We present an Alternating Direction Method of Multipliers (ADMM) algorithm for solving optimization problems with an $l_1$ regularized least-squares cost function subject to recursive equality constraints. The optimization problem has applications in control, such as $l_1$ regularized MPC, [1].

$l_1$ Regularized MPC

Control objective: Drive the output $y$ to zero, using few changes of the input $u$.

MPC idea: A cost function is minimized with respect to $u(t+i)$ for some time horizon $i = 0, \ldots, H_u - 1$. Only $u(t)$ is applied to the system. The cost function is minimized again in an iterative manner for each time step $i$.

Cost function:

$$V(t) = \sum_{i=1}^{H_u} ||y(t+i-1)||_2^2 + \lambda \sum_{i=1}^{H_u} ||\Delta u(t+i-1)||_1,$$

where $\Delta u(t+i) = u(t+i) - u(t+i - 1)$. The $l_1$-norm of $\Delta u$ promotes sparse $\Delta u$.

Optimization problem solved in each time step $t$:

minimize $V(t)$, subject to $x(t+i) = Ax(t+i-1) + Bu(t+i-1)$, $y(t+i-1) = Cx(t+i-1)$, $i = 1, \ldots, H_p$.

ADMM

Optimization problem: ADMM is an algorithm for solving

minimize $f(x)$, subject to $x \in \mathcal{C}$,

subject to $x = x_c$,

where $f(x)$ is a convex function, $\mathcal{C}$ is a convex set and $I_{\mathcal{C}}(x_c)$ is the indicator function of $\mathcal{C}$.

ADMM at iteration $k$:

Step 1 $x^{k+1} := \arg\min_x \{f(x) + (\rho/2)||x - x^k + x_{d}^k||_2^2\}$.

Step 2 $x_{c}^{k+1} := \arg\min_{x_c} \{I_{\mathcal{C}}(x_c) + (\rho/2)||x^{k+1} - x_c + x_{d}^{k+1}||_2^2\}$.

Step 3 $x_{d}^{k+1} := x_{d}^{k} + (x^{k+1} - x_{c}^{k+1})$.

Here $x_{d}$ is the dual variable of (2) scaled by $1/\rho$, $\rho > 0$.

Stopping criteria: ADMM is iterated until stopping criteria based on the norms of the primal and dual residuals of (2) are fulfilled. The residuals are

$e_p^k = (x^k - x_{c}^k)$, $e_d^k = -\rho(x_{d}^k - x_{d}^{k-1})$.

$l_1$ Regularized MPC and ADMM: The optimization problem (1) can be formulated as (2), and solved using ADMM. Step 1 and Step 3 are straightforward to perform. Step 2 is a projection, which we solve using a Riccati recursion.

Control of Tank Process

A tank process is controlled using $l_1$ regularized MPC. The MPC controls the pump voltages, $u_1$ and $u_2$. The water levels of the two lower tanks, $y_1$ and $y_2$, are the outputs.

The control objective is to drive $y_1$ and $y_2$ to the equilibrium points, using few changes of $u_1$ and $u_2$.

Conclusion

- $l_1$ Regularized MPC.
- ADMM is easy to implement.
- ADMM converges fast to moderate accuracy.
- ADMM enables parallel execution.

Further Reading