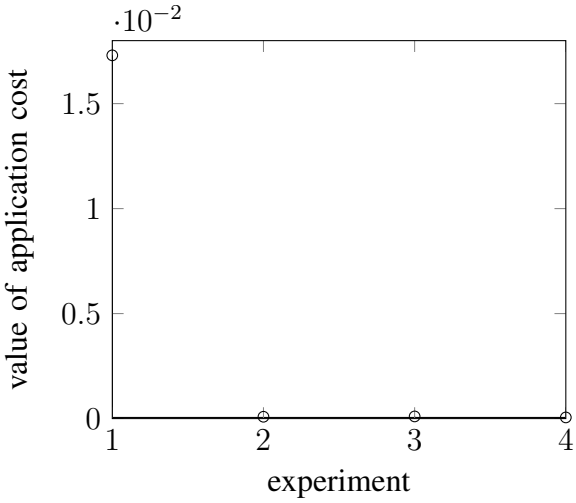

Figure 57. Example 3 – Application requirement evaluated in simulation on system based on θ^0. The second order approximation of the application cost for the different white estimates (○) are shown.

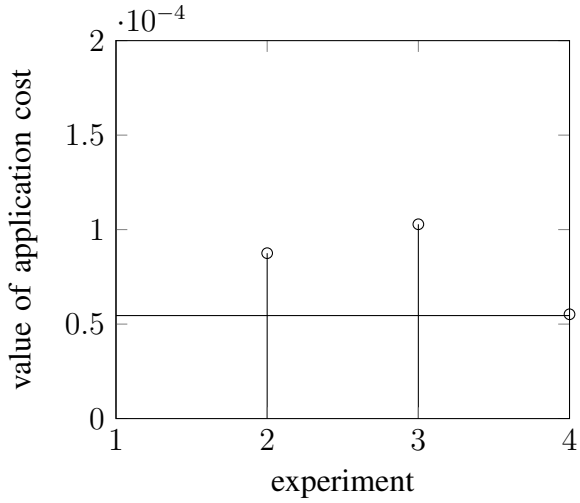

Figure 58. Example 3 – Application requirement evaluated in simulation on system based on θ^0 zoomed in. The fourth estimate almost fulfills the degradation limit $1/\gamma$ (▬).

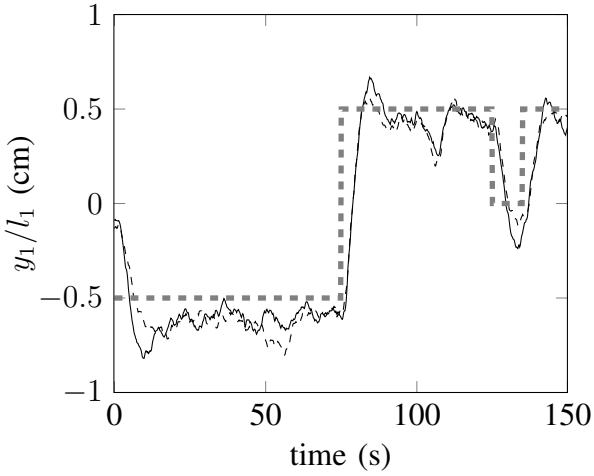

Figure 59. Example 3 – Process output y_1. The output signal y_1/l_1 for the first (▬) and fourth (▪▪▪) estimates are shown. The reference signal is denoted (▪▪▪).

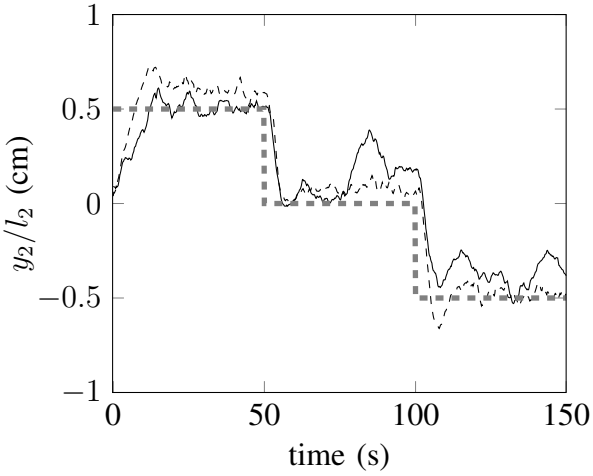

Figure 60. Example 3 – Process output y_2. The output signal y_2/l_2 for the first (▬) and fourth (▪▪▪) estimates are shown. The reference signal is denoted (▪▪▪).

Example 4 - Estimating $\theta = [\gamma_1 \ k_1]^{\mathrm{T}}$ with integral action in the MPC

We redo Example 1 but with $Q_u = 0I_{2\times 2}$. The MPC has integral action.

Check of initial estimates

We use as initial estimates the values from the process specification, that is $\theta^{\text{init}} = [0.7 \ 1.6]^{\mathrm{T}}$. The initial estimates do not fulfill the application requirement when evaluated in simulation.

Application ellipsoid

We calculate the same type of application ellipsoids as in Example 1. The ellipsoids are depicted in Figure 61. In Table XVIII, we give the range of each parameter value that may fulfill the second order approximation of the application requirement based on and centered around θ^{init}. In Table XIX, we give the range of each parameter value that may fulfill the second order approximation of the application requirement when based on θ^{init} and centered around θ^0, and based on and centered around θ^0.

We get the same important directions in terms of absolute measure as in Example 1. Meaning, they both yield the interpretation that γ_1 is more important to know with high accuracy than k_1.

However, in terms of relative measure, the result vary if the ellipsoids are based on θ^{init} centered around θ^0 or θ^{init}. This highlights an issue with using the application ellipsoid in an

absolute sense and simply shifting the application ellipsoid based on θ^{init} from being centered at θ^{init} to θ^0. The relative ranges based on θ^{init} and centered at θ^{init} are 24 % for γ_1 and 28 % for k_1. That is, γ_1 is more important to estimate correctly than k_1, but the relative ranges based on θ^{init} and centered at θ^0 are 23 % for γ_1 and 19 % for k_1. We see that the important directions in the former case coincide with those in the true design, while the important directions in the latter case do not coincide with those in the true design.

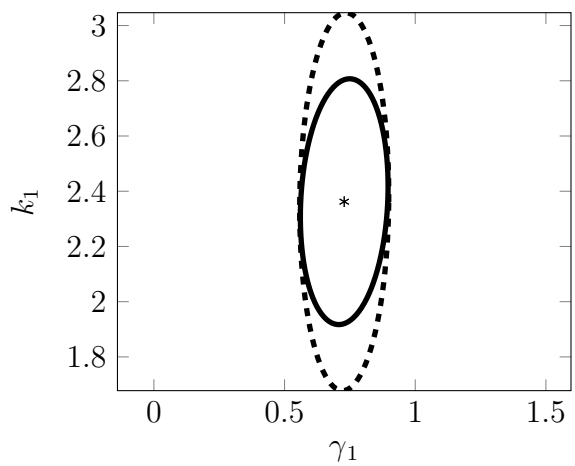

Figure 61. Example 4 – Application ellipsoids. The application ellipsoids based on θ^{init} and θ^0 are displayed as (▬) and (- - -), respectively. The true parameter values are denoted (*).

TABLE XVIII

EXAMPLE 4 – POSSIBLE RANGE OF EACH PARAMETER VALUE IN THE INITIAL DESIGN.

θ	θ^{init}	Possible range based on θ^{init}	
γ_1	0.7	0.7 ± 0.1678	$(\pm 24\%)$
k_1	1.6	1.6 ± 0.4450	$(\pm 28\%)$

TABLE XIX

EXAMPLE 4 – POSSIBLE RANGE OF EACH PARAMETER VALUE.

θ	θ^0	Possible range based on θ^{init}		Possible range based on θ^0	
γ_1	0.7283	0.7283 ± 0.1678	$(\pm 23\%)$	0.7283 ± 0.1704	$(\pm 25\%)$
k_1	2.362	2.362 ± 0.4450	$(\pm 19\%)$	2.362 ± 0.6850	$(\pm 29\%)$

Optimal spectrum

We calculate the optimal spectrum using 40 parameters, $c_0, \ldots, c_{39}$, where $c_i \in R^{2\times 2}$. The optimal spectrum is based on θ^{init}. We get approximately the same spectrum as in Example 1, see Figure 62. The design based on θ^{init} requires a higher power, trace$(c_0) = 0.0031$, than the design based on θ^0, where trace$(c_0) = 0.0013$.

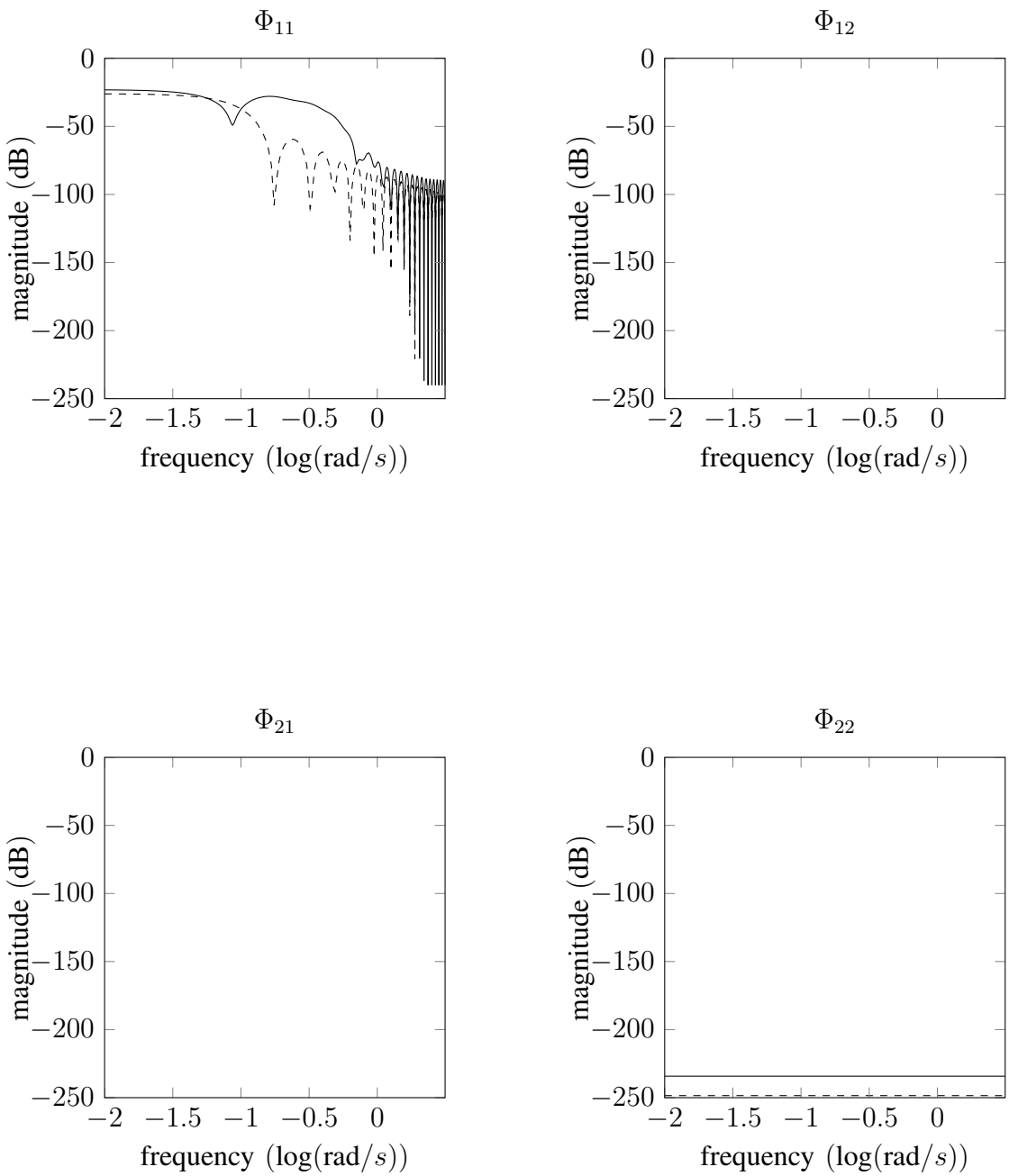

Figure 62. Example 4 – Optimal spectrum. The optimal spectrum based on θ^{init} (▬) and θ^0 (▪▪▪) are shown. We see that u_1 and u_2 are uncorrelated and the variance of u_2 is numerically zero for both spectra, as in Example 1.

Identification ellipsoid

We calculate the same kind of identification ellipsoids as in Example 1. The ellipsoids are shown in Figure 63. Note that the two ellipsoids are similar in shape and size, and that we have centered them around θ^0.

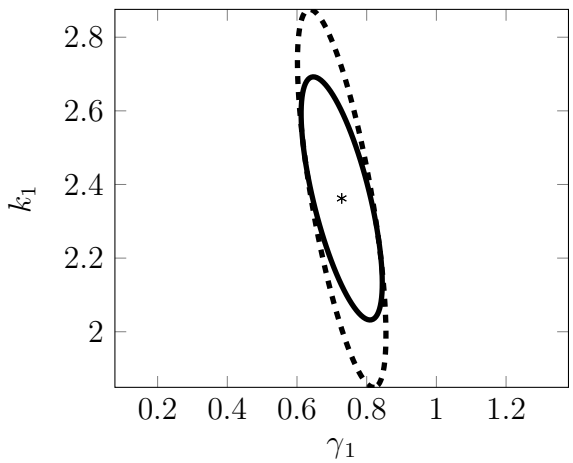

Figure 63. Example 4 – Identification ellipsoids. The identification ellipsoids based on θ^{init} and θ^0 are displayed as (▬) and (- - -), respectively. The true parameter values are denoted ($*$).

Estimates

We estimate θ using the optimal input signal and a white input signal. The white input signal has the same power as the optimal input signal, but divided equally between u_1 and u_2. We perform ten identification experiments for each type of signal. A new realization of the input signal is used in each experiment.

The resulting white estimates are shown in Figure 64, along with the identification ellipsoid based on the white spectrum and θ^0. The application ellipsoids based on θ^{init} and θ^0

are also shown for comparison. The white estimates are quite scattered, and only two estimates fulfill the second order approximation of the application requirement based on θ^{init}. The same two estimates along with one more fulfill the second order approximation of the application requirement based on θ^0.

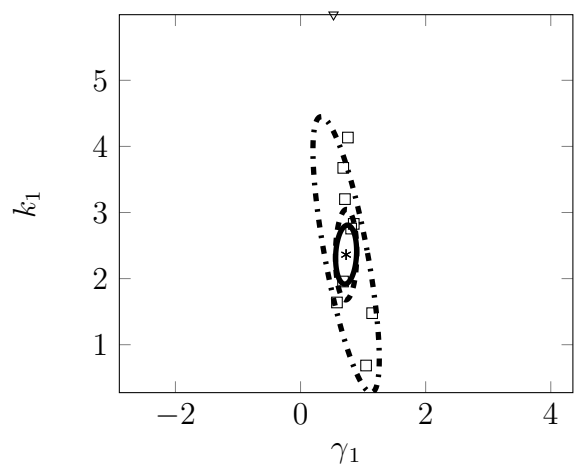

Figure 64. Example 4 – White estimates. The white estimates, denoted (□) and (▽) are shown, along with the application ellipsoid based on θ^{init} (▬) and on θ^0 (▪▪▪). Also, the identification ellipsoid based on the white spectrum and θ^0 is shown (·▪·▪). The true parameter values are denoted (∗).

The estimate denoted (▽) in Figure 64 corresponds to Experiment 4 and gives a particularly high application cost.

The resulting optimal estimates are shown in Figures 65 and 66. We see that the optimal estimates are more gathered than the white ones. Note also that they are spread out more in the k_1-direction than in the γ_1-direction. Meaning, the optimal estimates follow the shape of the application ellipsoid. Eight of the optimal estimates fulfill the second order approximation of the application requirement based on θ^{init}, see Figure 65. The same eight optimal estimates fulfill

the second order approximation of the application requirement based on θ^0, see Figure 66.

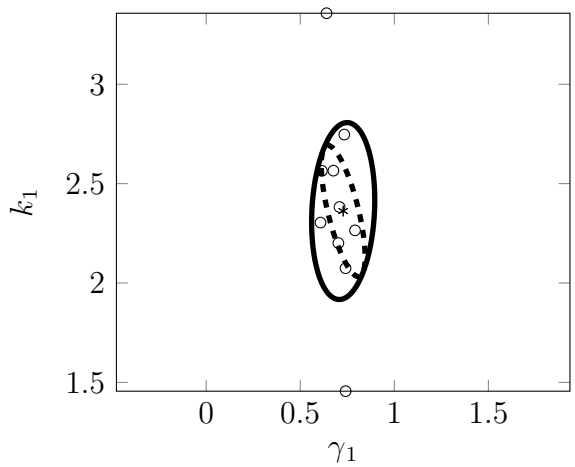

Figure 65. Example 4 – Optimal estimates and ellipsoids based on θ^{init}. The optimal estimates (○) are shown, along with the application ellipsoid (▬) and identification ellipsoid (- - -) based on θ^{init}. The true parameter values are denoted (∗).

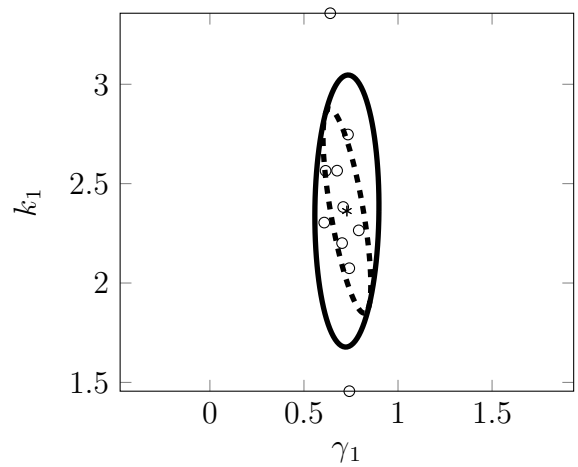

Figure 66. Example 4 – Optimal estimates and ellipsoids based on θ^0. The optimal estimates (○) are shown, along with the application ellipsoid (▬) and identification ellipsoid (- - -) based on θ^0. The true parameter values are denoted (∗).

As in Example 1, we also estimate θ using a white input signal with its covariance equal to the optimal c_0. The estimates are shown in Figure 67. We see that the estimates are still spread out. In fact, only two of the estimates fulfill the application requirement, both when based on θ^{init} and when based on θ^0.

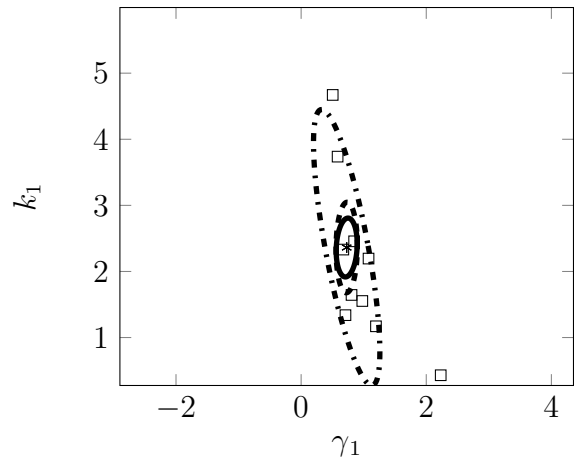

Figure 67. Example 4 – White estimates using optimal power distribution and ellipsoids based on θ^{init}. The white estimates (○) are shown, along with the application ellipsoid based on θ^{init} (▬) and on θ^0 (▪▪▪). Also, the identification ellipsoid based on the white spectrum and θ^0 is shown (·▪·▪). The true parameter values are denoted (∗)).

Application cost

We check the application cost for all white and optimal estimates. We evaluate the application cost on the process, see Figure 68 and 69. We also evaluate the application cost in simulation based on both θ^{init} and θ^0, see Figures 70–73.

We see in Figure 69 that none of the estimates fulfill the application cost evaluated on the process. However, the optimal estimates in general outperform the white estimates. One of the optimal estimates and none of the white estimates fulfill the application cost evaluated in simulation based on θ^{init}, see Figure 71. In Figure 73, we see that seven of the optimal estimates and one of the white estimates fulfill the application cost evaluated in simulation based on θ^0. The result is better in the latter case since the application cost based on θ^0 gives a larger set

of acceptable parameters than the cost based on θ^{init} does, see the ellipsoidal approximations in Figure 61.

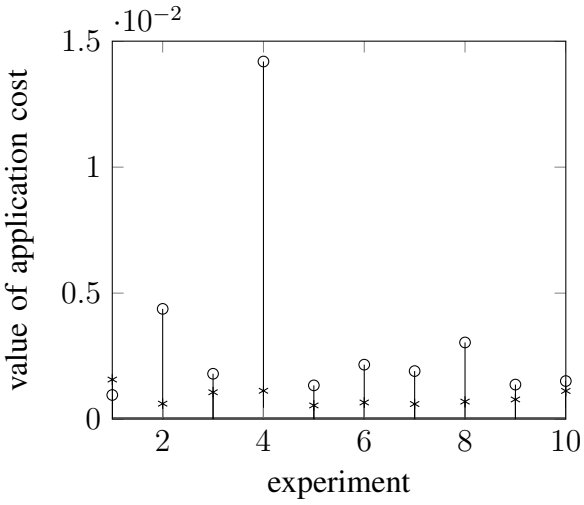

Figure 68. Example 4 – Application cost evaluated on process. The application cost for optimal estimates (∗) and white estimates (○) are shown.

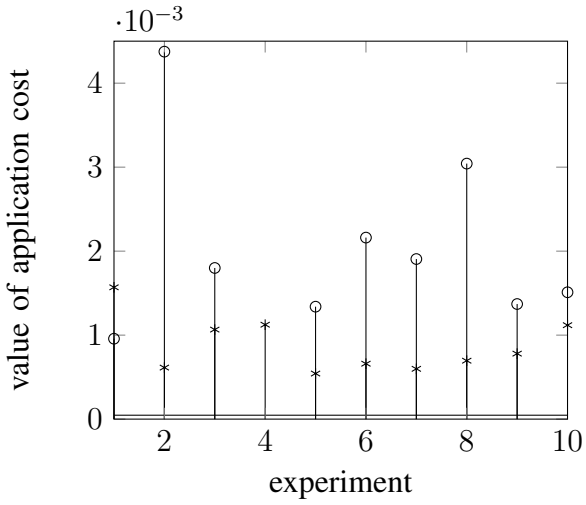

Figure 69. Example 4 – Application cost evaluated on process excluding white Experiment 4. We see that the optimal estimates in general outperforms the white estimates. However, none of the estimates fulfill the requirement of an application cost lower than $1/\gamma$, denoted (▬).

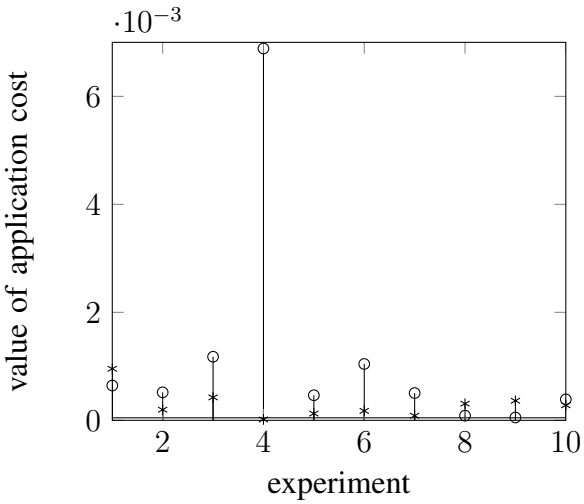

Figure 70. Example 4 – Application cost evaluated in simulation on system based on θ^{init}. The application cost for the optimal estimates ($*$) and white estimates ($\circ$) are shown.

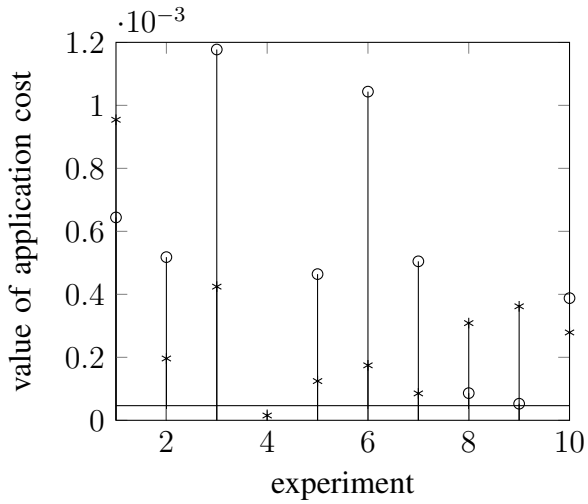

Figure 71. Example 4 – Application cost evaluated in simulation on system based on θ^{init} excluding white Experiment 4. We see that the optimal estimates in general outperforms the white estimates. One of the optimal estimates and none of the white estimates fulfill the requirement of an application cost lower than $1/\gamma$, denoted (▬).

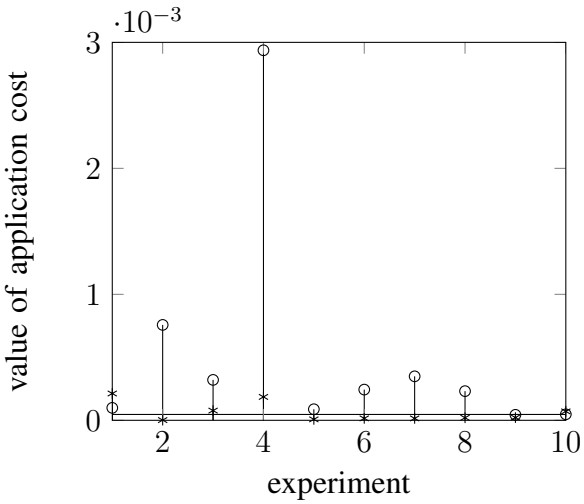

Figure 72. Example 4 – Application cost evaluated in simulation on system based on θ^0. The application cost for the optimal estimates ($*$) and white estimates ($\circ$) are shown.

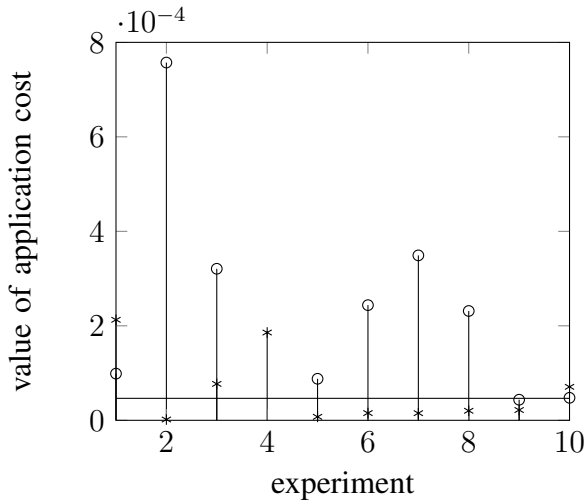

Figure 73. Example 4 – Application cost evaluated on system based on θ^0 excluding white Experiment 4. We see that the optimal estimates in general outperforms the white estimates. Seven of the optimal estimates and one of the white estimates fulfill the requirement of an application cost lower than $1/\gamma$, denoted (—).

Signals from process

The output signals from the process when evaluating the application cost are shown in Figures 74 and 75 for the optimal estimates and Figures 76 and 77 for the white estimates. The output signals when θ^0 are used in the MPC model are shown in Figures 78 and 79. All outputs fulfill the constraints imposed on the water tank levels.

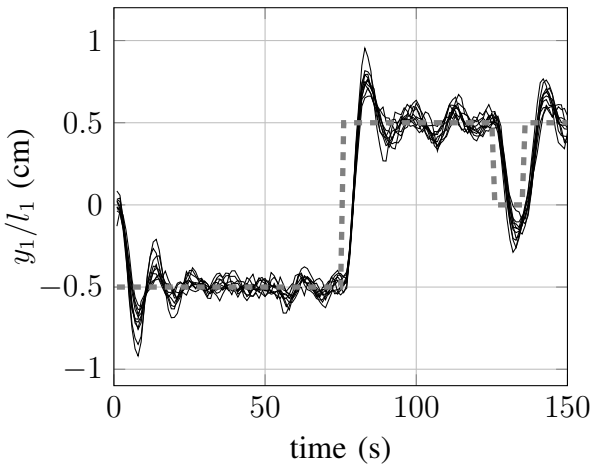

Figure 74. Example 4 – Optimal y_1/l_1 from application evaluation. The optimal y_1/l_1 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

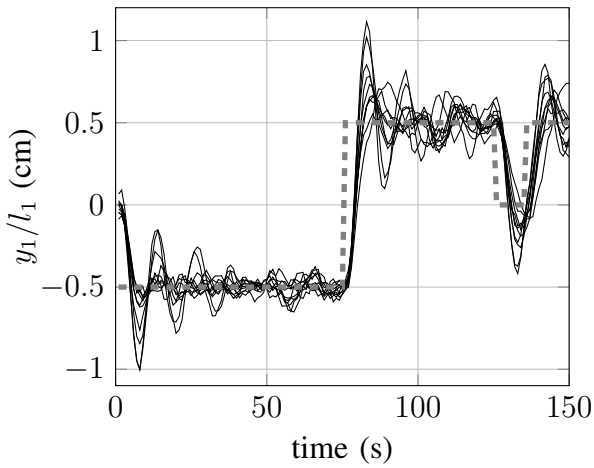

Figure 75. Example 4 – White y_1/l_1 from application evaluation. The white y_1/l_1 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

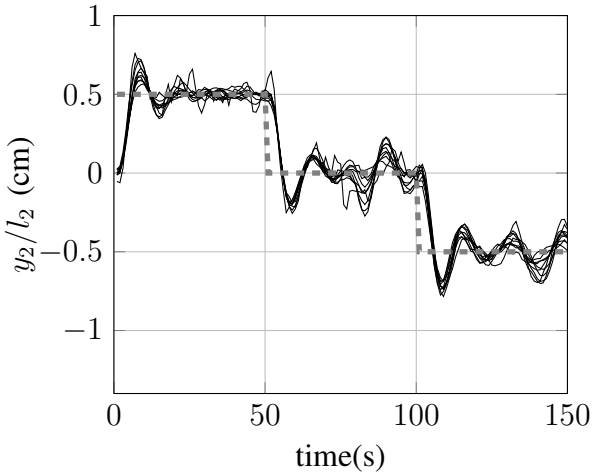

Figure 76. Example 4 – Optimal y_2/l_2 from application evaluation. The optimal y_2/l_2 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

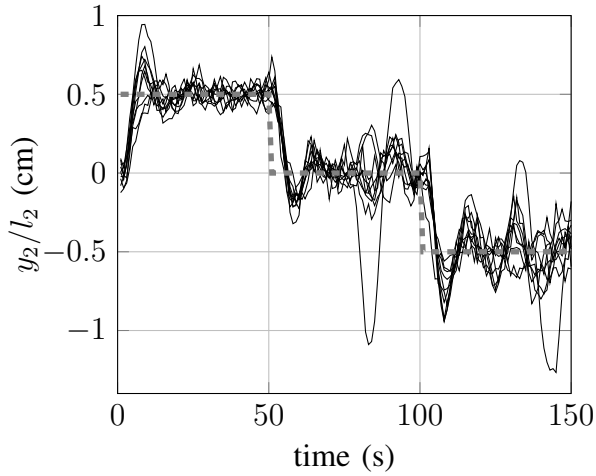

Figure 77. Example 4 – White y_2/l_2 from application evaluation. The white y_2/l_2 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

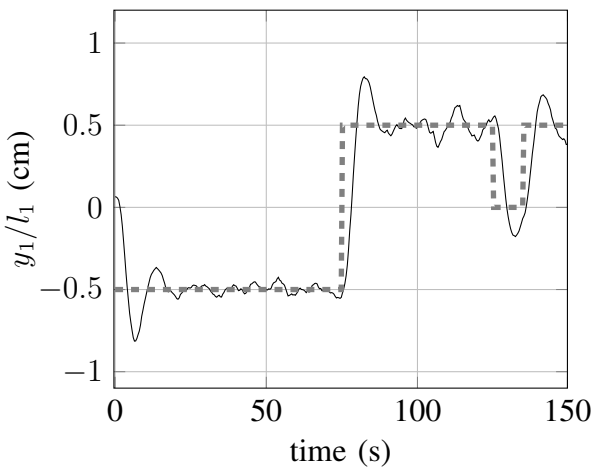

Figure 78. Example 4 – True y_1/l_1 used in application evaluation. The output y_1/l_1 (▬) is obtained from the process using θ^0 in the MPC model. The reference signal is denoted (- - -).

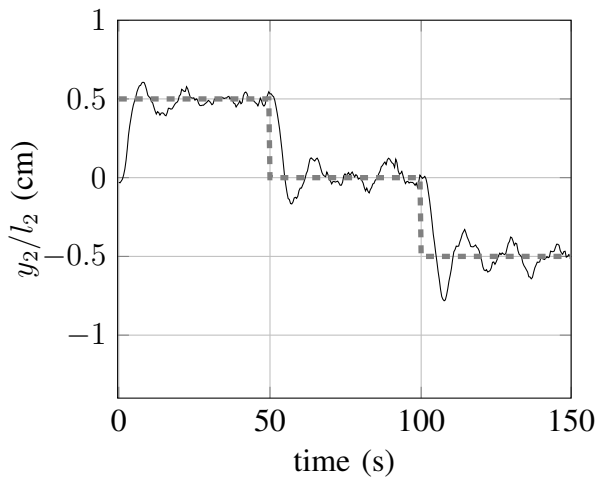

Figure 79. True y_2/l_2 used in application evaluation. The output y_2/l_2 (▬) is obtained from the process using θ^0 in the MPC model. The reference signal is denoted (- - -).

For comparison, we also checked the output signals from the process when using models obtained with the white noise with optimal covariance matrix c_0. That is, the models based on the estimates in Figure 67. The output signals when evaluating the application cost are shown in Figures 80 and 81. We see that, as with the other white noise experiments, the resulting control performance is worse than for the optimal models.

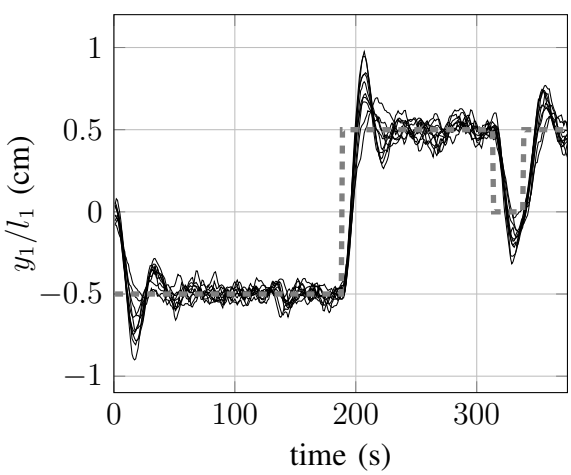

Figure 80. Example 4 – White y_2/l_2 from application evaluation (based on estimates using optimal c_0). The white y_2/l_2 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

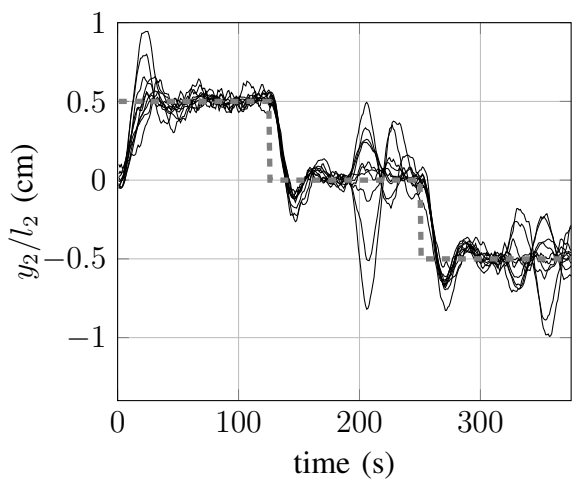

Figure 81. Example 4 – White y_2/l_2 from application evaluation (based on estimates using optimal c_0). The white y_2/l_2 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

Experiment length

We evaluate the four different experiment lengths as in the previous examples. The values of the experiment lengths are shown in Table XX. The white input signal requires, according to theory, a longer experiment length to achieve the same performance as the optimal input signal. The white input signal is approximately 42 times longer than the optimal input signal in both cases. Note also that $N_{\text{white}}^{\text{initial}}$ and N are larger than $N_{\text{white}}^{\text{true}}$ and N^{true}, respectively, which relates to the fact that the application ellipsoid based on θ^0 is larger than the application ellipsoid based on θ^{init}, see Figure 61.

TABLE XX

EXAMPLE 4 – EXPERIMENT LENGTHS.

Experiment length	Value in samples	Value in minutes
N	300	5
N^{true}	125	2
$N_{\text{white}}^{\text{initial}}$	12550	209
$N_{\text{white}}^{\text{true}}$	5194	87

Example 5 - Estimating $\theta = [a_1\ a_2\ a_3\ a_4\ \gamma_1\ \gamma_2\ k_1\ k_2]^{\mathrm{T}}$ with integral action in the MPC

We redo Example 2 but with $Q_u = 0I_{2\times 2}$. The MPC has integral action.

Check of initial estimates

We use the same intial estimates as in Example 3, that is, $\theta^{\text{init}} = [0.03\ \ 0.03\ \ 0.05\ \ 0.05\ \ 0.7\ \ 0.7\ \ 1.6\ \ 1.6]^{\mathrm{T}}$. However, with these values of θ^{init} `cvx` returns an inaccurate solution when calculating the optimal spectrum. As suggested in the `cvx`-manual, we check that the estimates are accurate enough for our purpose. We do so by checking that the linear matrix inequality constraint is approximately active, and it is. The smallest eigenvalue of the matrix should be zero and it is of the order of 10^{-7}. Before performing the input design, we check if the initial estimates fulfill the application requirement. In this check, we get that θ^{init} do not fulfill the application requirement.

Application ellipsoid

As in the other examples, we calculate two types of application ellipsoids. The first type is evaluated in simulation using a system and MPC model based on θ^{init}. The second type is evaluated in simulation using a system and MPC model based on θ^0.

In Tables XXI and XXII, we give the range of each parameter value that may fulfill the second order approximation of the application requirement centered at θ^{init} and θ^0, respectively.

As in Example 3, we get that the range of γ_1 is larger based on θ^{init} than based on θ^0. Consequently, the input design might not give estimates of these values that are accurate enough for the application in mind.

We can rank the parameters in terms of increasing importance based on absolute deviation and θ^{init} as k_2, k_1, γ_2, γ_1, a_3, a_4, a_2 and a_1, which is the same ranking as in Example 3. The ranking based on θ^0 is a_3, k_2, k_1, γ_2, a_4, γ_1, a_2 and a_1.

The order of increasing importance is different in terms of relative deviation than absolute deviation. The ranking based on relative deviation and the initial design centered at θ^{init} (Table XXI) is a_3, a_4, a_2, a_1, k_2, γ_2, k_1 and γ_1. The ranking based on relative deviation, θ^{init} and centered at θ^0 (Table XXII) is k_2, a_4, γ_2, a_2, a_3, a_1, γ_1 and k_1. The ranking based on relative deviation, θ^0 and centered at θ^0 (Table XIII) is a_3, a_4, k_2, a_2, a_1, k_1, γ_2 and γ_1.

Despite the different rankings, they all state that a_1 and a_2 are more important than a_3 and a_4, and γ_1 and γ_2 are more important than k_1 and k_2 to estimate with higher accuracy. Except the ranking based on relative deviation, θ^{init} and centered at θ^{init} (Table XXII). Here, a_3 is deemed more important than a_2.

TABLE XXI

EXAMPLE 5 – POSSIBLE RANGE OF EACH PARAMETER VALUE IN THE INITIAL DESIGN.

θ	θ^{init}	Possible range based on θ^{init}	
a_1	0.03	0.03 ± 0.0201	($\pm 67\%$)
a_2	0.03	0.03 ± 0.0256	($\pm 85\%$)
a_3	0.05	0.05 ± 0.1002	($\pm 200\%$)
a_4	0.05	0.05 ± 0.0643	($\pm 129\%$)
γ_1	0.7	0.7 ± 0.2207	($\pm 32\%$)
γ_2	0.7	0.7 ± 0.3147	($\pm 45\%$)
k_1	1.6	1.6 ± 0.6553	($\pm 41\%$)
k_2	1.6	1.6 ± 1.0428	($\pm 65\%$)

TABLE XXII

EXAMPLE 5 – POSSIBLE RANGE OF EACH PARAMETER VALUE.

θ	θ^0	Possible range based on θ^{init}		Possible range based on θ^0	
a_1	0.0649	0.0649 ± 0.0201	($\pm 31\%$)	0.0649 ± 0.0346	($\pm 53\%$)
a_2	0.0594	0.0594 ± 0.0256	($\pm 43\%$)	0.0594 ± 0.0422	($\pm 71\%$)
a_3	0.3211	0.3211 ± 0.1002	($\pm 31\%$)	0.3211 ± 3.2994	($\pm 1028\%$)
a_4	0.1353	0.1353 ± 0.0643	($\pm 48\%$)	0.1353 ± 0.2423	($\pm 179\%$)
γ_1	0.7283	0.7283 ± 0.2207	($\pm 30\%$)	0.7283 ± 0.1771	($\pm 24\%$)
γ_2	0.7271	0.7271 ± 0.3147	($\pm 43\%$)	0.7271 ± 0.3049	($\pm 42\%$)
k_1	2.362	2.362 ± 0.6553	($\pm 28\%$)	2.362 ± 1.0093	($\pm 43\%$)
k_2	1.797	1.797 ± 1.0428	($\pm 58\%$)	1.797 ± 1.2946	($\pm 72\%$)

Optimal spectrum

We calculate the optimal spectrum using 20 parameters $c_0, \ldots, c_{19}$, where $c_i \in R^{2\times 2}$. The optimal spectrum is shown in Figure 82, along with the optimal spectrum based on θ^0. The design based on θ^{init} requires a smaller power, trace$(c_0) = 0.027$, than the design based on θ^0, where trace$(c_0) = 0.0513$.

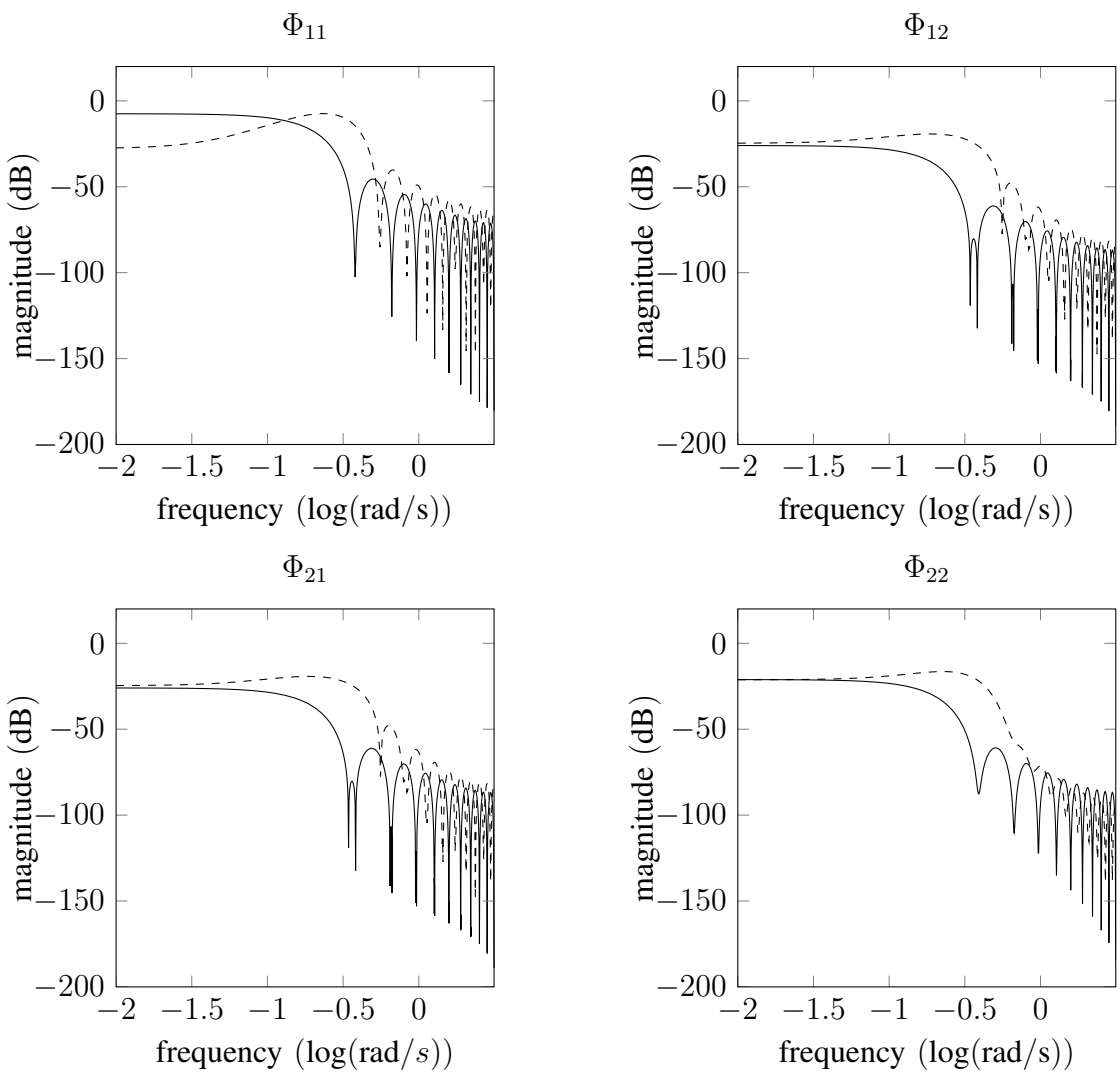

Figure 82. Example 5 – Optimal spectrum. The optimal spectrum based on θ^{init} (▬) and θ^0 (- - -) are shown. We see that u_1 and u_2 are correlated.

Identification ellipsoid

We calculate the same two kinds of identification ellipsoids as in the other examples.

In Table XXIII, we give the possible range of each estimated parameter value according to the identification ellipsoid.

We can rank the parameters in order of increasing importance. In terms of absolute deviation both spectra give the same ranking, that is, a_3, k_2, k_1, a_4, γ_2, γ_1, a_2 and a_1, which is the same ranking as in Example 3. In terms of relative deviation the optimal spectrum gives a_4, a_3, a_2, k_2, a_1, γ_2, k_1 and γ_1. The white spectrum gives a_3, a_4, a_2, k_2, k_1, a_1, γ_2 and γ_1.

The rankings differ from those obtained from the application ellipsoids. However, they all state that a_1 and a_2 are more important than a_3 and a_4, and γ_1 and γ_2 are more important than k_1 and k_2 to estimate with higher accuracy. Except for the optimal spectra where k_1 is deemed slightly more important than γ_2. We also see that the identification ellipsoid based on the optimal spectrum has a smaller possible range of each parameter than the ellipsoid based on the white spectrum. The volume of the white identification ellipsoid is 10617 times larger than the volume of the optimal identification ellipsoid.

TABLE XXIII

EXAMPLE 5 – POSSIBLE RANGE OF EACH ESTIMATED PARAMETER VALUE.

θ	θ^0	Possible range based on θ^0 and optimal spectrum		Possible range based on θ^0 and white spectrum	
a_1	0.0649	0.0649 ± 0.0128	$(\pm 20\%)$	0.0649 ± 0.0275	$(\pm 42\%)$
a_2	0.0594	0.0594 ± 0.0238	$(\pm 40\%)$	0.0594 ± 0.0503	$(\pm 85\%)$
a_3	0.3211	0.3211 ± 1.5253	$(\pm 48\%)$	0.3211 ± 2.6996	$(\pm 841\%)$
a_4	0.1353	0.1353 ± 0.2317	$(\pm 171\%)$	0.1353 ± 0.8238	$(\pm 609\%)$
γ_1	0.7283	0.7283 ± 0.0659	$(\pm 9\%)$	0.7283 ± 0.2092	$(\pm 29\%)$
γ_2	0.7271	0.7271 ± 0.1127	$(\pm 16\%)$	0.7271 ± 0.2536	$(\pm 35\%)$
k_1	2.362	2.362 ± 0.3540	$(\pm 15\%)$	2.362 ± 1.0865	$(\pm 46\%)$
k_2	1.797	1.797 ± 0.5278	$(\pm 29\%)$	1.797 ± 1.1000	$(\pm 61\%)$

Estimates

We estimate θ using the optimal input signal and the white input signal. We perform ten identification experiments for each type of signal. A new realization of the input signal is used in each experiment.

As in Example 2, it is difficult to estimate reasonable values of a_3 using the optimal signal. In five of the optimal estimates and three of the white estimates it is larger than 10^3. We believe that this is due to the small water tank level in steady state of Tank 3.

All ten optimal estimates are displayed in Table XXIV.

TABLE XXIV

EXAMPLE 5 – ESTIMATED PARAMETERS USING OPTIMAL INPUT SIGNAL.

θ	Exp. 1	Exp. 2	Exp. 3	Exp. 4	Exp. 5	Exp. 6	Exp. 7	Exp. 8	Exp. 9	Exp. 10
a_1	0.0712	0.0504	0.0790	0.0736	0.0512	0.0450	0.0627	0.0496	0.0799	0.0431
a_2	0.0403	0.0395	0.0640	0.0551	0.0654	0.0622	0.0448	0.1252	0.1100	0.0372
a_3	0.0037	0.0151	5×10^6	7×10^6	2×10^5	0.0019	0.0300	3×10^5	9×10^3	0.0026
a_4	0.0945	0.0231	0.0441	0.0454	0.0666	0.0342	0.0560	0.0379	0.0415	0.0903
γ_1	0.7657	0.7287	0.6939	0.7221	0.7094	0.6048	0.6790	0.5794	0.7160	0.7110
γ_2	0.5060	0.3980	0.7576	0.6801	0.8279	0.5780	0.9082	0.7953	0.6665	0.4096
k_1	1.9743	1.7920	2.7837	2.4012	2.4329	2.0632	2.0718	2.5144	2.9544	2.1190
k_2	1.7799	2.4986	1.6727	2.0309	1.6630	2.5898	1.4053	2.4632	2.6151	3.3532

Application cost

We check the application cost for all white and optimal estimates. We evaluate the application cost on the process, see Figures 83. We also evaluate the application cost in simulation based on both θ^{init} and θ^0, see Figures 84 and 85. The requirement on the application cost is not fulfilled in any of the experiments, but the optimal models generally get a lower application cost than the white models do. In addition, none of the models fulfill the second order approximation of the application requirement when based on θ^{init}, but one of the optimal models fulfills it when based on θ^0.

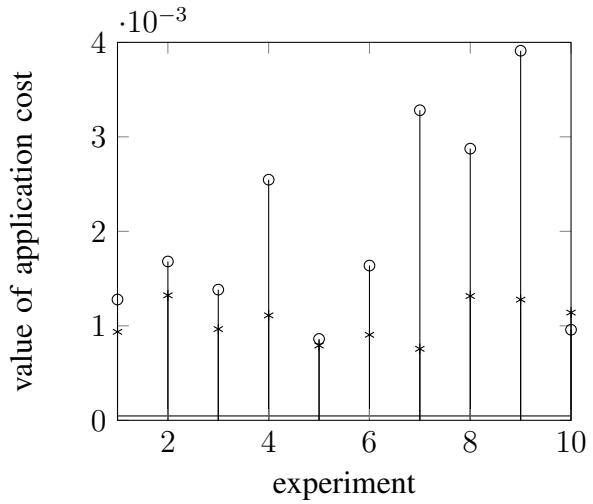

Figure 83. Example 5 – Application cost evaluated on process. The application cost for optimal estimates ($*$) and white estimates ($\circ$) are shown.

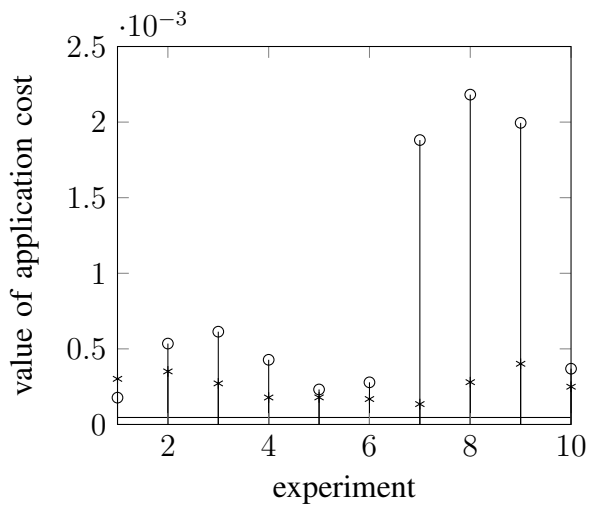

Figure 84. Example 5 – Application cost evaluated in simulation on system based on θ^{init}. The application cost for the optimal estimates ($*$) and white estimates ($\circ$) are shown.

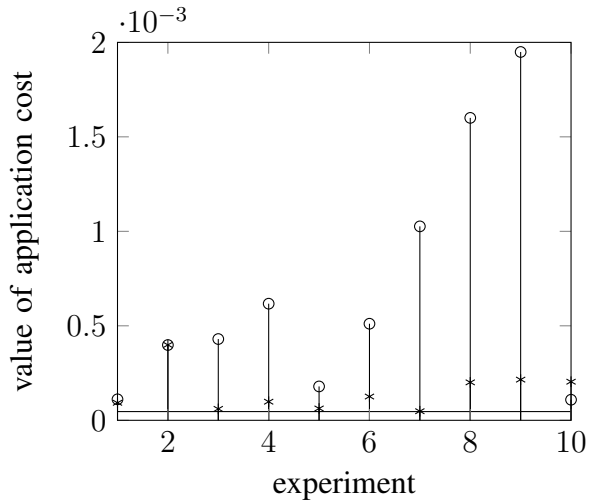

Figure 85. Example 5 – Application cost evaluated in simulation on system based on θ^0. The application cost for the optimal estimates ($*$) and white estimates ($\circ$) are shown.

Signals from process

The output signals from the process when evaluating the application cost are shown in Figures 86 and 88 for the optimal estimates and Figures 87and 89 for the white estimates. The output signals from the process when using θ^0 in the MPC model are shown in Figures 90 and 91. We see that the MPC models based on optimal estimates performs better than MPC models based on white estimates, especially when considering y_2. All outputs fulfill the constraints imposed on the water tank levels.

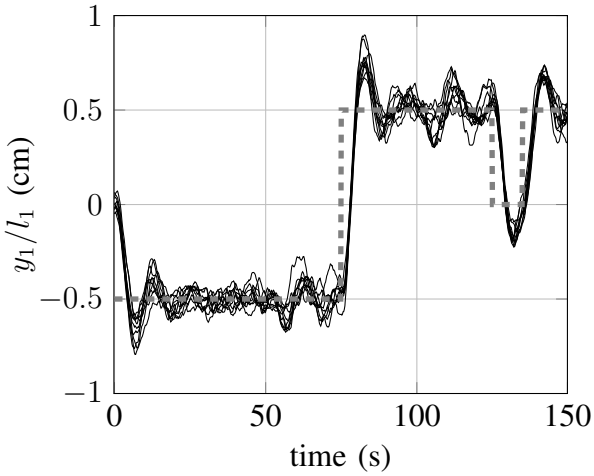

Figure 86. Example 5– Optimal y_1 from application evaluation. The optimal output y_1/l_1 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

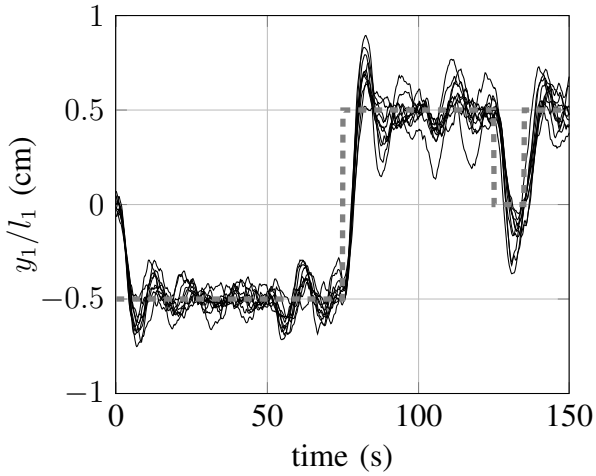

Figure 87. Example 5 – White y_1 from application evaluation. The white output y_1/l_1 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

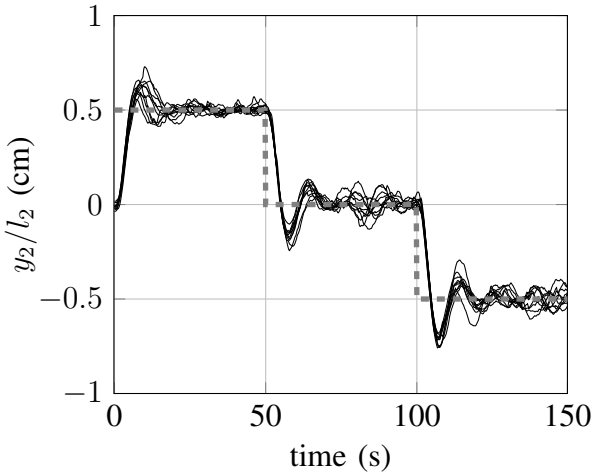

Figure 88. Example 5 – Optimal y_2 from application evaluation. The optimal output y_2/l_2 (▬) from ten application evaluations on the process are shown. The reference signal is denoted (- - -).

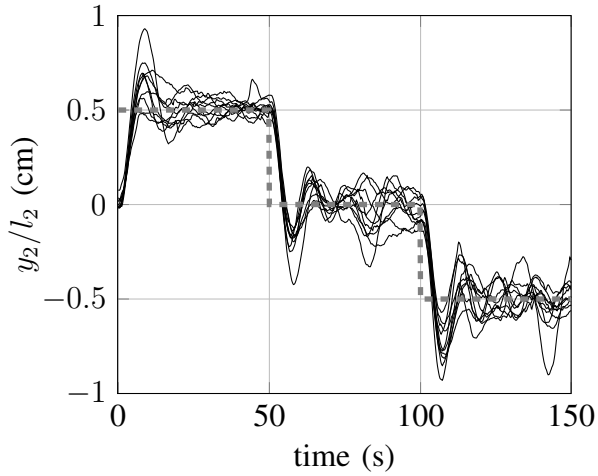

Figure 89. Example 5 – White y_2 from application evaluation. The white output y_2/l_2 (▬) ten application evaluations on the process are shown. The reference signal is denoted (- - -).

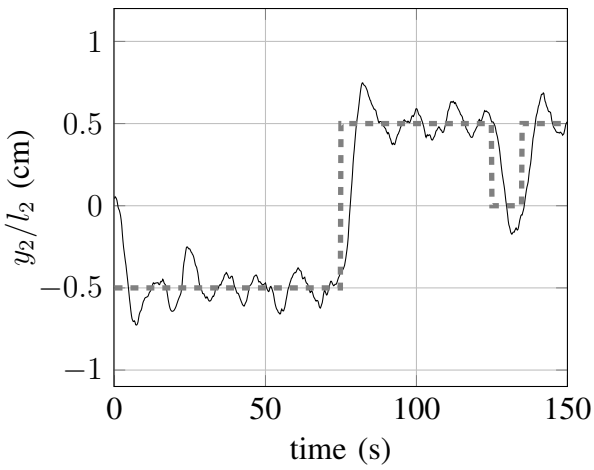

Figure 90. Example 5 – True y_1 used in application evaluation. The true output y_2/l_2 (▬) from the process is shown. The reference signal is denoted (- - -).

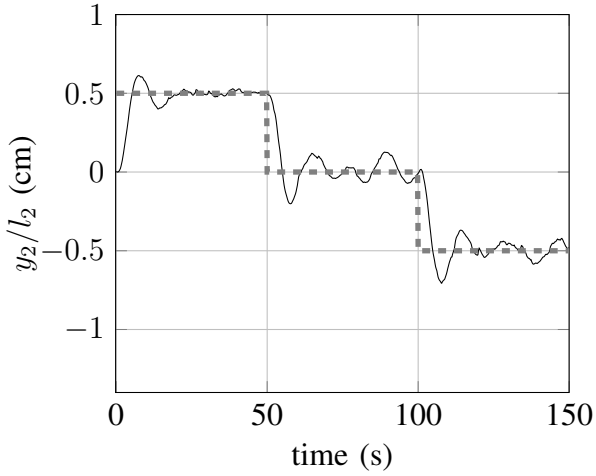

Figure 91. Example 5 – True y_2 used in application evaluation. The true output y_2/l_2 (▬) from the process is shown. The reference signal is denoted (- - -).

Experiment length

As in the previous examples, we evaluate four different experiment lengths. The values of the experiment lengths are shown in Table XXV. We see that the white input signal requires, according to theory, a longer experiment length to achieve the same performance as the optimal input signal. The white input signal is approximately 15 times longer than the optimal input signal when based on the initial estimates and 3.5 times longer when based on the true parameter values.

TABLE XXV

EXAMPLE 5 – EXPERIMENTAL LENGTHS.

Experiment length	Value in samples	Value in minutes
N	300	5
$N_{\text{white}}^{\text{initial}}$	4486	75
$N_{\text{white}}^{\text{true}}$	3205	53
N^{true}	927	15

Comments

We give some comments and reflections regarding the result obtained.

Important directions

In applications oriented input design we only consider parameter directions in an absolute sense. As seen in the examples, the important parameters can change when considering relative measure instead of absolute measure. Consequently, one should have models with parameter values of the same order of magnitude.

The initial estimate used in the design might have parameter values of wrong magnitude. This could lead to a design based on important directions that are different from the actual important directions. A remedy for this scenario is to use adaptive applications oriented input design. That is, to do the design and the identification experiment based on the initial estimate, and then to redo the whole procedure but with the new estimate as initial estimate. As we use more and more data in the identification experiment, we expect the parameter estimate to get better and better.

Initial estimate

The initial estimate greatly affect the input design, as seen in Example 2 and Example 3. In Example 2, we do not gain much from designing the input spectrum compared to using white noise of equal power. However, in all the experiments made we never got a worse control performance from designing the spectra compared to using white noise.

As suggested in the previous section, we can compensate for the input design's dependency on the accuracy of the initial estimate by using adaptive input design.

Approximative design

Applications oriented input design is based on a second order approximation of the application requirement. Depending on the shape of the application cost, the application ellipsoid might be much bigger or much smaller than the actual region that gives acceptable application cost. One could increase the value of γ to get a better second order approximation, but that also puts higher demand on the application performance. One could also use the scenario approach instead of the ellipsoidal approximation, see [10].

Application oriented input design is also based on a particular linear model structure. Of course, there are no true parameter values that ensures that the model captures the real nonlinear system. Consequently, this will also lead to an approximative design.

Noise

Applications oriented input design does not take noise into account. As seen in the plots of the application costs evaluated on the process, the noise increases the value of the application cost. It is not evident how to compensate for this increase in the design.

Stability

We have not taken closed-loop stability into account. However, this could be done (implicitly and approximately) when defining the application ellipsoid. That is, that one ensures that all values inside the application ellipsoid gives a closed-loop stable system for the given controller.

Completely wrong estimates

In Examples 2, 3 and 5, we have problems estimating either a_3 or a_4. Even though some of the estimates are completely wrong and the second order approximation of the application requirement for those estimates are several magnitudes larger than the acceptable level, they still give good performance when considering the output signals of the process. Meaning, the accuracy of the model is not important – the actual performance of the model is!

Application cost

In theory the application cost is a formal measure of the important quality of the considered system, and γ is a formal upper limit on the degradation of the quality. However, this is not the case when used in practice. All the issues stated above ensure that the application requirement is most likely *not* fulfilled when evaluated in practice (that is, not in simulation on a simulated process). Therefore, when applying applications oriented input design in practice, one should think of the application cost and γ as tunable parameters in an identification method. They are part of a tool that helps the user find models with good application performance, not

necessarily models that fulfill $V_{\text{app}}(\theta) \leq \frac{1}{\gamma}$.

References

[1] K. H. Johansson, "The quadruple-tank process: A multivariable laboratory process with an adjustable zero," *IEEE Transactions on Control Systems Technology*, vol. 8, no. 3, pp. 456–465, 2000.

[2] L. Ljung, *System Identification: Theory for the User (2nd Ed.)*. Prentice Hall, 1999.

[3] H. Hjalmarsson, "System identification of complex and structured systems," *European Journal of Control*, vol. 15, no. 3, pp. 275–310, 2009.

[4] C. Larsson, M. Annergren, and H. Hjalmarsson, "On optimal input design in system identification for model predictive control," in *50th IEEE Conference on Decision and Control*, 2010.

[5] J. M. Maciejowski, *Predictive Control with Constraints*. Prentice-Hall, 2002.

[6] M. Grant and S. Boyd, "CVX: Matlab software for disciplined convex programming, version 2.1," 2014. Available at www.cvxr.com/cvx.

[7] M. Grant and S. Boyd, "Graph implementations for nonsmooth convex programs," in *Recent Advances in Learning and Control*, Lecture Notes in Control and Information Sciences, pp. 95–110, Springer-Verlag Limited, 2008.

[8] K. C. Toh, M. Todd, R. Ttnc, and R. H. Tutuncu, "Sdpt3 - a matlab software package for semidefinite programming," *Optimization Methods and Software*, vol. 11, pp. 545–581, 1998.

[9] K. R. Muske and J. B. Rawlings, "Model predictive control with linear models," *American*

Institute of Chemical Enigneers' Journal, vol. 39, no. 2, pp. 262–287, 1993.

[10] C. A. Larsson, C. R. Rojas, and H. Hjalmarsson, "MPC oriented experiment design," in *Proceedings of the 18th IFAC World Congress*, (Milano), 2011.